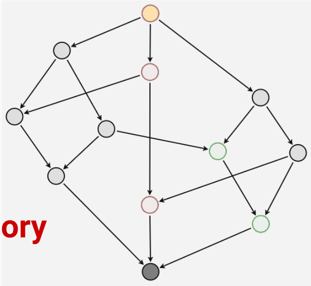
Combinatorics of lattices of subcategories of a module category



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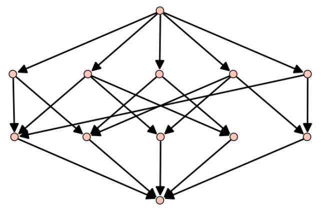

2 September 2022,
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Introduction

Overview

For a ring Λ,

$$\{\mathcal{C} \mid \mathcal{C} \text{ is a collection of } \Lambda\text{-modules satisfying (*)}\}$$

is a poset by inclusion for a given condition (*).

$\rightsquigarrow$ obtain various posets by changing (*).

Question

How these posets are combinatorially related?

Some important posets in combinatorics can be realized in this way.

$\rightsquigarrow$ Applications in combinatorics!

Example

k: a field (only consider f.d. modules).

Consider collections $\mathcal{C}$ of k-modules which is

- closed under direct sums and direct summands
  $(X, Y \in \mathcal{C} \Leftrightarrow X \oplus Y \in \mathcal{C})$

Such $\mathcal{C}$: only 0 and $\operatorname{mod} k$ (all k-modules).

$\rightsquigarrow$ poset: $0 < \operatorname{mod} k$.

Example

$k^2 = k \times k$: a product of field (only consider f.d. modules).

Consider collections $\mathcal{C}$ of k^2-modules satisfying:

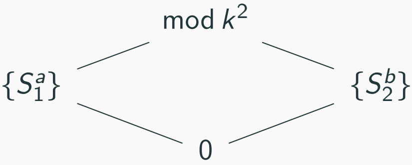
- closed under direct sums and direct summands

Every k^2-module M is decomposed as

$$M = S_1^a \oplus S_2^b$$

for unique $a, b \in \mathbb{N}$, where $S_1 := k \oplus 0$ and $S_2 := 0 \oplus k$.

Such $\mathcal{C} \leftrightarrow$ choosing S_1 and S_2



Example

k^n: a product of field (only consider f.d. modules).

Consider collections $\mathcal{C}$ of k^n-modules satisfying:

- closed under direct sums and direct summands

Every k^n-module M is decomposed uniquely as

$$M = S_1^{a_1} \oplus S_2^{a_2} \oplus \cdots \oplus S_n^{a_n}$$

where $S_i := 0 \oplus \cdots \oplus 0 \oplus k \oplus 0 \oplus \cdots \oplus 0$.

Such $\mathcal{C} \leftrightarrow$ subset of $\{S_1, \ldots, S_n\}$!

Thus the poset $\cong$ powerset $2^{\{1,\ldots,n\}}$ of $\{1, \ldots, n\}$.

Today's talk

Today: Two class of collections of Λ-modules.

1. Torsion class $\rightsquigarrow$ the poset $\operatorname{tors}\Lambda$
2. Wide subcategory $\rightsquigarrow$ the poset $\operatorname{wide}\Lambda$

Then construct $\operatorname{wide}\Lambda$ from $\operatorname{tors}\Lambda$ combinatorially!

Torsion classes and wide subcategories

Subcategories of the module category

- Λ: a finite-dimensional k-algebra over a field k.
- $\operatorname{mod}\Lambda$: the category of f.g. Λ-modules.

Collections of Λ-modules $=$ Subcategories of $\operatorname{mod}\Lambda$

Theorem (Krull-Schmidt)

Every $M \in \operatorname{mod}\Lambda$ is decomposed uniquely as

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$$

such that each M_i is indecomposable.

Subcategories (closed under direct sums & summands)
$=$ Sets of indecomposable Λ-modules.

Torsion classes

Definition (Dickson 1966)

A subcategory $\mathcal{T}$ of $\operatorname{mod}\Lambda$ is a torsion class
$:\Leftrightarrow$ closed under extensions and quotients: for any

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0,$$

$L, N \in \mathcal{T} \Rightarrow M \in \mathcal{T}, \quad$ and $M \in \mathcal{T} \Rightarrow N \in \mathcal{T}$.
$\operatorname{tors}\Lambda$: the poset of torsion classes (by inclusion)

By Krull-Schmidt, we only have to consider *indecomposable* modules in a torsion class.

- $\operatorname{tors} k = \{0, \operatorname{mod} k\}$.
- Λ: semisimple $\Rightarrow$ torsion class = subcat closed under direct sum & summands, $\operatorname{tors}\Lambda$ is the powerset of the set of indecomposable (=simple) modules.

Torsion classes: Working example

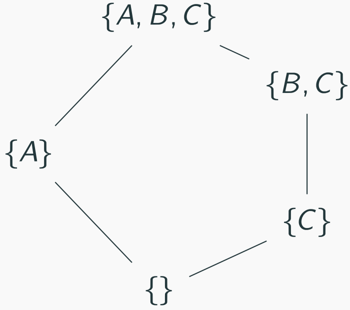
torsion class = closed under ext & quotients

Consider $\Lambda := \begin{bmatrix} k & k \\ 0 & k \end{bmatrix} = \{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in k\}$.

Then $\exists$ 3 indec. Λ-modules A, B, C with exact seq

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

tors Λ:



Torsion classes: Remarks

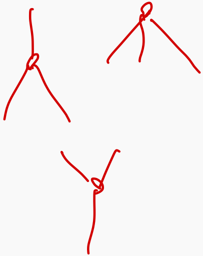
- A torsion class is a part of a torsion pair:
- A torsion pair $(\mathcal{T}, \mathcal{F})$ is a pair of subcats of $\operatorname{mod} \Lambda$ which *divides* $\operatorname{mod} \Lambda$ *into two parts*:
  torsion part $\mathcal{T}$ and torsion-free part $\mathcal{F}$.
  Classicaly: torsion groups and torsion-free groups.
- Naturally appears when considering the derived category and t-structures.

Torsion classes: Combinatorial properties

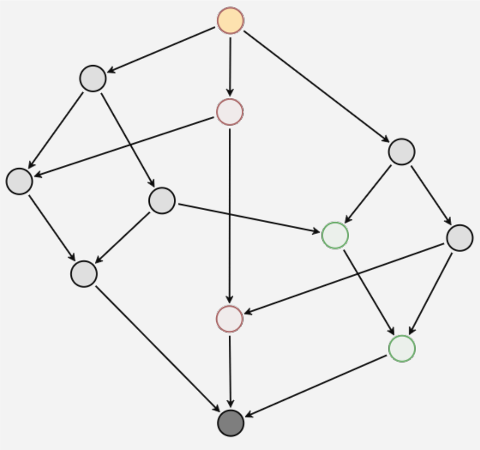
In the rest, we assume $\operatorname{tors}\Lambda$ is finite.

$\operatorname{tors}\Lambda$ has the following properties:

- A complete lattice: for any family $\mathcal{T}_i$ of torsion classes, $\exists$ $\bigvee \mathcal{T}_i$ and $\bigwedge \mathcal{T}_i (= \bigcap \mathcal{T}_i)$ in $\operatorname{tors}\Lambda$.
- The Hasse diagram is an n-regular graph (n is the number of simple Λ-modules).
- $\#\{\mathcal{T} \mid \mathcal{T}$ covers i elements$\}$ $= \#\{\mathcal{T} \mid \mathcal{T}$ is covered by i elements$\}$
- Semiditributive.



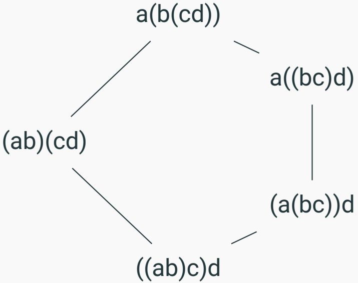
Torsion classes: Rank 3 Example



tors $k(1 \xrightarrow{a} 2 \xrightarrow{b} 3)/(ab)$,
calculated by Geuenich's String Applet

There is an algebra Λ such that $\operatorname{tors}\Lambda$ is:

- Tamari lattice Tam: the poset of order of binary operation:



- Dynkin variants of Tamari lattice (*Cambrian lattice*).
- Finite Coxeter groups with weak order.

Wide subcategories (module cat inside module cat!)

Definition (Hovey 2001)

A subcategory $\mathcal{W}$ of $\operatorname{mod}\Lambda$ is wide if

- closed under kernels and cokernels: for every $f : W_1 \to W_2$ with $W_i \in \mathcal{W}$, we have $\operatorname{Ker} f, \operatorname{Coker} f \in \mathcal{W}$.
- closed under extensions.

$\operatorname{wide}\Lambda$: the poset of wide subcategories of $\operatorname{mod}\Lambda$.

An wide subcategory is an abelian subcategory.

Example

$\Lambda \twoheadrightarrow \Gamma$: a *nice* ring surj $\Rightarrow \operatorname{mod}\Gamma \hookrightarrow \operatorname{mod}\Lambda$: wide subcat.

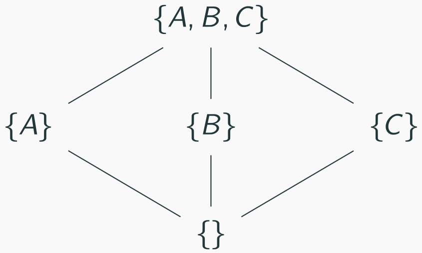
Indeed, $\operatorname{wide}\Lambda \cong \{\Lambda \to \Gamma \mid \text{a nice ring epi}\}$.

Wide subcategories: Example

- Λ: semisimple $\rightsquigarrow$ $\operatorname{wide}\Lambda = \operatorname{tors}\Lambda \cong$ powerset.
- Working example: $\Lambda := \begin{bmatrix} k & k \\ 0 & k \end{bmatrix} = \{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in k\}$.
  3 indec. modules A, B, C with exact seq

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

$\operatorname{wide}\Lambda$



Wide subcategories: Combinatorial properties

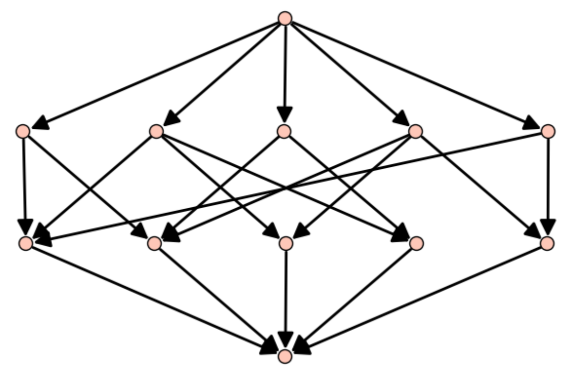
wide Λ has the following properties:

- A complete lattice.
- Ranked (graded).
- Rank-symmetric:

$$a_i := \#\{\mathcal{W} \mid \mathcal{W} \text{ is rank } i\} = \#\{\mathcal{W} \mid \mathcal{W} \text{ is rank } n-i\}.$$

- Rank-unimodal ($a_0 \leq a_1 \leq \cdots \leq a_{n/2} \geq \cdots \geq a_{n-1} \geq a_n$) [Aoki-Higashitani-Iyama-Kase-Mizuno 2022]

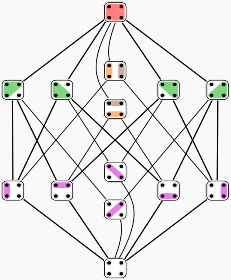
Wide subcategories: Rank 3 Example



wide $k(1 \xrightarrow{a} 2 \xrightarrow{b} 3)/(ab)$

There is an algebra Λ such that wide Λ is:

- Non-crossiong partition lattice NC:



from Wikipedia, T. Piesk (CC BY 3.0)

- Dynkin variants of NC.
- The shard intersection order, a lattice structure on a finite Coxeter group defined recently [Reading 2011], using combinatorics of hyperplane arrangement

Main result

Main result: Overview

For a given algebra Λ, we have two posets:

- $\operatorname{tors}\Lambda$, the poset of torsion classes.
- $\operatorname{wide}\Lambda$, the poset of wide subcategories.

We will construct $\operatorname{wide}\Lambda$ from $\operatorname{tors}\Lambda$ purely combinatorially!

Example

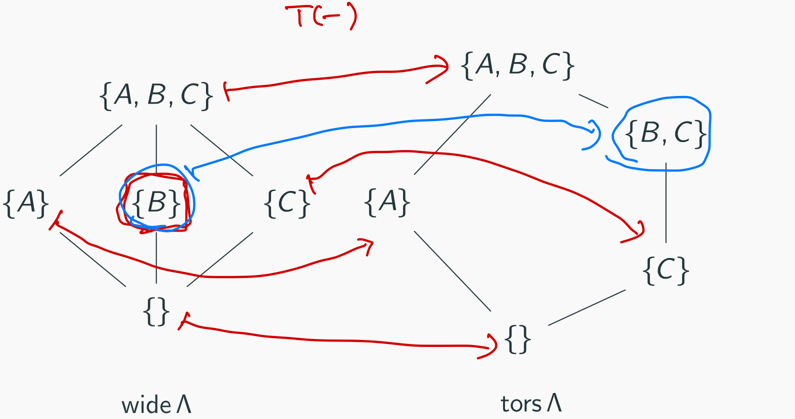
Λ: an upper-triangular matrix algebra, then

- $\operatorname{tors}\Lambda \cong \mathsf{Tam}$: Tamari lattice
- $\operatorname{wide}\Lambda \cong \mathsf{NC}$: non-crossing partition lattice

$\rightsquigarrow$ Non-trivial relation between them!

Working Example

$\Lambda := \left[\begin{smallmatrix} k & k \\ 0 & k \end{smallmatrix}\right]$, 3 indec. modules A, B, C.



Non-isomorphic posets, but the number is same!

Marks–Šťovíček's bijection

Theorem (Marks–Šťovíček 2017)

We have a bijection

$$\mathsf{T}\colon \operatorname{wide}\Lambda \xrightarrow{\simeq} \operatorname{tors}\Lambda$$

where $\mathsf{T}(\mathcal{W})$ *is the smallest torsion class containing* $\mathcal{W}$.

Problem

$\mathsf{T}\colon \operatorname{wide}\Lambda \to \operatorname{tors}\Lambda$ is order-preserving, but not a poset isom.

Strategy

Give a new poset str $\leq_\kappa$ on $\operatorname{tors}\Lambda$ such that

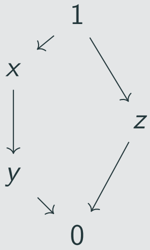
$$\mathsf{T}\colon \operatorname{wide}\Lambda \cong (\operatorname{tors}\Lambda, \leq_\kappa).$$

Setting

We are given tors Λ just as an abstract lattice.

Working Example

$\Lambda := \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$. The Hasse quiver of tors Λ:



Hasse quiver: $a \to b$ if $a > b$ and $\nexists c$ with $a > c > b$.

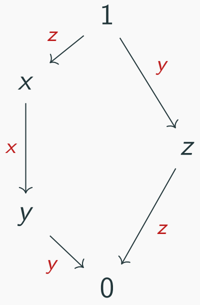
The join-irreducible labeling

Definition

For each Hasse arrow $a \to b$ in $\operatorname{tors}\Lambda$, we label its arrow by

$$\min\{x \in \operatorname{tors}\Lambda \mid b \vee x = a\}.$$

This is well-defined and called the join-irreducible label.



Kappa map κ

[Barnard–Todorov–Zhu 2021] introduced $\kappa\colon \operatorname{tors}\Lambda \xrightarrow{\sim} \operatorname{tors}\Lambda$:

Definition

For each $x \in \operatorname{tors}\Lambda$, there's unique $\kappa(x) \in \operatorname{tors}\Lambda$ satisfying

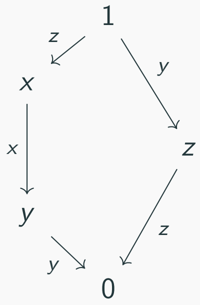
$$
\begin{aligned}
&\{\text{labels of Hasse arrows starting at } x\} \\
=&\{\text{labels of Hasse arrows ending at } \kappa(x)\}
\end{aligned}
$$

(well-defined for semidistributive lattices.)

Kappa map κ: Example

Red: the orbit of κ.

labels from a = labels into $\kappa(a)$



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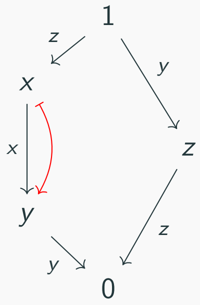

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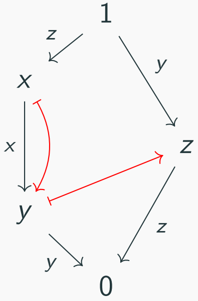

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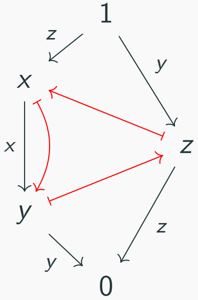

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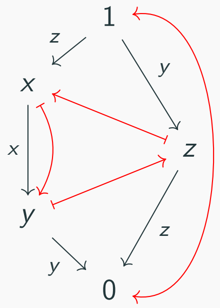


The kappa order and Main Theorem

Definition (The kappa order)

For $x, y \in \operatorname{tors}\Lambda$, define

$$x \leq_\kappa y :\Longleftrightarrow x \leq y \text{ and } \kappa(x) \geq \kappa(y).$$

Theorem (E)

$\mathsf{T}\colon \operatorname{wide}\Lambda \to \operatorname{tors}\Lambda$ induces a poset isomorphism:

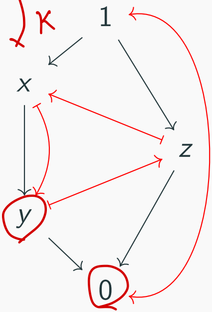
$$\operatorname{wide}\Lambda \cong (\operatorname{tors}\Lambda, \leq_\kappa)$$

Working Example

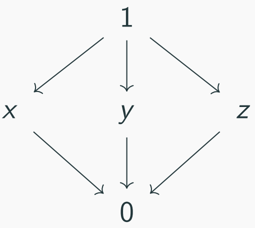
$\Lambda := \left[\begin{smallmatrix} k & k \\ 0 & k \end{smallmatrix}\right]$

$$x \leq_\kappa y :\Longleftrightarrow x \leq y \text{ and } \kappa(x) \geq \kappa(y).$$

$\operatorname{tors}\Lambda$



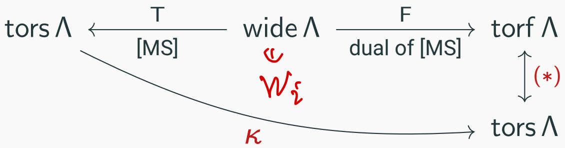
$(\operatorname{tors}\Lambda, \leq_\kappa) \cong \operatorname{wide}\Lambda$



A sketch of proof

κ has a natural representation-theoretic meaning! $(\mathcal{T}, \mathcal{F})$ tors pair

We can dually consider torsion-free class (torf).



(*) is poset anti-isom.

Then for $\mathcal{W}_1, \mathcal{W}_2 \in \operatorname{wide}\Lambda$, we have

$$
\begin{aligned}
\mathsf{T}(\mathcal{W}_1) \leq_\kappa \mathsf{T}(\mathcal{W}_2) :&\iff \mathsf{T}(\mathcal{W}_1) \subseteq \mathsf{T}(\mathcal{W}_2) \text{ and } \kappa\mathsf{T}(\mathcal{W}_1) \supseteq \kappa\mathsf{T}(\mathcal{W}_2) \\
&\iff \mathsf{T}(\mathcal{W}_1) \subseteq \mathsf{T}(\mathcal{W}_2) \text{ and } \mathsf{F}(\mathcal{W}_1) \subseteq \mathsf{F}(\mathcal{W}_2) \\
&\iff \mathcal{W}_1 \subseteq \mathcal{W}_2
\end{aligned}
$$

$\mathcal{W} = \mathsf{T}(\mathcal{W}) \cap \mathsf{F}(\mathcal{W})$

Combinatorial applications

By applying $\mathsf{wide}\,\Lambda \cong (\mathsf{tors}\,\Lambda, \leq_{\kappa})$, we obtain:

- Non-crossing partition lattice $\cong$ Tamari lattice with $\leq_{\kappa}$.
- Dynkin variants of the above.
- The shard intersection order on a Coxeter group W coincide with $\leq_{\kappa}$ w.r.t. the weak order.