Classifications of exact structures and Cohen-Macaulay-finite algebras

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In this talk, I will discuss a classification of exact structures on a given additive category and its application, based on [1]. Exact categories, in the sense of Quillen, have been playing an important role in the representation theory of algebras. In general, an additive category has many exact structures. Recently, Rump [3] showed that every additive category has the largest exact structures, but no general description of exact structures was known. We give an explicit description of all exact structures on a given additive category $\mathcal{E}$ by using particular modules over $\mathcal{E}$ (equivalently, modules over the Auslander algebra of $\mathcal{E}$).

Let k be a field. For simplicity, all algebras are assumed to be finite dimensional over k. To this end, the following condition for simple modules plays an indispensable role.

Definition 1. Let Γ be an algebra and S a simple Γ-module. We say that S *satisfies the* 2*-regular condition* if the following conditions are satisfied.

(1) The projective dimension of S is equal to 2.
(2) $\mathrm{Ext}^i_\Gamma(S,\Gamma)=0$ for $i=0,1$.
(3) $\mathrm{Ext}^2_\Gamma(S,\Gamma)$ is simple Γ^{op}-module.

Surprisingly, the following shows that categorical notion (exact structures) is deeply related to homological condition (2-regular conditions). Also this can be seen as a classification of exact categories with finitely many indecomposables.

Theorem 2. *Let $\mathcal{E}$ be an idempotent complete Hom-finite additive k-category with finitely many indecomposables, and let Γ be its Auslander algebra. Then there exists a bijection between the following two classes.*

(1) *Exact structures on $\mathcal{E}$.*
(2) *Sets of simple Γ-modules satisfying the* 2*-regular condition.*
(3) *Sets of dotted arrows in the translation quiver $Q(\Gamma)$ associated with Γ.*

As an application, we give the Auslander-type correspondence for Cohen-Macaulay-finite Iwanaga-Gorenstein algebras. We say that an algebra Λ is *Iwanaga-Gorenstein* if the left and right injective dimension of Γ itself is finite. For such an algebra Λ, a finitely generated Γ-module X is called *Cohen-Macaulay* if $\mathrm{Ext}^i_\Lambda(X,\Lambda)=0$ for all $i>0$. We say that an Iwanaga-Gorenstein algebra is *Cohen-Macaulay-finite* (CM-finite) if there exist finitely many Cohen-Macaulay modules up to isomorphism. By using the previous theorem and the results in [2], we proved the following.

Theorem 3. *There exists a bijection between the following two classes.*

(1) *Morita-equivalence classes of CM-finite Iwanaga-Gorenstein algebras.*
(2) *Equivalence classes of pairs (Γ,X), where Γ is an algebra with finite global dimension and X is a union of stable τ-orbits in the translation quiver $Q(\Gamma)$.*

Moreover, we give an explicit method to construct a CM-finite algebra from the pair (Γ,X). This gives a systematic method to construct CM-finite Iwanaga-Gorenstein algebras, and all such algebras are obtained in this way. Thus our result reduces the classification problem of CM-finite Iwanaga-Gorenstein algebras to that of algebras with finite global dimension.

References

[1] H. Enomoto, *Classifications of exact structures and Cohen-Macaulay-finite algebras*, arXiv:1705.02163.
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[3] W. Rump, *On the maximal exact structure on an additive category*, Fund. Math. **214** (2011), no. 1, 77–87.

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