From the lattice of torsion classes to wide and ICE-closed subcategories.

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References

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<u>Setting</u> Λ : a f.d. k-alg

$\operatorname{mod}\Lambda$: the cat of f.g. right Λ-modules.

Moviation

$\{ \mathcal{C} \subseteq \operatorname{mod} \Lambda \mid \mathcal{C} \text{ is closed under } \square \}$

is a poset (usually complete lattice)

Poset

Ex

(1) $\square$:= quotients & ext.

$\rightsquigarrow$ torsion class $\quad \operatorname{tors} \Lambda$

(2) $\square$:= kernel, & coker & ext

$\rightsquigarrow$ wide subcat $\quad \operatorname{wide} \Lambda$

(3) $\square$:= Images & Coker & Ext.

$\rightsquigarrow$ ICE-closed subcat, $\quad \operatorname{ice} \Lambda$

Rem

$$\begin{array}{ccc} \operatorname{tors} \Lambda & \subseteq & \operatorname{ice} \Lambda. \\ \operatorname{wide} \Lambda & \subseteq & \end{array}$$

Q

How are these posets related?

Ex Q: Dynkin quiver

W: its Weyl grp

	tors	wide
KQ	Cambrian lattice of W	non-crossing partition lattice of W
Π_Q preproj alg	W with right weak order.	sharding intersection order of W

[Rea]

?

Answer

We can construct

$$\operatorname{wide} \Lambda \leftarrow \operatorname{ice} \Lambda$$

only from the lattice $\operatorname{tors} \Lambda$.

§ 1. Torsion hearts

Def

- $[\mathcal{U}, \mathcal{T}]$: itv (interval) in $\operatorname{tors}\Lambda$

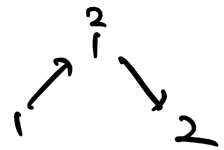
  (i.e., $\mathcal{U} \subseteq \mathcal{T}$) " $\mathcal{T} - \mathcal{U}$ "

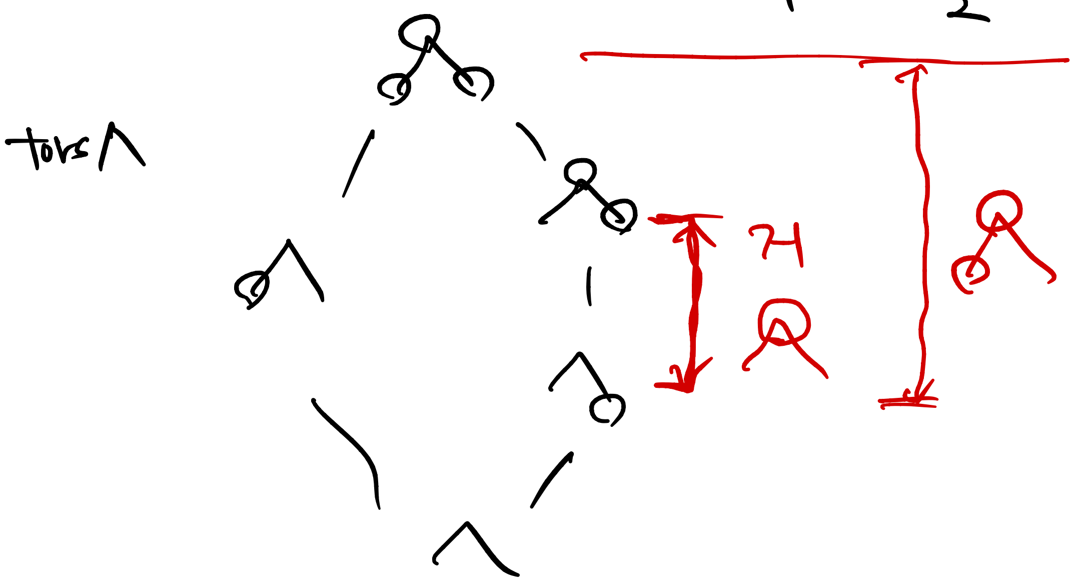
  $\rightsquigarrow \mathcal{H}_{[\mathcal{U},\mathcal{T}]} := \mathcal{T} \cap \mathcal{U}^{\perp}$ ($=$ " $\mathcal{T} - \mathcal{U}$ ")

  (the heart of $[\mathcal{U}, \mathcal{T}]$)

- A <u>torsion heart</u> is a subcat. arising in this way.

Ex mod $k(1 \leftarrow 2)$





- Every tors & torf is a torsion heart.

Thm [AP] [ES]

Every wide subcat & ICE-closed subcat is a tors. heart.

Moreover, we can check when $\mathcal{H}_{[\mathcal{U},\mathcal{T}]}$ is wide or ICE-closed lattice-theoretically.

Fact. [DIRRT]

For $\mathcal{H}_1, \mathcal{H}_2$: torsion hearts,

$\mathcal{H}_1 \subseteq \mathcal{H}_2$

$\iff \operatorname{brick}\mathcal{H}_1 \subseteq \operatorname{brick}\mathcal{H}_2$,

where $\operatorname{brick}\mathcal{H}$

$= \{ B \in \mathcal{H} \mid \underline{B \text{ is a brick}} \} / \cong$

$\Updownarrow$ ⁚

$\operatorname{End}_\Lambda B$ is a division alg.

Fact [DIRRT, BCZ]

$$\mathrm{brick}\,\Lambda \xrightarrow[\sim]{T(-)} \text{j-irr}(\mathrm{tors}\,\Lambda) \quad \text{is bij,}$$

$$B \longmapsto T(B)$$

$T(B)$: the smallest tors containing B.

where, for a complete lattice L,

$$\text{j-irr}\,L := \{ j \in L \mid j = \bigvee_{a \in A} a \Rightarrow j \in A \}$$

join-irred. elem.

$\lrcorner$

wide itvs. wide Λ

$$\begin{tikzcd} \{\text{itvs in } \mathrm{tors}\,\Lambda\} \arrow[r, two heads, "\mathcal{H}(-)"] \arrow[d, dashed, "?"] & \{\text{tors heart}\} \arrow[d, hook, "\mathrm{brick}(-)", "\text{poset emb.}"'] \\ 2^{\text{j-irr}(\mathrm{tors}\,\Lambda)} \arrow[r, equal, "\sim"] & 2^{\mathrm{brick}\,\Lambda} \end{tikzcd}$$

$2^{(\cdot)}$: the power set.

$$T(B) \longleftrightarrow B$$

For $[\mathcal{U}, \mathcal{T}]$: itv in $\mathrm{tors}\,\Lambda$,
B : brick,

when $B \in \mathcal{H}_{[\mathcal{U},\mathcal{T}]}$?

($T(B)$, $\mathcal{U}$, $\mathcal{T} \in \mathrm{tors}\,\Lambda$)

§2. Kappa map.

Def L : completely semidistributive lattice.

(e.g. $L := \mathrm{tors}\,\Lambda$)

For $j \in \text{j-irr}\,L$,

$$\kappa(j) := \max\{ m \in L \mid j \wedge m = j_* \}$$

where $j \to j_*$: Hasse arrow.

$$\rightsquigarrow \kappa : \text{j-irr}\,L \xrightarrow{\sim} \text{m-irr}\,L = \{\text{meet irred.}\} \ : \text{bijection.}$$

Prop [BTZ]

The following commutes

$$\begin{tikzcd}[column sep=small]
 & \mathrm{brick}\,\Lambda \arrow[dl, "T(-)"', "\sim"] \arrow[dr, "F(-)", "\sim"'] & \\
\text{j-irr}(\mathrm{tors}\,\Lambda) \arrow[drr, "\kappa"'] \arrow[rr, phantom, "@."] & & \text{j-irr}(\mathrm{torf}\,\Lambda) \arrow[d, "{}^{\perp}(-)", "\wr"'] \\
 & & \text{m-irr}(\mathrm{tors}\,\Lambda)
\end{tikzcd}$$

That is,

$$\kappa\, T(B) = {}^{\perp}F(B)$$

Thm $[\mathcal{U},\mathcal{T}]$: itv in tors Λ
B : brick.

$\rightsquigarrow$ $B \in \mathcal{H}[\mathcal{U},\mathcal{T}]$

$$\Longleftrightarrow \begin{cases} T(B) \leq \mathcal{T} \\ \kappa\, T(B) \geq \mathcal{U} \end{cases} \text{ in tors } \Lambda.$$

(!) $B \in \mathcal{H}[\mathcal{U},\mathcal{T}] = \mathcal{T} \cap \mathcal{U}^{\perp}$

$$\Longleftrightarrow \begin{cases} \underline{B \in \mathcal{T}} \Leftarrow \\ B \in \mathcal{U}^{\perp} \Longleftrightarrow F(B) \leq \mathcal{U}^{\perp} \text{ in torf } \Lambda \end{cases} \quad \square$$

$\mathcal{U}^{\perp}$: torf

Cor.

We construct wide Λ, ice Λ, {tors. hearts} from tors Λ.

(!) Define

$$\text{j-brick} : \{\text{itvs in tors}\Lambda\} \longrightarrow 2^{\text{j-irr}(\mathrm{tors}\Lambda)}$$

by

$$\text{j-brick}\,[\mathcal{U},\mathcal{T}] := \{ j \mid j \leq \mathcal{T},\ \kappa(j) \geq \mathcal{U} \}.$$

$\rightsquigarrow$

$$\text{wide}\Lambda \cong \text{j-brick}\{\text{wide itvs}\} \subseteq 2^{\text{j-irr}\bigcirc}$$

ice

ICE itv.

§ 3. More on wide Λ

Thm [MS]

$T(-)$: wide $\Lambda \hookrightarrow$ tors Λ

is injective.

Q

(1) Im $T(-)$?

(2) Poset str ?

$$w_1 \le w_2 \xRightarrow[\;\not\Longleftarrow\;]{} T(w_1) \le T(w_2)$$

(1)

L : a complete lattice.

$x \in L$.

$$x = \bigvee_{a \in A} a \; : \text{ a canonical join representation.} \quad \text{CJR}$$

if this expression is "universally minimal"

∘ $\mathrm{tors}_0 \Lambda := \{ T \mid T \text{ has a CJR} \}$

Prop We have a bij

$$\mathrm{wide}\,\Lambda \xrightarrow[\sim]{T(-)} \mathrm{tors}_0 \Lambda$$

Rem

∘ For a semibrick $\mathcal{S}$.

$$T(\mathcal{S}) = \bigvee_{S \in \mathcal{S}} T(S)$$

is a CJR and every CJR in $\mathrm{tors}\,\Lambda$ is of this form.

∘ If $\mathrm{tors}\,\Lambda$ is finite,

$$\mathrm{tors}_0 \Lambda = \mathrm{tors}\,\Lambda.$$

The extended kappa map

$$\begin{tikzcd}
 & \mathrm{wide}\,\Lambda \arrow[dl, "T(-)"', "\sim"] \arrow[dr, "F(-)", "\sim"'] & \\
\mathrm{tors}_0 \Lambda \arrow[drr, "\overline{\kappa}", "\sim"'] & & \mathrm{torf}_0 \Lambda \arrow[d, "{}^{\perp}(-)", "\wr"'] \\
 & & \mathrm{tors}^0 \Lambda
\end{tikzcd}$$

$$\mathrm{tors}^0 \Lambda = \{ T \mid T \text{ has CMR} \}$$

Thm

$\bar{\kappa}$ can be computed lattice theoretically.

$$\left(\bar{\kappa}(\mathcal{X}) = \bar{\kappa}\Big(\bigvee_{\mathrm{CJR}} a\Big) := \bigwedge \kappa(a) \right)$$

$\rightsquigarrow$ $\bar{\kappa}\, T(\mathcal{W}) = {}^{\perp}F(\mathcal{W})$

Thm

Define $\underset{\kappa}{\leq}$ on $\mathrm{tors}_0\Lambda$ by

$$\mathcal{U} \underset{\kappa}{\leq} \mathcal{T} :\Longleftrightarrow \begin{cases} \mathcal{U} \leq \mathcal{T} \\ \bar{\kappa}\,\mathcal{U} \geq \bar{\kappa}\,\mathcal{T}. \end{cases}$$

Then we have in $\mathrm{tors}\Lambda$ a poset isom

$$(\mathrm{wide}\Lambda, \subseteq) \xrightarrow[\sim]{T(-)} (\mathrm{tors}_0\Lambda, \underset{\kappa}{\leq})$$

(!)

$$T(\mathcal{W}_1) \underset{\kappa}{\leq} T(\mathcal{W}_2)$$

$$\Longleftrightarrow \begin{cases} T(\mathcal{W}_1) \leq T(\mathcal{W}_2) \\ \bar{\kappa}\, T(\mathcal{W}_1) \geq \bar{\kappa}\, T(\mathcal{W}_2) \end{cases}$$

$$\begin{array}{ccc} \bar{\kappa}\, T(\mathcal{W}_1) & \geq & \bar{\kappa}\, T(\mathcal{W}_2) \\ \| & & \| \\ {}^{\perp}F(\mathcal{W}_1) & \Updownarrow & {}^{\perp}F(\mathcal{W}_2) \end{array}$$

$F(\mathcal{W}_1) \leq F(\mathcal{W}_2)$ in $\mathrm{torf}\Lambda$.

$\Longleftrightarrow$ $\mathcal{W}_1 \subseteq \mathcal{W}_2$

(*) follows since

$\mathcal{C} = T(\mathcal{C}) \wedge F(\mathcal{C})$

if $\mathcal{C}$ is closed under images & ext.

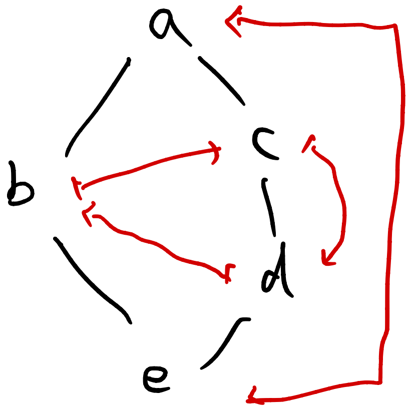
Ex

$k(1 \leftarrow 2)$

$\bar{x}$

$\mapsto$

tors Λ



$(\operatorname{tors}\Lambda, \leq_x)$

$\cong$

wide Λ