The lattice of wide subcategories

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@ 代数学若手研究会

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§1. Intro

Λ : ring

$$\boxed{\begin{array}{l}\text{Representation theory of } \Lambda \\ \qquad = \text{Study } \Lambda\text{-modules}\end{array}}$$

?

One direction : Study "good" subcategories of the cat. of Λ-modules

$\rightsquigarrow$ We obtain various posets (ordered by inclusion)

Today

Consider 2 classes of subcats of $\text{mod}\,\Lambda$:

1. Torsion classes
2. Wide subcategories

and describe the relation.

1. Torsion classes $\xrightarrow{\text{posets of}}$ tors Λ

2. Wide subcats $\xrightarrow[\text{subcats}]{}$ wide Λ

Main Theorem

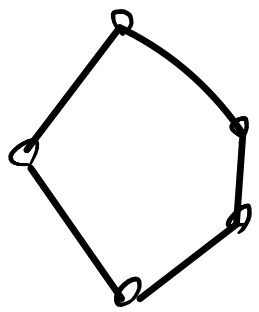
The poset wide Λ can be computed from the poset tors Λ

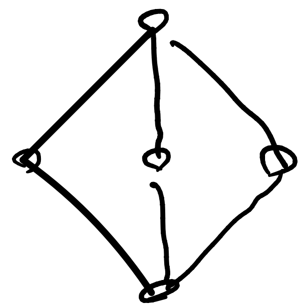
Example

$$\Lambda = \begin{bmatrix} k & 0 \\ k & k \end{bmatrix}$$

2 "combinatorial invariants"

tors Λ $\xrightarrow{\text{Main Thm}}$ wide Λ





§2. Torsion classes and wide subcategories

Torsion class

Setting

- k : a field
- Λ : a fin. dim k-algebra
- $\mathrm{mod}\,\Lambda$: the category of finitely generated Λ-modules

  ($\rightsquigarrow$ abelian category)
- "subcat" = full additive subcat, closed under isom

Def (Dickson 1966)

$\mathcal{T} \subseteq \mathrm{mod}\,\Lambda$ is a torsion class

if (i) $\mathcal{T}$ is closed under extensions , i.e.,

$$\forall\ 0 \to L \to M \to N \to 0 : \text{s.e.s},\ L, N \in \mathcal{T} \Rightarrow M \in \mathcal{T}$$

(ii) $\mathcal{T}$ is closed under quotients , i.e.,

$$\forall\ M \twoheadrightarrow N : \text{surj},\ M \in \mathcal{T} \Rightarrow N \in \mathcal{T}$$

Torsion class ($\Leftarrow$ closed under ext, quotient)

Def

$\text{tors}\,\Lambda := \{\text{torsion classes in } \text{mod}\,\Lambda\}$

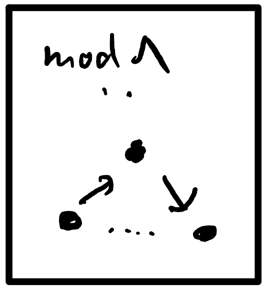
$\rightsquigarrow$ poset (partially ordered set) with inclusion.

Example

- $0, \text{mod}\,\Lambda \in \text{tors}\,\Lambda$
- $\{\text{torsion abelian grps}\} \subset \text{mod}\,\mathbb{Z}$: torsion class
- $\text{tors}\,k = \{0, \text{mod}\,k\}$

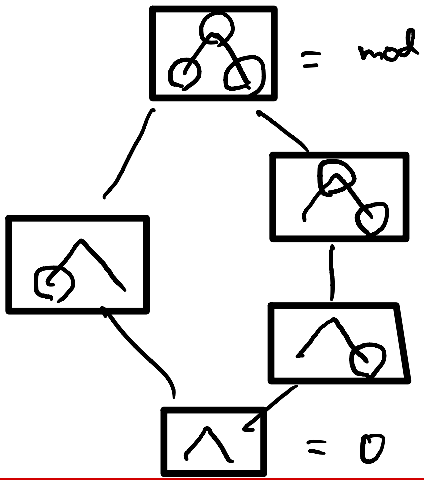
Running example

$\Lambda := T_2(k) := \begin{bmatrix} k & 0 \\ k & k \end{bmatrix} \rightsquigarrow$



3 indecs

$\rightsquigarrow$ $\text{tors}\,\Lambda$:



$= \text{mod}\,\Lambda$

$= 0$

(Hasse diagram)

Torsion class

Remark

$$T_n(K) := \overbrace{\begin{bmatrix} K & & 0 \\ \vdots & \ddots & \\ K & \cdots & K \end{bmatrix}}^{n} \rightsquigarrow \text{tors}\, T_n(K) \cong \text{the Tamari lattice}$$

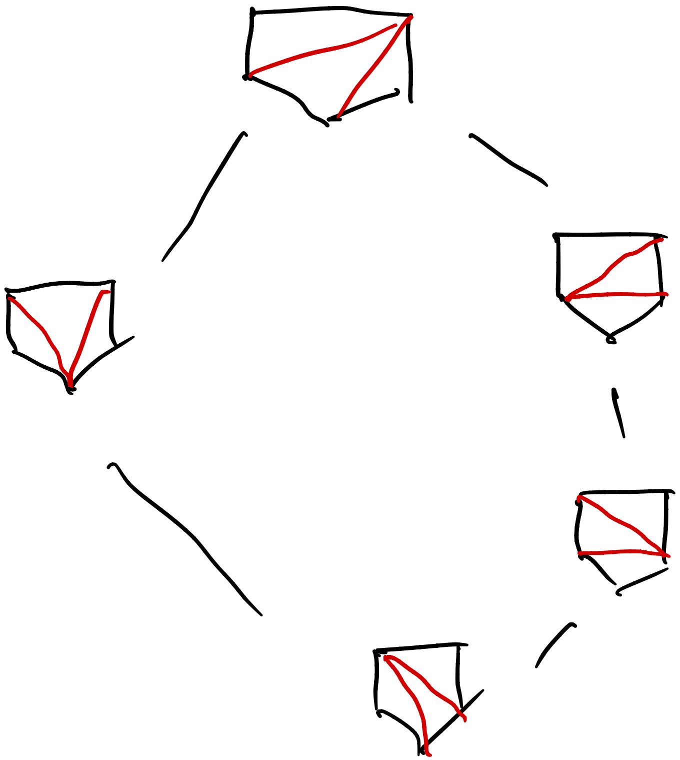
("Catalan family" in combinatorics)

$$\# \text{tors}\, T_n(K) = (n+1)\text{-th Catalan number}$$

$$\frac{1}{n+2}\binom{2n+2}{n+1}$$

e.g.

$n=2 \rightsquigarrow$ tors $\underset{T_n(K)}{T_2(K)} \cong$ poset of triangulation of $\underset{(n+3)\text{-gon}}{(2+3)\text{-gon}}$



$\rightsquigarrow$ related to cluster alg, ...

Torsion class

Remark

- Torsion classes are one of the main topics in the recent study of rep. theory of alg.
- "Mutation" of τ-tilting modules [Adachi-Iyama-Reiten 2014] gives method to compute the poset tors Λ. (in good situation)

Wide subcategories

Def [Hovey 2001]

- $\mathcal{W} \subseteq \operatorname{mod}\Lambda$ is a **wide subcategory** if

  (i) $\mathcal{W}$ is closed under extensions

  (ii) $\mathcal{W}$ is closed under kernels and cokernels

  exact abelian subcat

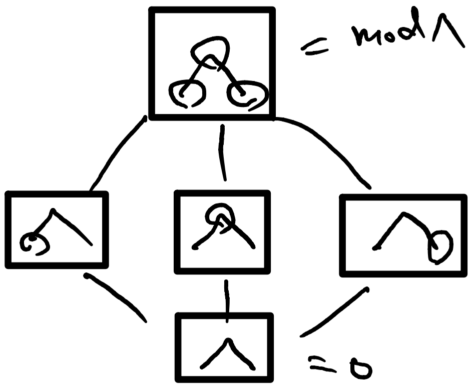
- $\operatorname{wide}\Lambda := \{\text{wide subcats of } \operatorname{mod}\Lambda\} \rightsquigarrow$ poset by inclusion.

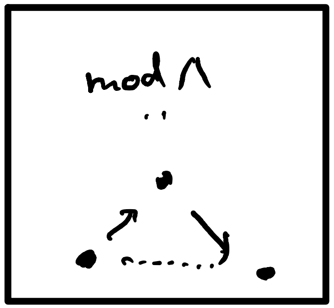
Example

- $0,\ \operatorname{mod}\Lambda \in \operatorname{wide}\Lambda$
- $\operatorname{wide} K = \{0, \operatorname{mod} K\} = \operatorname{tors} K$

Running Example

$\operatorname{wide}\begin{bmatrix} K & 0 \\ K & K \end{bmatrix}$:





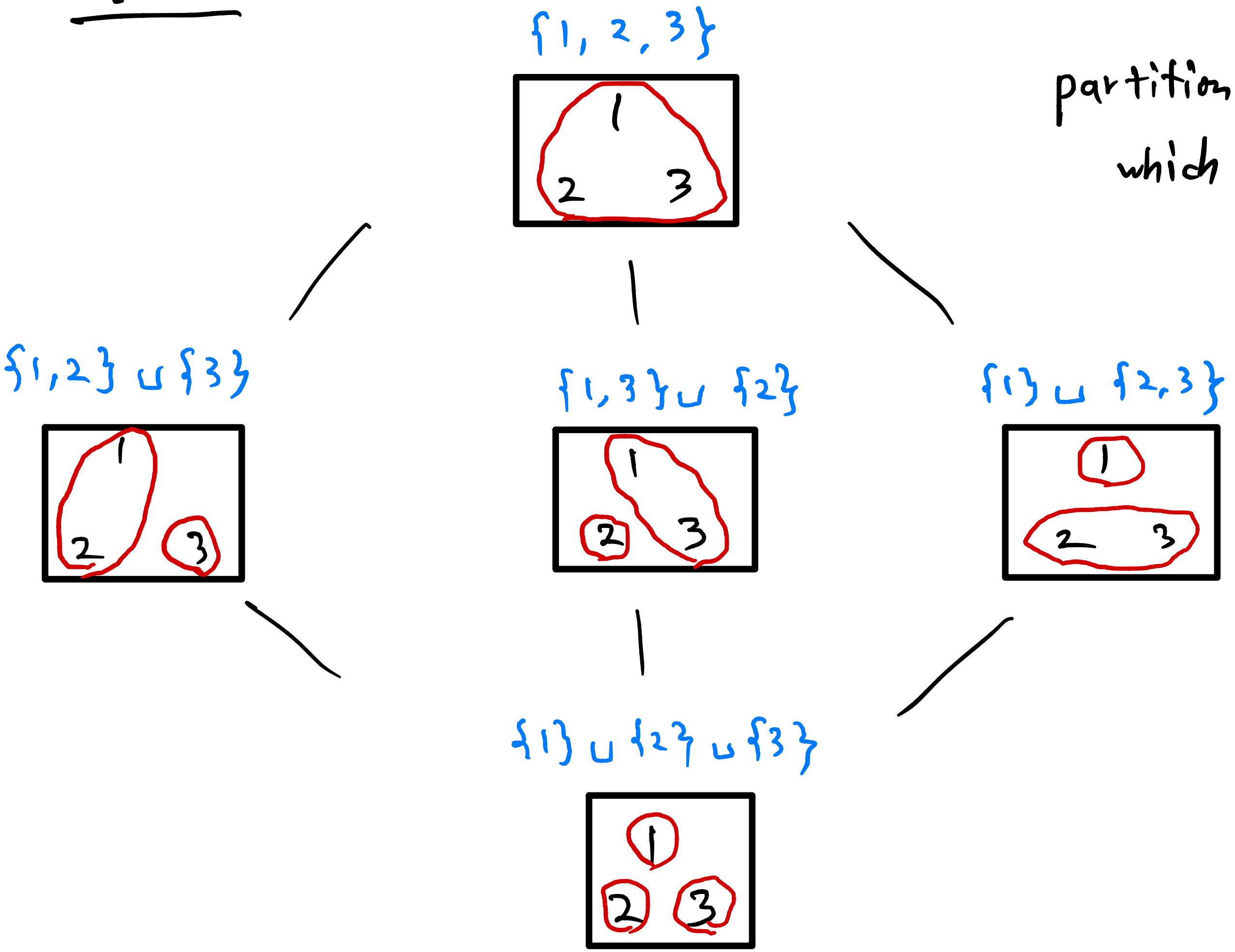
Wide subcategories

Remark $\qquad T_n(k) = \overbrace{\begin{bmatrix} k & & 0 \\ & \ddots & \\ k & & k \end{bmatrix}}^{n}$

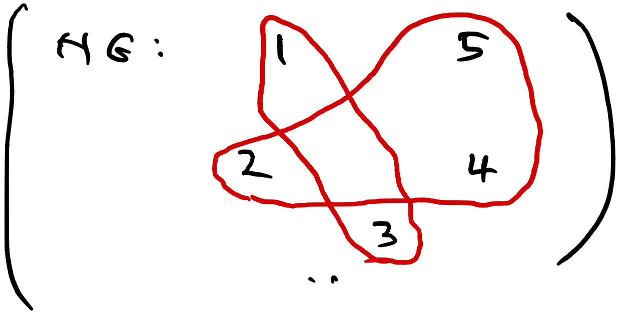
wide $T_n(k) \cong$ "non-crossing partition lattice"

another "Catalan family"

$n=2$



partition of $\{1, 2, \cdots, n, n+1\}$ which is "non-crossing"



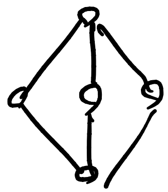
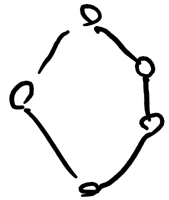
$\{1,3\} \sqcup \{2,4,5\}$

Torsion classes and wide subcategories

Observation

$\# \text{ wide } T_n(k) = \# \text{ tors } T_n(k) = $ Catalan number

But not isomorphic as posets

 $\not\cong$ 

wide $T_2(k)$ tors $T_2(k)$

What's the relation?

$\boxed{\text{Marks - Šťovíček's bijection}}$

For simplicity, assume $\# \text{ tors } \Lambda < \infty$

Example

Λ : representation-finite $\left(\overset{!}{\Longleftrightarrow} \# \{ \text{indecomposable } \Lambda\text{-mod} \} < \infty \right)$

e.g. rep. of Dynkin quiver $\left(A_n \longleftrightarrow T_n(k) \right)$

Torsion classes and wide subcategories

Marks - Stovicek's bijection

Def

For a subcat $\mathcal{E} \subseteq \mathrm{mod}\,\Lambda$,

$T(\mathcal{E})$: the smallest torsion class containing $\mathcal{E}$.

Theorem [Marks - Stovicek 2017]

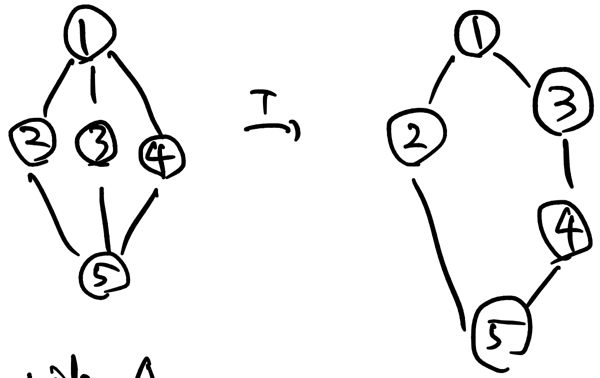
Assume $\#\,\mathrm{tors}\,\Lambda < \infty$. Then

$$T : \mathrm{wide}\,\Lambda \longrightarrow \mathrm{tors}\,\Lambda$$

is a bijection. ($\rightsquigarrow$ $\#\,\mathrm{wide} = \#\,\mathrm{tors}$)

(BUT not poset isom in general)

Ex $\Lambda = \begin{bmatrix} K & 0 \\ K & K \end{bmatrix}$



wide Λ tors Λ

Strategy

Introduce new poset str $\underset{K}{\leq}$ on $\mathrm{tors}\,\Lambda$ so that

$$\mathrm{wide}\,\Lambda \xrightarrow[\sim]{T} (\mathrm{tors}\,\Lambda, \underset{K}{\leq}) \quad : \text{poset isom}$$

§ 3. Main Results

Kappa map

- $\mathrm{tors}\,\Lambda$ is a lattice, i.e., for $\mathcal{T}, \mathcal{U} \in \mathrm{tors}\,\Lambda$,

$$\exists\ \overset{\text{meet}}{\mathcal{T} \wedge \mathcal{U}} \ (= \overset{\text{intersection}}{\mathcal{T} \cap \mathcal{U}}), \quad \exists\ \overset{\text{join}}{\mathcal{T} \vee \mathcal{U}}$$

$$\begin{tikzcd}[column sep=small] & \mathcal{T} \vee \mathcal{U} \arrow[dl, dash] \arrow[dr, dash] & \\ \mathcal{T} \arrow[dr, dash] & & \mathcal{U} \arrow[dl, dash] \\ & \mathcal{T} \cap \mathcal{U} & \end{tikzcd}$$

- In what follows, let L ($= \mathrm{tors}\,\Lambda$ in mind) be a finite lattice.

Def [Barnard-Todorov-Zhu 2021]

(1) For a cover relation $a \lessdot b$ in L

- $a < b$
- $\nexists c$ s.t. $a < c < b$

$$\begin{tikzcd} b \arrow[d, dash] \\ a \end{tikzcd}$$

$$\kappa(a \lessdot b) := \max\{ m \in L \mid b \wedge m = a \}$$

(if exists)

(2) For $x \in L$,

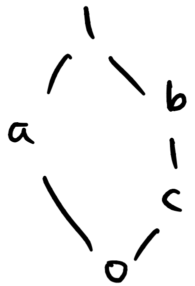
$$\begin{tikzcd}[column sep=small] & x \arrow[dl, dash, "/"] \arrow[d, dash, "/"] \arrow[dr, dash, "/"] & \\ a_1 & a_2 \ \cdots & a_n \end{tikzcd}$$

$$\overline{\kappa}(x) := \bigwedge_{a \lessdot x} \kappa(a \lessdot x)$$

Kappa map

$$\boxed{\begin{aligned} \kappa(a \lessdot b) &= \max\{m \mid b \wedge m = a\} \\ \bar{\kappa}(x) &= \bigwedge_{a \lessdot x} \kappa(a \lessdot x) \end{aligned}}$$

Example

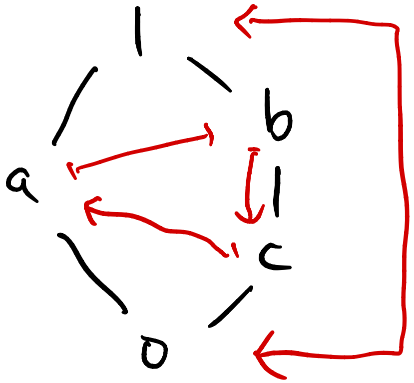
L : 

$\kappa(0 \lessdot a) = b$ $\quad \therefore \; \bar{\kappa}(a) = b$

$\kappa(c \lessdot b) = c$ $\quad \therefore \; \bar{\kappa}(b) = c$

$\kappa(a \lessdot 1) = a$ $\quad \therefore \; \bar{\kappa}(1) = 0$

$\kappa(b \lessdot 1) = b$



red : Kappa orbit

Kappa order

Proposition [Barnard-Todorov-Zhu 2021]

If $\# \operatorname{tors} \Lambda < \infty$,

then $\overline{\kappa} : \operatorname{tors} \Lambda \longrightarrow \operatorname{tors} \Lambda$ is well-defined.

(and bijective !)

Def (E 2022)

Let $(L, \leq)$ be a finite lattice s.t. $\overline{\kappa} : L \to L$ is well-defined.

For $x, y \in L$, define

$$x \leq_{\kappa} y \; :\Longleftrightarrow \; x \leq y \text{ and } \overline{\kappa}(x) \geq \overline{\kappa}(y).$$

$\rightsquigarrow$ another poset structure on L : the kappa order

Main Result

Theorem (E 2022)

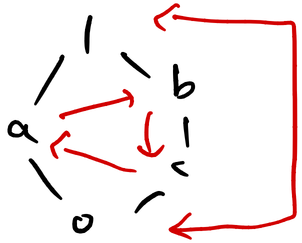
Suppose $\# \operatorname{tors}\Lambda < \infty$.

Then $T : \operatorname{wide}\Lambda \longrightarrow \operatorname{tors}\Lambda$ gives

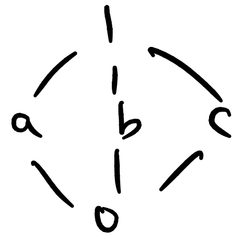
a poset isom $(\operatorname{wide}\Lambda, \subseteq) \xrightarrow{\sim} (\operatorname{tors}\Lambda, \leq_{\kappa})$.

Example $\Lambda = \begin{bmatrix} k & 0 \\ k & k \end{bmatrix}$

$\operatorname{tors}\Lambda$:



$\rightsquigarrow (\operatorname{tors}\Lambda, \leq_{\kappa})$:



$\cong \operatorname{wide}\Lambda$

$\left(\text{e.g. } b \leq c, \text{ but } \text{NOT } b \leq_{\kappa} c \text{ since } \underset{\overset{\shortparallel}{c}}{\bar{\kappa}(b)} \not\geq \underset{\overset{\shortparallel}{a}}{\bar{\kappa}(c)} \right)$

Application ("Categorification of combinatorics")

- $\Lambda = T_n(k) = \begin{bmatrix} k & & 0 \\ \vdots & \ddots & \\ k & \cdots & k \end{bmatrix}$

$\rightsquigarrow$ $\operatorname{tors} \Lambda$ = Tamari lattice Tam

$\operatorname{wide} \Lambda$ = non-crossing partition lattice NC

$\therefore$ $NC \cong (\mathrm{Tam}, \leq_{\kappa})$

(Can be generalized to any Dynkin type)

- Π : preproj alg of Dynkin type, W : Weyl grp

$\rightsquigarrow$ $\operatorname{tors} \Pi \cong (W, \text{"weak Bruhat order"})$ ← classical

$\operatorname{wide} \Pi \cong (W, \text{"shard intersection order"})$ ← new! [Reading 2011]

$\therefore$ shard int. order = Kappa order w.r.t. weak Bruhat order.

Application

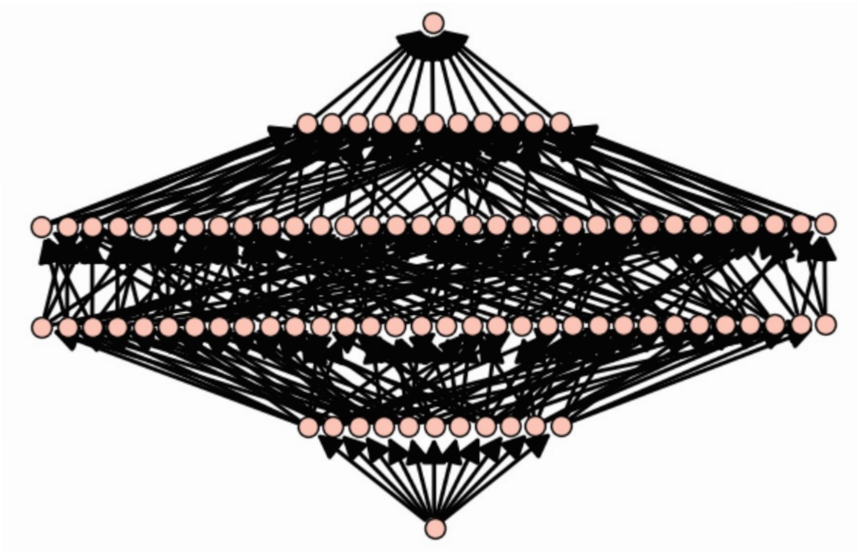
A computer can compute the poset tors Λ

for a good alg (e.g. rep-fin special biserial alg)

https://www.math.uni-bielefeld.de/~jgeuenich/string-applet/

$\rightsquigarrow$ Can compute also wide Λ using my result.

$\rightsquigarrow$ Can check and conjecture properties of wide Λ !



wide Λ for

$$\boxed{\Lambda = k\left[\begin{tikzcd}[column sep=small, row sep=tiny] & & & 4 \\ 1 \arrow[r, "a"] & 2 \arrow[r, "b"] & 3 \arrow[ur, "c"] \arrow[dr, "d"'] & \\ & & & 5 \end{tikzcd}\right] \Big/ (ab, bd)}$$

(88 wide subcats !)

Sketch of proof

Recall $T : \operatorname{wide}\Lambda \longrightarrow \operatorname{tors}\Lambda$ is bijective.

$$\boxed{\text{Enough to show !} \quad \mathcal{W}_1 \subseteq \mathcal{W}_2 \iff T(\mathcal{W}_1) \underset{\kappa}{\leq} T(\mathcal{W}_2) \quad \text{for } \mathcal{W}_1, \mathcal{W}_2 \in \operatorname{wide}\Lambda.}$$

(Fact $\bar{\kappa}\, T(\mathcal{W}_1) = {}^{\perp}\mathcal{W}_1$ [Barnard - Todorov - Zhu 2021])

$$\begin{aligned}
T(\mathcal{W}_1) \underset{\kappa}{\leq} T(\mathcal{W}_2) &\iff T(\mathcal{W}_1) \subseteq T(\mathcal{W}_2) \text{ and } \bar{\kappa}\, T(\mathcal{W}_1) \supseteq \bar{\kappa}\, T(\mathcal{W}_2) \\
&\iff T(\mathcal{W}_1) \subseteq T(\mathcal{W}_2) \text{ and } {}^{\perp}\mathcal{W}_1 \supseteq {}^{\perp}\mathcal{W}_2 \\
&\iff T(\mathcal{W}_1) \subseteq T(\mathcal{W}_2) \text{ and } F(\mathcal{W}_1) \subseteq F(\mathcal{W}_2) \\
&\overset{(*)}{\iff} \mathcal{W}_1 \subseteq \mathcal{W}_2
\end{aligned}$$

$(-)^{\perp}$

torsion-free closure

($(*)$ $(\Leftarrow)$ Clear

$(\Rightarrow)$ $\mathcal{W} = T(\mathcal{W}) \cap F(\mathcal{W})$ for any $\mathcal{W} \in \operatorname{wide}\Lambda$)

$\square$