Maximal self-orthogonal modules and a new generalization of tilting modules

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§0. Background
§1. (Projectively) Wakamatsu tilting
§2. Results
§3. Sketch of proofs.

W-tilt
||
Wakamatsu-tilting module.

Setting

Λ : f.d. k-alg (k : field)

$\mathrm{mod}\Lambda$: the cat of f.g. right Λ-modules

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proj side	dual	inj side
Λ : progenerator	$\longleftrightarrow$ **dual**	$D\Lambda$: inj cogenerator
$\rotatebox{90}{$\in$}$		$\rotatebox{90}{$\in$}$
$\{$tilting mod, pd $\le 1\}$ $\subset$ sτ-tilt.		$\{$cotilting, id $\le 1\}$ $\subset$ sτ-tilt.
$\cap$		$\cap$
$\{$tilting mods, $\boxed{\text{pd} < \infty}\}$		$\{$cotilting, $\boxed{\text{id} < \infty}\}$
$\cap$		$\cap$
Today: $\{$projectively W-tilt$\}$ $\boxed{\text{pd} = \infty}$		$\{$injectively W-tilt$\}$ $\boxed{\text{id} = \infty}$
$\cap$		$\cap$

$\{$W-tilt$\}$

self-dual.

§0. (SO)

<u>Def</u> $M \in \mathrm{mod}\,\Lambda$: self-orthogonal

$:\Longleftrightarrow$ $\mathrm{Ext}_\Lambda^{>0}(M, M) = 0$

(i.e $\mathrm{Ext}_\Lambda^i(M,M)=0 \ \forall i>0$)

Very elementary, but

$\exists$ open conjectures :

<u>Conj 1.</u> (Boundedness Conj [Happel])

$M_\Lambda : SO \Rightarrow |M| \leq |\Lambda|$

$$|M| \overset{\cdot\cdot}{=} \#\{\text{indec summands of } M\}/\cong.$$

<u>Conj 2.</u> (Auslander-Reiten Conj)

$\Lambda \oplus M : SO$

$\Rightarrow M \in \mathrm{add}\,\Lambda = \mathrm{proj}\,\Lambda.$

($\Leftarrow$: Λ_Λ is maximal SO)

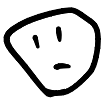
<u>Def</u> T : maximal SO $\in \mathrm{mod}\,\Lambda$

$:\Longleftrightarrow \begin{cases} T : SO \\ T \oplus X : SO \Rightarrow X \in \mathrm{add}\,T. \end{cases}$ $\lrcorner$

Observation

Prop M : SO, $\mathrm{pd}\, M \leq 1$

$\Rightarrow$ $|M| \leq |\Lambda|$. ┘

 $\rightsquigarrow$ M : partial tilt.

$\rightsquigarrow$ $\exists N$, $M \oplus N$: tilt ($\mathrm{pd} \leq 1$)

① (Bongartz completion)

$\rightsquigarrow$ $|M| \leq |M \oplus N| \underset{\text{tilt}}{=} |\Lambda|$ ②

$\square$

Rem

For M : SO, $\mathrm{pd}\, M < \infty$

$\Rightarrow$ "completion" of M may not exist.

e.g.

$$\Lambda : \begin{tikzcd} & 2 \arrow[d, shift right, "b"'] \arrow[d, shift left, "c"] \\ 1 \arrow[ur, "a"] & 3 \arrow[l, "d"] \end{tikzcd} \Big/ \langle ab, cd, da \rangle$$

$\rightsquigarrow$ $S(1)$: $\mathrm{pd} = 2$.

maximal SO. [Rickard-Schofield]

Rem

Conj 1 & Conj 2 are open even if $\mathrm{pd}\, M < \infty$. ┘

(<u>Today</u> Conj 1, 2: true for $\underline{\Lambda : \text{rep-fin}}$.)

<u>§1.</u>

<u>Def</u> Let $T \in \operatorname{mod}\Lambda$.

(1) $T^{\perp} := T^{\perp_{>0}} \subseteq \operatorname{mod}\Lambda$

$= \{ X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^{>0}_{\Lambda}(T, X) = 0 \}$

(2) $\mathcal{Y}_T \subseteq T^{\perp}$

$:= \left\{ X \in T^{\perp} \;\middle|\; \begin{array}{l} \exists \cdots \to \overbrace{T_1 \to T_0}^{\operatorname{add} T} \to X \to 0 : \text{ex} \\ \text{s.t. } (T, -)\text{-exact.} \end{array} \right\}$

(3) In general, For $\mathcal{C} \subseteq \operatorname{mod}\Lambda$: ext-closed

$P \in \mathcal{C}$ is a progenerator of $\mathcal{C}$.

$:\Longleftrightarrow \begin{cases} \circ \ \operatorname{Ext}^{1}_{\Lambda}(P, \mathcal{C}) = 0 \\ \circ \ \forall C \in \mathcal{C}, \ \exists \text{ s.e.s.} \\ \quad 0 \to C' \to P_0 \to C \to 0 \\ \quad \text{s.t. } C' \in \mathcal{C}, \ P_0 \in \operatorname{add} P. \end{cases}$

(Dually for inj cogen of $\mathcal{C}$). ┘

Prop Let $T \in \operatorname{mod}\Lambda$ be **SO**.

(1) $D\Lambda \in T^{\perp}$: inj cogen of $T^{\perp}$. $(T \in T^{\perp})$

(2) $T \in \mathcal{Y}_T$: progen of $\mathcal{Y}_T$. ┘

T : SO	$\mathcal{Y}_T$	$\subseteq T^{\perp}$
progen	T	?? $\boxed{T} \Longleftrightarrow T$: pw-tilt
inj cogen	?? $\boxed{D\Lambda} \Longleftrightarrow T$: w-tilt.	$D\Lambda$

Ex $T \in \operatorname{mod}\Lambda$: tilting (pd $< \infty$)

$$:\Longleftrightarrow \left\{\begin{array}{l} (1)\ T : \mathrm{SO} \\ (2)\ \mathrm{pd}\, T < \infty \\ (3)\ \exists\ 0 \to \Lambda \to \underbrace{T^0 \to \cdots \to T^d}_{\text{add } T} \to 0 \ : \text{ex} \end{array}\right.$$

$\Longrightarrow \mathcal{Y}_T = T^{\perp}$ (by [Auslander-Reiten])

(hence $T^{\perp}$ has progen T, inj cogen $D\Lambda$)

In general, $\mathcal{Y}_T \subsetneq T^{\perp}$

(for P : indec proj.
$\rightsquigarrow$ $P^\perp = \mathrm{mod}\,\Lambda$, but
$\mathcal{Y}_P$: small
$\subsetneq \mathrm{mod}\,\Lambda$)

Def T : projectively W-tilt.
(pW-tilt)

$:\Longleftrightarrow$ (i) T : SO.
(ii) T : progen of $T^\perp$. $\lrcorner$

Ex
$\{\text{tilt}\} \subseteq \{\text{pW-tilt}\}$ $\lrcorner$

Prop (equiv. char.)
TFAE for $T \in \mathrm{mod}\,\Lambda$: SO.
(1) T : pW-tilt
(2) $T^\perp \subseteq \mathrm{Fac}\,T$. e.g.
$\forall X \in T^\perp,\ \exists\, T' \twoheadrightarrow X$: surj.
(3) $\mathcal{Y}_T = T^\perp$. $\lrcorner$

(**Rem** Dually $\mathcal{X}_T \subseteq {}^\perp T$,
T : iW-tilt.).

Def

$T \in \mathrm{mod}\,\Lambda$ is W-tilt

if (1) T : SO

(2) $D\Lambda \in \mathcal{Y}_T$.

i.e., $\exists \cdots \to T_1 \to T_0 \to D\Lambda \to 0$

$\underbrace{\cdots \to T_1 \to T_0}_{\text{add } T}$: $(T,-)$-exact.

Prop TFAE for $T \in \mathrm{mod}\,\Lambda$: SO

(1) T : W-tilt

(2) $\mathcal{Y}_T$ has inj cogen $D\Lambda$

(3) $(DT)^{\wedge}$: W-tilt

(4) $\Lambda_\Lambda \in \mathcal{X}_T$ i.e.,

$0 \to \Lambda \to T^0 \to T^1 \to \cdots$

: ex s.t. $(-,T)$-exact. ⌟

Cor T : W-tilt

$\Rightarrow$ $\mathcal{Y}_T$ has progen T,

inj cogen $D\Lambda$. ⌟

Cor

$\{\text{pW-tilt}\} \subseteq \{\text{W-tilt}\}$.

Rem

$\forall$ (P, I) : "progen - inj cogen pair" of exact cat (Hom-fin K S)

$(P, I) \sim (T, D\Lambda)$ "equiv" for some f.d. alg Λ.

[E] T_Λ: W-tilt.

So W-tilt is natural from exact cat.

Q $\{pW\text{-tilt}\} \subseteq \{W\text{-tilt}\}$

Are they actually different?

A Yes in general, but

No if $\boxed{\Lambda : \text{rep-fin.}}$

Q Easy systematic method for W-tilt, but NOT pW-tilt ??

§2.

Thm 1. ("completion")

Let $M \in \mathrm{mod}\,\Lambda$: SO,

suppose $\# \mathrm{ind}(M^{\perp}) < \infty$.

(e.g. Λ : rep-fin)

Then $\exists\, N \in \mathrm{mod}\,\Lambda$. s.t.

$$M \oplus N : \underbrace{p W\text{-tilt}}_{(W\text{-tilt})}.$$

Thm 2.

Let $T \in \mathrm{mod}\,\Lambda$ with $\# \mathrm{ind}(T^{\perp}) < \infty$.

(e.g. Λ : rep-fin)

TFAE

(1) T : pW-tilt

(2) T : W-tilt

(3) T : SO and $|T| = |\Lambda|$

(4) T : maximal SO.

Cor. (BC) Boundedness Conj

Let $M \in \mathrm{mod}\Lambda$: SO

If $\# \operatorname{ind}(M^{\perp}) < \infty$ (e.g. Λ: rep-fin),

then $|M| \leq |\Lambda|$ ⌟

(Pf) By Thm 1. $\exists N \in \mathrm{mod}\Lambda$

s.t. $M \oplus N$: W-tilt.

Then $(M \oplus N)^{\perp} \subseteq M^{\perp}$.

$\operatorname{ind} < \infty \Leftarrow \operatorname{ind} < \infty$

So Thm 2 implies.

$$|M \oplus N| = |\Lambda|$$

Thus $|M| \leq |M \oplus N| \leq |\Lambda|$ □

Cor (ARC) [Auslander-Reiten]

Λ : rep-fin $\Rightarrow$ Λ_{Λ} : max. SO. ⌟

Cor Computer can list up all W-tilt. modules! (if Λ : rep-fin).

[FD Applet]

Problem (Nakayama)

For a given class of algs,

count $\#\{W\text{-tilt}\}$!

(or $\#\{\text{tilt mod } (\mathrm{pd} < \infty)\}$)

(Difficult) Conjectures

T_Λ: SO.

$$\begin{tikzcd}
p W\text{-tilt} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] \arrow[dr, Leftrightarrow] & W\text{-tilt} \arrow[d, Rightarrow] \arrow[dl, Leftrightarrow] \\
|T| = |\Lambda| & T : \text{max. SO.} \arrow[l, Rightarrow, "\times"] \arrow[u, Rightarrow, "\times"']
\end{tikzcd}$$

[RS]

$\Leftrightarrow$: All open.

(BC) : T : SO $\Rightarrow$ $|T| \leq |\Lambda|$

T: W-tilt $\Rightarrow |T| = |\Lambda|$ W-tilt $\Rightarrow$ max. SO.

|| ||

$$\Downarrow \qquad\qquad\qquad\qquad\qquad\qquad \Downarrow$$

T: pW-tilt $\Rightarrow |T| = |\Lambda|$ $\qquad$ pW-tilt $\Rightarrow$ max SO.

$(t = 2''_L)$ $\Uparrow$ $\Downarrow$ $\qquad\qquad$ $\not\swarrow$ equiv.

(ARC)

Λ_n : max SO.

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§3. Sketch of Proof

Thm 1 Suppose $M \in \operatorname{mod} \Lambda$: SO with $\# \operatorname{ind}(M^\perp) < \infty$. $\quad$ $M^\perp$: cov. fin.

Then $\Lambda \longrightarrow N$: left min $(M^\perp)$-approx

$\Rightarrow$ N : proj in $M^\perp$ (split) [Auslander–Smalo]

($\underbrace{\text{minimal cover}}$)

$\forall X \in M^\perp \ \exists\ N^l \longrightarrow X$: surj.

Claim $M \oplus N$: pW-tilt with $M^\perp = (M \oplus N)^\perp$.

$\square$

Thm2.

Key Lem ($\Leftarrow M^{\perp}$)

$\mathcal{E}$: Hom-fin KS exact cat

$\#\, \mathrm{ind}\, \mathcal{E} < \infty$.

$\Longrightarrow$ $\#\{\text{indec proj in } \mathcal{E}\}$

$= \#\{\text{— inj —}\}$.

(∵) ∃ AR seq

有限 $\leadsto$ # indec non-proj

= # indec non-inj

$\Longrightarrow$ Claim follows.

Conj

$\mathcal{E}$: Hom. fin
(AR 列もつ)

$\Longrightarrow$ 一般に # proj
= # inj

Rem

pW-tilt + pd < ∞

$\Updownarrow$ OK

tilt.

W-tilt + pd < ∞

$\Updownarrow$?? OK

tilt

Wakamatsu tilt conj.