

Orthogonal modules and

(projectively) Wakamatsu tilting modules

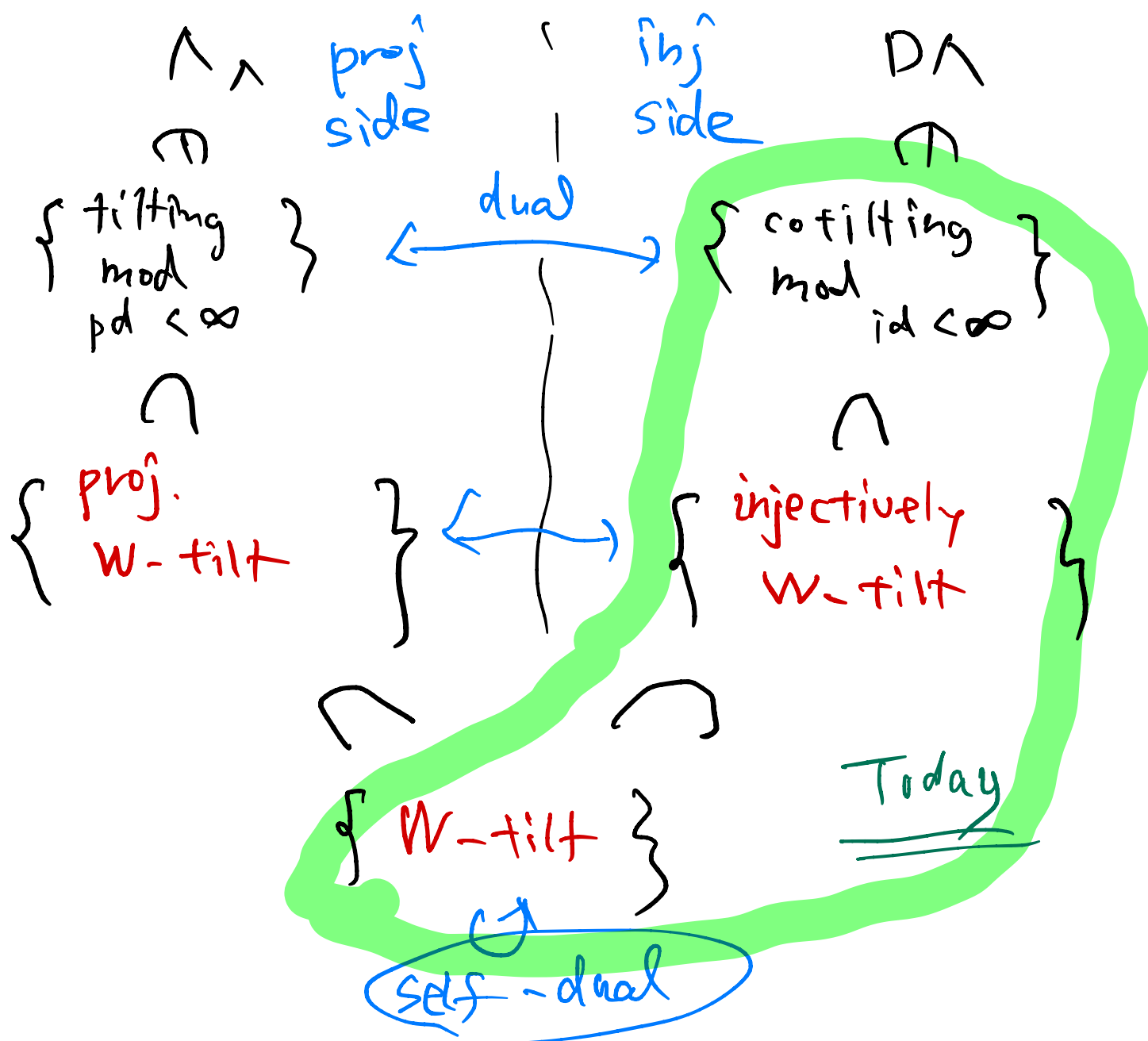
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injectively

W-tilt

## Overview

Hierarchy of "full rank" ortho. modules.



## Setting

$\Lambda$  : f.d k-alg    /  $k$  : field

$\text{mod } \Lambda$  : the cat of f.g. right  $\Lambda$ -modules

Def Let  $\mathcal{C} \subseteq \text{mod } \Lambda$  be a subcat.

(1)  $P \in \mathcal{C}$  is progenerator of  $\mathcal{C}$ .

$$\Leftrightarrow \text{(i)} \text{Ext}_{\Lambda}^1(P, \mathcal{C}) = 0$$

$$\text{(ii)} \forall X \in \mathcal{C}, \exists \text{ s.e.s.}$$

$$0 \rightarrow X' \rightarrow P_0 \rightarrow X \rightarrow 0$$

$$\text{s.t. } X' \in \mathcal{C}, P_0 \in \text{add } P.$$

(2) Dually for inj. cogen. of  $\mathcal{C}$ .

## W-tilting theory

$$\text{Ext}_{\Lambda}^{>0}(T, T) = 0$$



Def Let  $T \in \text{mod } \Lambda$  be orthogonal.

Define  $X_T \subseteq {}^{\perp} T \subseteq \text{mod } \Lambda$ .

$$(1) X \in {}^{\perp} T : \Leftrightarrow \text{Ext}_{\Lambda}^{>0}(X, T) = 0.$$

$$(2) X \in X_T$$

$$\Leftrightarrow \text{(i)} X \in {}^{\perp} T \quad (*)$$

(ii)  $\exists 0 \rightarrow X \rightarrow T_0 \rightarrow T_1 \rightarrow \dots$   
 s.t.  $T_i \in \text{add } T$   
 and  $\text{Hom}_\Lambda(X, T)$  is  
 exact.

Ex  
 $\bullet X_{D\Lambda} = {}^\perp D\Lambda = \text{mod } \Lambda.$

$\bullet X_\Lambda = \text{GP } \Lambda (= \text{Gorenstein-proj}).$

Thm Let  $T \in \text{mod } \Lambda$  : ortho.

(1)  $\Lambda_\Lambda$  is a progen of  ${}^\perp T$ .

(2)  $T$  is an inj cogen. of  $X_T$ .

$T$ : ortho	$X_T \leq {}^\perp T$	
progen	<span style="border: 1px solid red; border-radius: 50%; padding: 2px;">??</span> $\Lambda$ iff $T$ : w-tilt.	$\Lambda_\Lambda$
inj cogen	$T$	<span style="border: 1px solid red; border-radius: 50%; padding: 2px;">??</span> $T$ iff $T$ : iw-tilt.

Def  $T \in \text{mod } \Lambda$  is **cotilting**  
 if (1)  $\text{id } T < \infty$

(2)  $T$  : ortho.

(3)  $\exists 0 \rightarrow T_n \rightarrow \dots \rightarrow T_0 \rightarrow 0 \in \text{mod } \Lambda$   
 with  $T_i \in \text{add } T$ .

Thm [Auslander-Reiten '91]

If  $T$  is cotilt, then

$$\mathcal{X}_T = {}^\perp T.$$

( $\rightarrow {}^\perp T$  has proj dim  $\leq 1$  and inj cog dim  $\leq 1$ )

Def  $T \in \text{mod } \Lambda$  is **W-tilt**.

if (1)  $T$  : ortho.

(2)  $\Lambda \in \mathcal{X}_T$

( $\Leftrightarrow \Lambda$  is a progen of  $\mathcal{X}_T$ )

Ex

(1)  $\{\text{cotilt}\} \subset \{\text{W-tilt}\}$

(2)  $T_\Lambda$  is W-tilt  $\Leftrightarrow {}_\Lambda (D T)$  is W-tilt

$$(3) \{ \text{tilt} \} \subset \{ W\text{-tilt} \}.$$

$$(4) \wedge_{\Lambda} \text{ is } W\text{-tilt}.$$

$$\leadsto \chi_{\Lambda} = \text{GP} \wedge$$



Rem

Consider a "pair"  $(P, I)$  s.t.

$P$  : progen,  $I$  : inj cogen of some

exact cat  $\left( \begin{array}{l} \mathcal{E} \subseteq \text{mod } \Gamma \\ \mathcal{P} \text{ ext.-closed} \end{array} \exists \Gamma : \text{f.d. alg.} \right)$

(1) If  $T_{\Lambda} : W\text{-tilt}$ ,

$(\wedge, T)$  is such a pair  
 $(\chi_T)$ .

(2)  $\forall (P, I) : \text{pair (+ assump)}$   
 $[E] (P, I) \underbrace{\text{"equiv"}} (\wedge, T)$

for some alg  $\wedge, T_{\Lambda} : W\text{-tilt}$ .

$W\text{-tilt} = \text{"universal inj cogen"}$

Def  $T \in \text{mod } \Lambda$  is *injectively*  
*W-tilt* (*iW-tilt*) if.

(1)  $T : \text{ortho}$ ,

(2)  $T$  is an inj cogem of  ${}^{\perp}T$ .

( $\Leftrightarrow \forall X \in {}^{\perp}T, \exists X \hookrightarrow T^n$ )

Prop

$T : \text{iW-tilt} \Leftrightarrow T : \text{W-tilt}$  s.t.

$$X_T = {}^{\perp}T.$$

Ex

(1)  $\{\text{cotilt}\} \subset \{\text{iW-tilt}\}$

(2)  $\Lambda_{\Lambda}$  is iW-tilt

$$\Leftrightarrow \begin{array}{c} X_{\Lambda} = {}^{\perp}\Lambda \\ \parallel \\ \text{GP } \Lambda \end{array}$$

$\Leftrightarrow : \Lambda : \text{weakly Gorenstein.}$

# Main Results

## Classical Result [Bongartz]

$T \in \text{mod } \Lambda$  : cttlt with  $\text{id} \leq 1$

$$\Leftrightarrow \begin{cases} \circ \text{id } T \leq 1 \\ \circ T : \text{ortho.} \\ \circ |T| = |\Lambda| \end{cases}$$

$$\cdots 0 \rightarrow T_1 \rightarrow T_0 \rightarrow \bigvee \Lambda \rightarrow 0$$

$\#$  of indec summands of  $T \} / 2$

This depends "completion"

which works for  $\text{id} \leq 1$

At least for  $\Lambda$  : rep-fin?

Thm Let  $\Lambda$  be rep-fin.

(1) For any  $M \in \text{mod } \Lambda$  : ortho,  
 $\exists M' \in \text{mod } \Lambda$ ,  $M \oplus M'$  : (i) W-tilt

(2) TFAE for  $T \in \text{mod } \Lambda$

(i)  $T$  : iW-tilt

(ii)  $T$  : W-tilt

(iii)  $T$  : ortho s.t.  $|T| = |\Lambda|$ .

(iv)  $T$ : maximal orthogonal

$$\left( \begin{array}{l} \Leftrightarrow \bullet T: \text{ortho.} \\ \bullet T \oplus M: \text{ortho} \\ \Rightarrow M \in \text{add } T. \end{array} \right)$$

Cor  $\Lambda$ : rep-fin. then.

$$(1) M: \text{ortho} \Rightarrow |M| \leq |\Lambda|$$

$$(2) T \in \text{add } \Lambda : \text{W-tilt}$$

$$\Rightarrow X_T = {}^\perp T$$

$$(3) \text{GP } \Lambda = {}^\perp \Lambda \quad (\Lambda: \text{weakly Gor})$$

### Related Conj

(Boundedness Conj [Happel])

$$M: \text{ortho} \Rightarrow |M| \leq |\Lambda|$$

(Proj = Inj Conj)

$$T_\Lambda : \text{W-tilt} \Rightarrow |T| = |\Lambda|$$

(weak)

(Maximal Ortho. Conj)

$$T_\Lambda : \text{W-tilt} \Rightarrow T: \text{maximal ortho.}$$

(i W-tilt)

$\Lambda$  is maximal ortho.

$$\begin{array}{lcl}
 (BC) \Rightarrow (PIC) \Rightarrow & \left( \begin{array}{l} \text{Auslander-Reiten} \\ \text{Conj} \end{array} \right) & \\
 \Rightarrow (MOC) \Rightarrow & (wMOC) & (GND) \\
 & \text{---} & 
 \end{array}$$

These are true if  $\Lambda$  is rep-fn.

$T, W\text{-tilt}$

$\perp T$  has inj cog

$\langle \perp \rangle$   $T: iw\text{-tilt}$

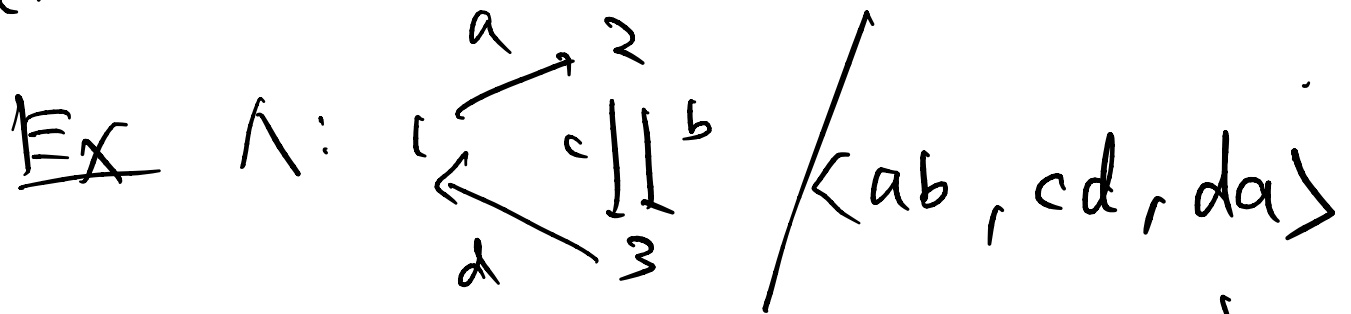
$$GP \Lambda = \perp \Lambda \quad \Lambda \wedge$$

$T: iw\text{-tilt} \Rightarrow W\text{-tilt}$

$\max$

$$|T| \approx \omega$$

[Rickard-Schofield]

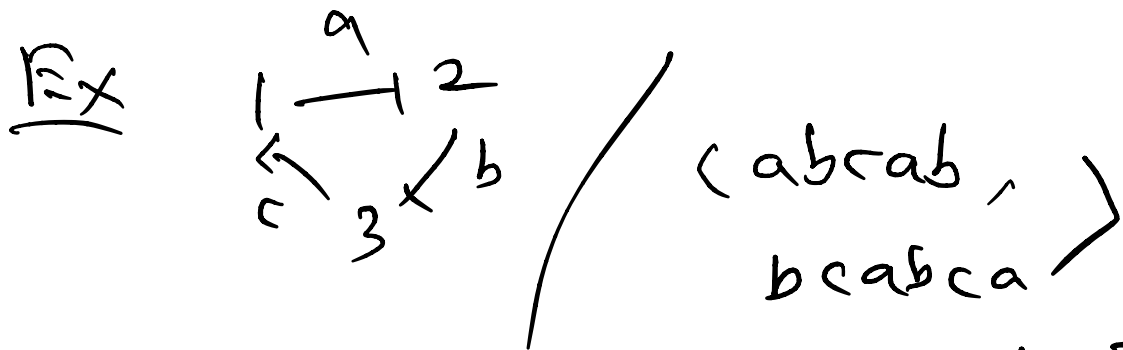


: rep-inf.

$$M \in {}^\perp \text{sc} \cap \text{sc}^\perp$$

$\text{sc}(1)$  : simple at 1

$\Rightarrow \text{sc}(1)$  : maximal ortho.



$$\underbrace{P(2) \oplus P(3)}_{\text{proj-inj}} \oplus X \quad \left( X : \begin{array}{cc} 1 & 2 \\ 2 & 3 \\ 3 & 1 \end{array} \right)$$

$\leadsto$  W-filt,  $\text{pd} = \text{id} = \infty$

Conj

$\Lambda$  : Iwanaga-Gorenstein

$\cap M$  : ortho.

$\Rightarrow \text{pd } M < \infty$

(This is true for rep-fini)

$\Lambda$  : self-inj

$M : W\text{-tilt} \iff \begin{matrix} M : \text{tilt} \\ \updownarrow \\ M : \text{rot}\text{-}ff \end{matrix}$

$\exists \Lambda \text{ s.t. } \left( \begin{array}{c} \Gamma \wedge \neq \text{GPA} \\ \parallel \\ X_{\Lambda} \end{array} \right)$   
 $\Downarrow$   
 $(\Lambda : \text{not weakly Gov})$

$\Lambda_1: W\text{-}t\text{'th}$

$$\wedge_n : iW - \text{fibre} \iff \perp_n = \text{GP} \wedge.$$