Orthogonal modules and (~~projectively~~ injectively) Wakamatsu tilting modules

W-tilt

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Overview

Hierarchy of "full rank" ortho. modules.

proj side | inj side

Λ | $D\Lambda$

$\cap$ | $\cap$

$\{\text{tilting mod, pd} < \infty\}$ $\xleftrightarrow{\text{dual}}$ $\{\text{cotilting mod, id} < \infty\}$

$\cap$ | $\cap$

$\{\text{proj. W-tilt}\}$ $\longleftrightarrow$ $\{\text{injectively W-tilt}\}$

$\cap$ | $\cap$

$\{\text{W-tilt}\}$ ↺ self-dual

Today

Setting

Λ : f.d k-alg / k : field

$\operatorname{mod}\Lambda$: the cat of f.g. right Λ-modules,

Def Let $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ be a subcat.

(1) $P \in \mathcal{C}$ is progenerator of $\mathcal{C}$.

$\Longleftrightarrow$ (i) $\operatorname{Ext}^1_\Lambda(P, \mathcal{C}) = 0$

(ii) $\forall X \in \mathcal{C}$, $\exists$ s.e.s.

$$0 \to X' \to P_0 \to X \to 0$$

s.t. $X' \in \mathcal{C}$, $P_0 \in \operatorname{add} P$.

(2) Dually for inj. cogen. of $\mathcal{C}$.

W-tilting theory

Def Let $T \in \operatorname{mod}\Lambda$ be orthogonal.

$\operatorname{Ext}^{>0}_\Lambda(T,T) = 0$

Define $\mathcal{X}_T \subseteq {}^\perp T \subseteq \operatorname{mod}\Lambda$.

(1) $X \in {}^\perp T :\Longleftrightarrow \operatorname{Ext}^{>0}_\Lambda(X, T) = 0.$

(2) $X \in \mathcal{X}_T$

$\Longleftrightarrow$ (i) $X \in {}^\perp T$

(*)

(ii) $\exists\ 0 \to X \to T_0 \to T_1 \to \cdots$ ex.

s.t. $T_i \in \operatorname{add} T$

and $\operatorname{Hom}_\Lambda((*), T)$ is exact. ∎

<u>Ex</u>

- $\mathcal{X}_{D\Lambda} = {}^{\perp}D\Lambda = \operatorname{mod}\Lambda$.
- $\mathcal{X}_\Lambda = \mathrm{GP}\Lambda$ (= Gorenstein-proj).

<u>Thm</u> Let $T \in \operatorname{mod}\Lambda$: ortho.

(1) Λ_Λ is a progen of ${}^{\perp}T$.

(2) T is an inj cogen. of $\mathcal{X}_T$.

T : ortho	$\mathcal{X}_T$	$\subseteq$ ${}^{\perp}T$
progen	?? Λ iff T: w-tilt.	Λ_Λ
inj cogen	T	?? T iff T: iw-tilt.

Def $T \in \operatorname{mod} \Lambda$ is cotilting

if (1) $\operatorname{id} T < \infty$

(2) T: ortho.

(3) $\exists\, 0 \to T_n \to \cdots \to T_0 \to D\Lambda \to 0$

: ex with $T_i \in \operatorname{add} T$. ┘

Thm [Auslander-Reiten '91]

If T is cotilt, then

$$\mathcal{X}_T = {}^{\perp}T.$$

$\left(\leadsto {}^{\perp}T \text{ has } \begin{matrix} \text{progen} & \Lambda \\ \text{inj cogen} & T \end{matrix} \right)$ ┘

Def $T \in \operatorname{mod} \Lambda$ is W-tilt.

if (1) T: ortho.

(2) $\Lambda \in \mathcal{X}_T$

($\Leftrightarrow \Lambda$ is a progen of $\mathcal{X}_T$)

Ex

(1) $\{\text{cotilt}\} \subset \{\text{W-tilt}\}$

(2) T_Λ is W-tilt $\Leftrightarrow {}_\Lambda(DT)$ is W-tilt

(3) $\{\text{tilt}\} \subset \{W\text{-tilt}\}$.

(4) Λ_Λ is W-tilt.

$\rightsquigarrow \mathcal{X}_\Lambda = \mathrm{GP}\Lambda$ $\lrcorner$

Rem

Consider a "pair" (P, I) s.t.

P : progen, I : inj cogen of some

exact cat ($\doteqdot$ $\mathcal{E} \subseteq \operatorname{mod}\Gamma$ $\exists\, \Gamma$: f.d.alg.)

ext-closed.

(1) If T_Λ : W-tilt,

(Λ, T) is such a pair

($\mathcal{X}_T$).

(2) $\forall\, (P, I)$: pair (+ assump)

[E]

(P, I) $\overset{\text{"equiv"}}{\sim}$ (Λ, T)

for some alg Λ, T_Λ : W-tilt.

W-tilt = "universal inj cogen"

Def $T \in \operatorname{mod}\Lambda$ is injectively W-tilt (iW-tilt) if

(1) T : ortho,

(2) T is an inj cogen of ${}^{\perp}T$.

($\iff \forall X \in {}^{\perp}T$, $\exists\ X \hookrightarrow T^{n}$)

Prop

T : iW-tilt $\iff$ T : W-tilt s.t. $\mathcal{X}_T = {}^{\perp}T$.

Ex

(1) $\{\text{cotilt}\} \subset \{\text{iW-tilt}\}$

(2) Λ_{Λ} is iW-tilt

$$\iff \underset{\substack{\| \\ \mathrm{GP}\Lambda}}{\mathcal{X}_{\Lambda}} = {}^{\perp}\Lambda$$

$\iff$: Λ : weakly Gorenstein.

Main Results

Classical Result [Bongartz]

$T \in \operatorname{mod}\Lambda$: cotilt with $\mathrm{id} \leq 1$

$\iff$ $\begin{cases} \cdot\ \mathrm{id}\,T \leq 1 \\ \cdot\ T : \text{ortho.} \\ \cdot\ |T| = |\Lambda| \end{cases}$

"$0 \to T_1 \to T_0 \to D\Lambda \to 0$"

$|T|$ = # { index summands of T } $/\cong$.

This depends "completion"

which works for $\mathrm{id} \leq 1$

At least for Λ : rep-fin ?

Thm Let Λ be rep-fin.

(1) For any $M \in \operatorname{mod}\Lambda$: ortho,
$\exists M' \in \operatorname{mod}\Lambda$, $M \oplus M'$: (i)W-tilt

(2) TFAE for $T \in \operatorname{mod}\Lambda$

(i) T : iW-tilt

(ii) T : W-tilt

(iii) T : ortho s.t. $|T| = |\Lambda|$.

(iv) T : maximal orthogonal

$\overset{\text{def}}{\iff}$ $\left(\begin{array}{l} \circ\ T : \text{ortho.} \\ \circ\ T \oplus M : \text{ortho} \\ \quad \Rightarrow M \in \operatorname{add} T. \end{array}\right)$ ⌟

<u>Cor</u> Λ : rep-fin. then.

(1) M : ortho $\Rightarrow$ $|M| \leq |\Lambda|$.

(2) $T \in \operatorname{mod}\Lambda$: W-tilt

$\Rightarrow$ $\mathcal{X}_T = {}^{\perp}T$.

(3) $\mathrm{GP}\Lambda = {}^{\perp}\Lambda$ (Λ : weakly Gor)

<u>Related Conj</u>

(Boundedness Conj [Happel])

M : ortho $\Rightarrow$ $|M| \leq |\Lambda|$.

(Proj = Inj Conj)

T_Λ : W-tilt $\Rightarrow$ $|T| = |\Lambda|$

(weak)
(Maximal Ortho. Conj)

T_Λ : W-tilt $\Rightarrow$ T : maximal ortho.

(iW-tilt)

Λ is maximal ortho.

$$\begin{array}{ccccc}
 & \Longrightarrow & (\mathrm{PIC}) & \Longrightarrow & (\text{Auslander-Reiten Conj}) \\
(\mathrm{BC}) & & & & \Uparrow \\
 & \Longrightarrow & (\mathrm{MOC}) & \Longrightarrow & (\mathrm{wMOC}) \quad (\mathrm{GNC})
\end{array}$$

These are true if Λ is rep-fin.

T : w-tilt

${}^{\perp}T$ has inj cog $\Longleftrightarrow$ (?) T : iw-tilt

$GP\Lambda = {}^{\perp}\Lambda$ $\quad \Lambda_\Lambda$

T : iw-tilt $\Longrightarrow$ w-tilt

max $\qquad |T| = |\Lambda|$

[Rickard-Schofield]

Ex Λ :
$$\begin{tikzcd} & 2 \arrow[d, shift left, "b"] \arrow[d, shift right, "c"'] \\ 1 \arrow[ur, "a"] & 3 \arrow[l, "d"] \end{tikzcd} \Big/ \langle ab, cd, da \rangle$$

: rep-inf.

$M \in {}^{\perp}S(1) \cap S(1)^{\perp}$

$S(1)$: simple at 1

$\not\Rightarrow$ $S(1)$: maximal ortho.

Ex
$$\begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[dl, "b"] \\ 3 \arrow[u, "c"] & \end{tikzcd} \Big/ \langle abcab, bcabca \rangle$$

$$\underbrace{P(2) \oplus P(3)}_{\text{proj-inj}} \oplus X \quad \left(X : \begin{smallmatrix} 1 \\ 2 \end{smallmatrix},\ \begin{smallmatrix} 2 \\ 3 \end{smallmatrix},\ \begin{smallmatrix} 3 \\ 1 \\ 2 \\ 3 \end{smallmatrix},\ \begin{smallmatrix} 1 \\ 2 \\ 3 \\ 1 \end{smallmatrix} \right)$$

$\rightsquigarrow$ W-filt, pd = id = ∞.

Conj

Λ : Iwanaga-Gorenstein

Λ : self-inj

, M : ortho.

$\Longrightarrow$ pd $M < \infty$

(This is true for rep-fin)

- M : W-tilt $\Longleftrightarrow$ M : tilt

$\Updownarrow$

M : cotilt

$\exists\ \Lambda_\Lambda$ s.t. ${}^\perp\Lambda \neq \mathrm{GP}\Lambda$
$\quad = \mathcal{X}_\Lambda$

$\Downarrow$ (Λ : not weakly Gor)

Λ_Λ : W-tilt,

Λ_Λ : iW-tilt $\Longleftrightarrow$ ${}^\perp\Lambda = \mathrm{GP}\Lambda$.