The Grothendieck monoid of an extriangulated category

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This talk is based on joint work with Shunya Saito (Nagoya University). The *Grothendieck group* is the classical and basic invariant for both a triangulated category and an exact category. For exact categories, the *Grothendieck monoid*, a natural monoid version of the Grothendieck group, has been recently studied by several authors [1, 2, 6].

In the representation theory of algebras, we often consider extension-closed subcategories of a triangulated category which are not exact nor triangulated. An *extriangulated category* introduced by Nakaoka–Palu [5] is a convenient framework to consider such subcategories. Extriangulated categories unify both exact categories and triangulated categories, and have the notion of *conflations*, which generalize conflations (short exact sequences) in an exact category and triangles in a triangulated category. We can naturally define the *Grothendieck monoid* $\mathsf{M}(\mathcal{C})$ of an extriangulated category $\mathcal{C}$ using conflations. In this talk, we give several results about it.

The first result is about the classifications of several classes of subcategories, which extends [6] and [7] respectively.

Theorem 1. *Let $\mathcal{C}$ be an extriangulated category. Then we have the following two bijections.*

1. *A bijection between the set of Serre subcategories of $\mathcal{C}$ and the set of faces of $\mathsf{M}(\mathcal{C})$.*
2. *A bijection between the set of dense 2-out-of-3 subcategories and the set of cofinal subtractive submonoids of $\mathsf{M}(\mathcal{C})$.*

The second result is about the localization of an extriangulated category. For a nice subcategory $\mathcal{N}$ of an extriangulated category $\mathcal{C}$, Nakaoka–Ogawa–Sakai [4] constructed the *exact localization* $\mathcal{C}/\mathcal{N}$, which generalizes the Verdier quotient of a triangulated category and the Serre quotient of an abelian category. We show that under some conditions, this *commutes with the Grothendieck monoid*:

Theorem 2. *Let $\mathcal{C}$ be an extriangulated category and $\mathcal{N}$ a subcategory of $\mathcal{C}$ satisfying some conditions. Then we have an isomorphism of monoids*

$$\mathsf{M}(\mathcal{C}/\mathcal{N}) \cong \mathsf{M}(\mathcal{C})/\mathsf{M}_{\mathcal{N}},$$

where the right hand side is the monoid quotient by $\mathsf{M}_{\mathcal{N}} := \{[N] \mid N \in \mathcal{N}\}$. This can be applied to the Verdier quotient of a triangulated category, the stable category of a Frobenius category, and the Serre quotient of an abelian category.

As a toy example, we consider an *intermediate subcategory* of the derived category $\mathsf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$, which is a subcategory $\mathcal{C}$ closed under extensions and direct summand satisfying $\mathcal{A} \subseteq \mathcal{C} \subseteq \mathcal{A}[1] * \mathcal{A}$. We show that an intermediate subcategory is precisely a subcategory of the form $\mathcal{F}[1] * \mathcal{A}$ for a torsionfree class $\mathcal{F}$ of $\mathcal{A}$, and then compute its Grothendieck group, classify Serre subcategories, and study the exact localization.

References

[1] A. Berenstein, J. Greenstein, *Primitively generated Hall algebras*, Pacific J. Math. **281** (2016), no. 2, 287–331.

[2] H. Enomoto, *The Jordan-Hölder property and Grothendieck monoids of exact categories*, Adv. Math, **396** (2022).

[3] H. Matsui, *Classifying dense resolving and coresolving subcategories of exact categories via Grothendieck groups*, Algebr. Represent. Theory **21** (2018), no. 3, 551–563.

[4] H. Nakaoka, Y. Ogawa, A. Sakai, *Localization of extriangulated categories*, arXiv:2103.16907.

[5] H. Nakaoka and Y. Palu, *Extriangulated categories, Hovey twin cotorsion pairs and model structures*, Cah. Topol. Géom. Différ. Catég. **60** (2019), no. 2, 117–193.

[6] S. Saito, *The spectrum of Grothendieck monoid: a new approach to classify Serre subcategories*, arXiv:2206.15271.

[7] R. W. Thomason, *The classification of triangulated subcategories*, Compositio Math. **105** (1997), no. 1, 1–27.

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