II.4. Smalø's symmetry

Thm $(\mathcal{T}, \mathcal{F})$: tors pair in mod$A$ とする

$\mathcal{T}$: fun. fin $\Longleftrightarrow$ $\mathcal{F}$: fun. fin.

Rem これは f.d. alg でないと成り立たない

(R : d.v.r.

(fl. R, proj R) : tors. pair in mod R

not fun. finite! / fun. fin.)

Proof : 講演原稿 1. の最後.

キーアイデア

Thm $\mathcal{T}$: tors in mod$A$. TFAE.

(1) $\mathcal{T}$: fun. fin.

(2) $|P(\mathcal{T})| = |I(\mathcal{T})|$.

一般に $\leq$, $|I(\mathcal{T})| = |\mathrm{supp}\,\mathcal{T}|$.

III. Wide interval
（広大区間）

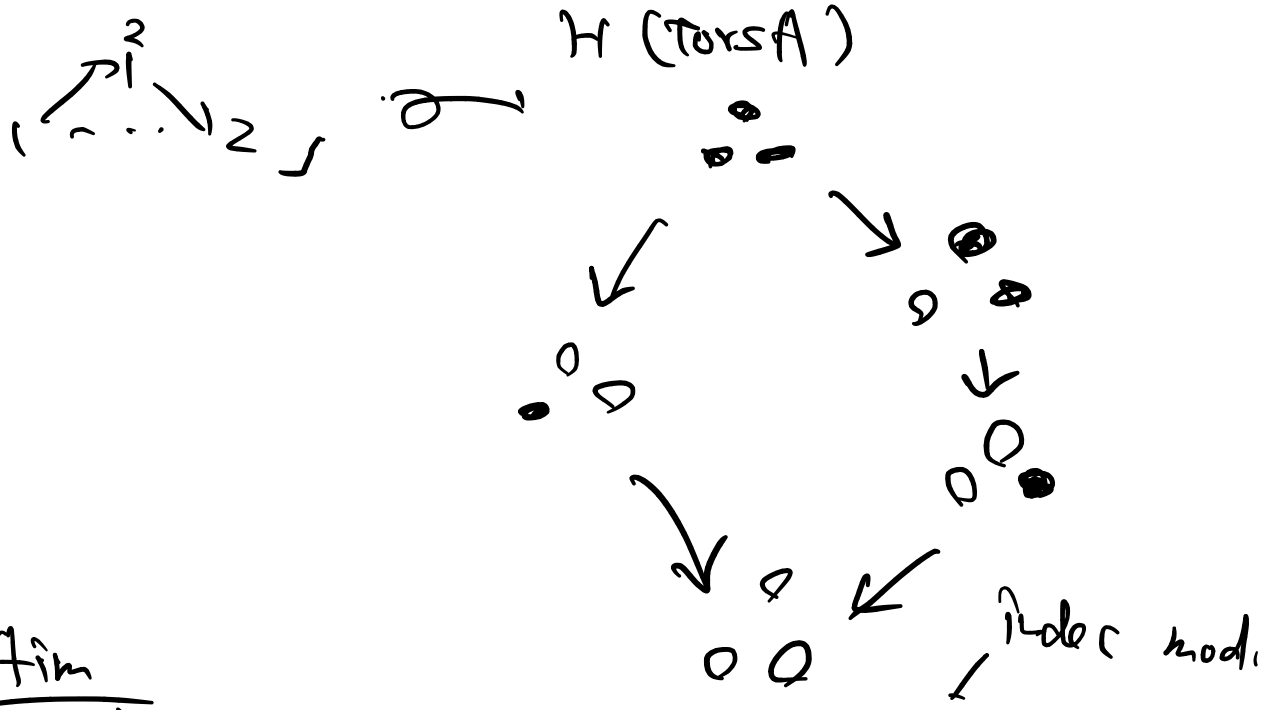
Def $\vec{H}(\mathrm{tors}A)$: Hasse quiver :

vtx : $\mathcal{T} \in \mathrm{tors}A$

arrow : $\mathcal{T} \to \mathcal{U}$

$:\Longleftrightarrow \begin{cases} \mathcal{T} \supsetneq \mathcal{U}, \\ \nexists\, \mathcal{C} \in \mathrm{tors}A, \\ \quad \mathcal{T} \supsetneq \mathcal{C} \supsetneq \mathcal{U}. \end{cases}$

Ex



Aim

$\vec{H}(\mathrm{tors}A)$の矢を"brick"と

sp-projを使って解釈する.

II.1. Heart.

Def $\mathcal{U}, \mathcal{T} \in \operatorname{tors} A$ s.t. $\mathcal{U} \subseteq \mathcal{T}$.

(1) $[\mathcal{U}, \mathcal{T}] := \{ \mathcal{C} \in \operatorname{tors} A \mid \mathcal{U} \subseteq \mathcal{C} \subseteq \mathcal{T} \}$

itv (interval)

(2) $\mathcal{H}_{[\mathcal{U},\mathcal{T}]} \subseteq \operatorname{mod} A$.

$$\mathcal{H}_{[\mathcal{U},\mathcal{T}]} = \mathcal{T} \cap \underbrace{\mathcal{U}^{\perp}}_{\text{torf.}} \quad (= \text{“} \mathcal{T} - \mathcal{U} \text{”})$$

$$\left(\begin{array}{ccc} (\mathcal{T}, & \mathcal{T}^{\perp}) & : \text{tors pair} \\ \cup\!\mid & \cap\!\mid & \\ (\mathcal{U}, & \mathcal{U}^{\perp}) & : [\mathcal{T}, \mathcal{U} \in \operatorname{tors} A,\ \mathcal{U} \subseteq \mathcal{T}] \end{array} \right.$$

Ex

- $\mathcal{H}_{[0,\mathcal{T}]} = \mathcal{T}$. ($\Leftrightarrow$ “$\mathcal{T} - 0 = \mathcal{T}$”)

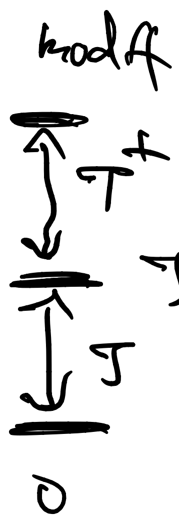
- $\mathcal{H}_{[\mathcal{T}, \operatorname{mod} A]} = \mathcal{T}^{\perp}$ (torf)

[浅井-Pfeifer]

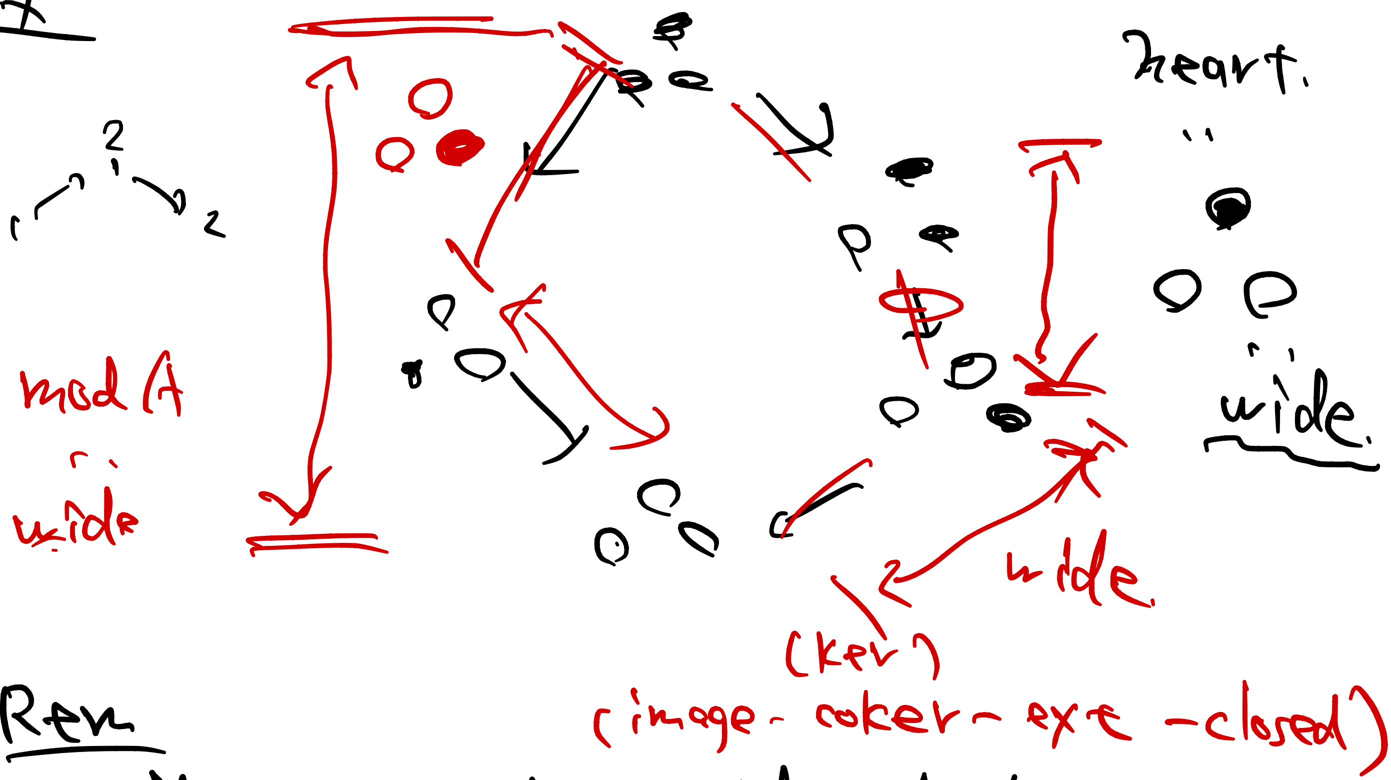
Def $[\mathcal{U}, \mathcal{T}]$: wide itv

:$\Longleftrightarrow$ $\mathcal{H}_{[\mathcal{U},\mathcal{T}]}$: wide subcat

(ker, cok, ext-closed)



Ex



Rem

$\forall\, \mathcal{C} \subseteq \operatorname{mod} A$: wide subcat は

必ず $\exists\, [\mathcal{U}, \mathcal{T}]$ s.t. $\mathcal{C} = \mathcal{H}_{[\mathcal{U},\mathcal{T}]}$.

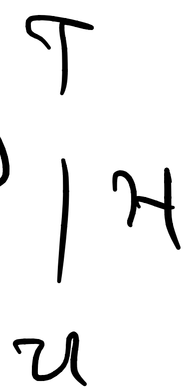
Prop

$[\mathcal{U}, \mathcal{T}]$: itv. with heart $\mathcal{H}$ とき

(1) $\mathcal{T} = \mathcal{U} * \mathcal{H}$ ("$\mathcal{T} = \mathcal{U} + \mathcal{H}$")

(2) $\mathcal{U} = \mathcal{T} \cap {}^{\perp}\mathcal{H}$ ("$\mathcal{U} = \mathcal{T} - \mathcal{H}$")

((3) $\mathcal{H} = \mathcal{T} \cap \mathcal{U}^{\perp}$)



(1)のみ, $\mathcal{U}, \mathcal{H} \subseteq \mathcal{T}$.

$\mathcal{T}$: ext-closed $\rightsquigarrow$ $\mathcal{U} * \mathcal{H} \subseteq \mathcal{T}$.

逆に $X \in \mathcal{T}$ ~~を~~.

$\rightsquigarrow$ $(\mathcal{U}, \mathcal{U}^{\perp})$: tors. pair.

$$\Rightarrow \quad 0 \to uX \to X \to gX \to 0$$

$$uX \in \mathcal{U}, \quad X \in \mathcal{T}, \quad gX \in \mathcal{U}^{\perp} \cap \mathcal{T} = \mathcal{H}$$

$\therefore$ $X \in \mathcal{U} * \mathcal{H}$. $\square$

Key

Prop $[\mathcal{U}, \mathcal{T}]$: itv. $\mathcal{H}$: そのheartとき.

$\mathcal{U}, \mathcal{T}, \mathcal{H}$ のうち 2つが fun. fin

$\Rightarrow$ 残りも fun. fin.

(2-out-of-3)

$(\mathcal{U}, \mathcal{H} \Rightarrow \mathcal{T})$

次のFactと $\mathcal{T} = \mathcal{U} * \mathcal{H}$ から従う:

Fact. $\mathcal{C}, \mathcal{D} \subseteq \operatorname{mod} A$ について

$\mathcal{C}, \mathcal{D}$: fun. fin $\Rightarrow$ $\mathcal{C} * \mathcal{D}$ も fun. fin.

$(\mathcal{T}, \mathcal{H} \Rightarrow \mathcal{U})$

上のことの双対と Smalø's symmetry から従う.

$$\left(\begin{array}{ccc} \mathcal{J} & \overset{\text{Smalo}}{\longrightarrow} & \mathcal{T}^{\perp} \\ \text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\supseteq$}} \\ \mathcal{U} & \xleftarrow{\text{Smalo}} & \mathcal{U}^{\perp} \end{array}\right.$$

$\mathcal{J}$: fun fin, $\mathcal{T}^{\perp}$: fun. fin.] $\mathcal{H}$] $\pm \delta_j$

tors A $\longleftarrow$ torf : fun. fin

$(\mathcal{T}, \mathcal{U} \Rightarrow \mathcal{H})$

$\mathcal{H} = \mathcal{T} \cap \underbrace{\mathcal{U}^{\perp}}$ \mathcal{G}

$\forall X \in \mathrm{mod} A$. $\qquad$ $= \mathcal{G}$: fun. fin

$\mathcal{G}$: torf

$\mathcal{G}$-approx (Smalo)

$\mathcal{G} \supseteq \mathcal{G}_X \to X$ の全射が

$\mathcal{T} \cap \mathcal{G} \ni t(\mathcal{G}_X) \to \mathcal{G}_X$ right $\mathcal{T}$-approx, right $\mathcal{H}$-approx.

$\mathcal{T} \cap \mathcal{G} = \mathcal{H}$.

$t(\mathcal{G}_X) \in \mathcal{T}$

□.

(Fact (=))

$\mathcal{C}, \mathcal{D}$: cov. fin. $X \in \mathrm{mod} A$.

$C \to \square \to D$

$X \longrightarrow \underline{D^X}$: left $\mathcal{D}$ approx とる.

$\oplus$

$A^n \to D^X$ surj

$\rightsquigarrow$

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "\text{left } \mathcal{C}\text{-approx.}"'] \arrow[dr, phantom, "\text{p.o.}"] & X \oplus A^2 \arrow[r] \arrow[d] & D^X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C^K \arrow[r] & \square \arrow[r] & D^X \arrow[r] & 0
\end{tikzcd}$$

$\in \mathcal{C} * \mathcal{D}$

ここが…

left $(\mathcal{C} * \mathcal{D})$-approx! (HW)

Thm [Asai-Pfeifer, Jasso]

$[\mathcal{U}, \mathcal{T}]$: wide itv. in tors A.

heart $\mathcal{W}$ とあると、

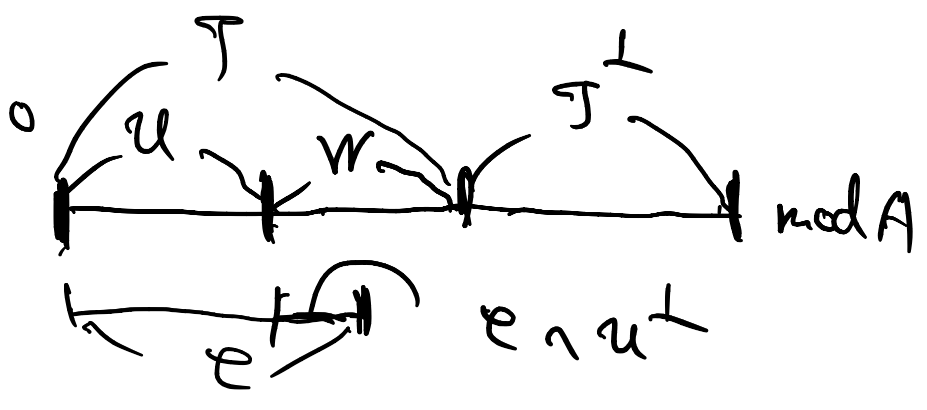
$$[\mathcal{U}, \mathcal{T}] \overset{1-1}{\longleftrightarrow} \underbrace{\operatorname{tors} \mathcal{W}}$$

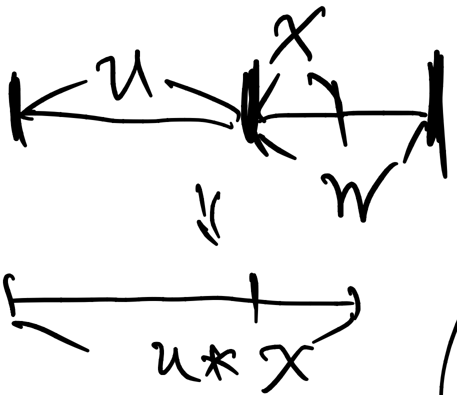
($\mathcal{W}$: abelian cat のだ その tors の poset)

$$\mathcal{C} \longmapsto \mathcal{C} \cap \mathcal{U}^{\perp}$$

$$\mathcal{U} * \mathcal{X} \longleftarrow\!\shortmid \mathcal{X}$$

poset isom.





「足立・E・塚本」

Rem

wide itv でなくても
高てっtrs 位で成立
完全圏の tors. pair

∵ **well-defined 性**

$\mathcal{C} \in [\mathcal{U}, \mathcal{T}]$

$\rightsquigarrow (\mathcal{C} \cap \mathcal{U}^{\perp}, \mathcal{T} \cap \mathcal{C}^{\perp})$: tors pair in $\mathcal{W}$.

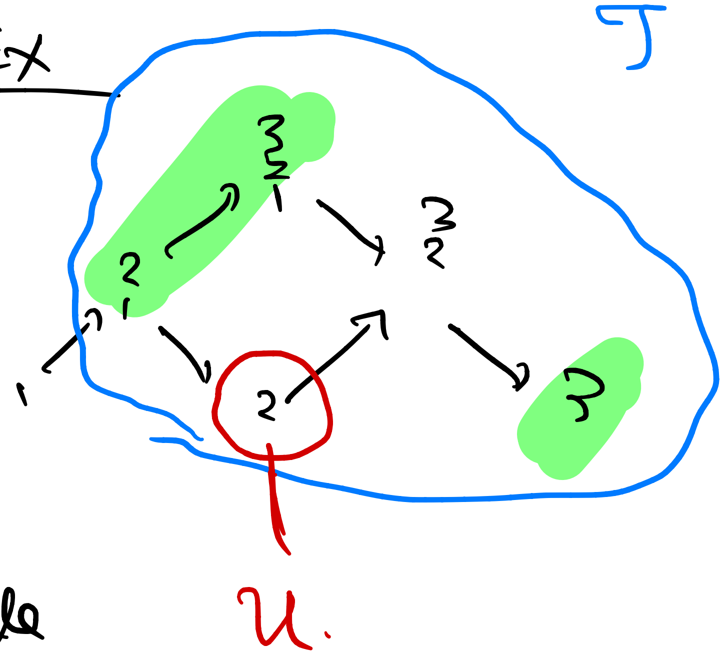
$\mathcal{X} \in \operatorname{tors} \mathcal{W}$, $(\mathcal{X}, \mathcal{Y})$: tors pair in $\mathcal{W}$

$\Rightarrow (\mathcal{U} * \mathcal{X}, \mathcal{Y} * \mathcal{T}^{\perp})$: tors pair in mod A.

$$\begin{aligned}(\ \operatorname{mod} A &= \underline{\mathcal{T}} * \mathcal{T}^{\perp} \\ &= \underline{\mathcal{U} * \mathcal{W}} * \mathcal{T}^{\perp} \\ &= \mathcal{U} * (\mathcal{X} * \mathcal{Y}) * \mathcal{T}^{\perp} \\ &\overset{(*:\text{ass})}{=} (\mathcal{U} * \mathcal{X}) * (\mathcal{Y} * \mathcal{T}^{\perp}) \quad \square\end{aligned}$$

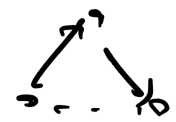
互いに逆写
やれば
わかる。

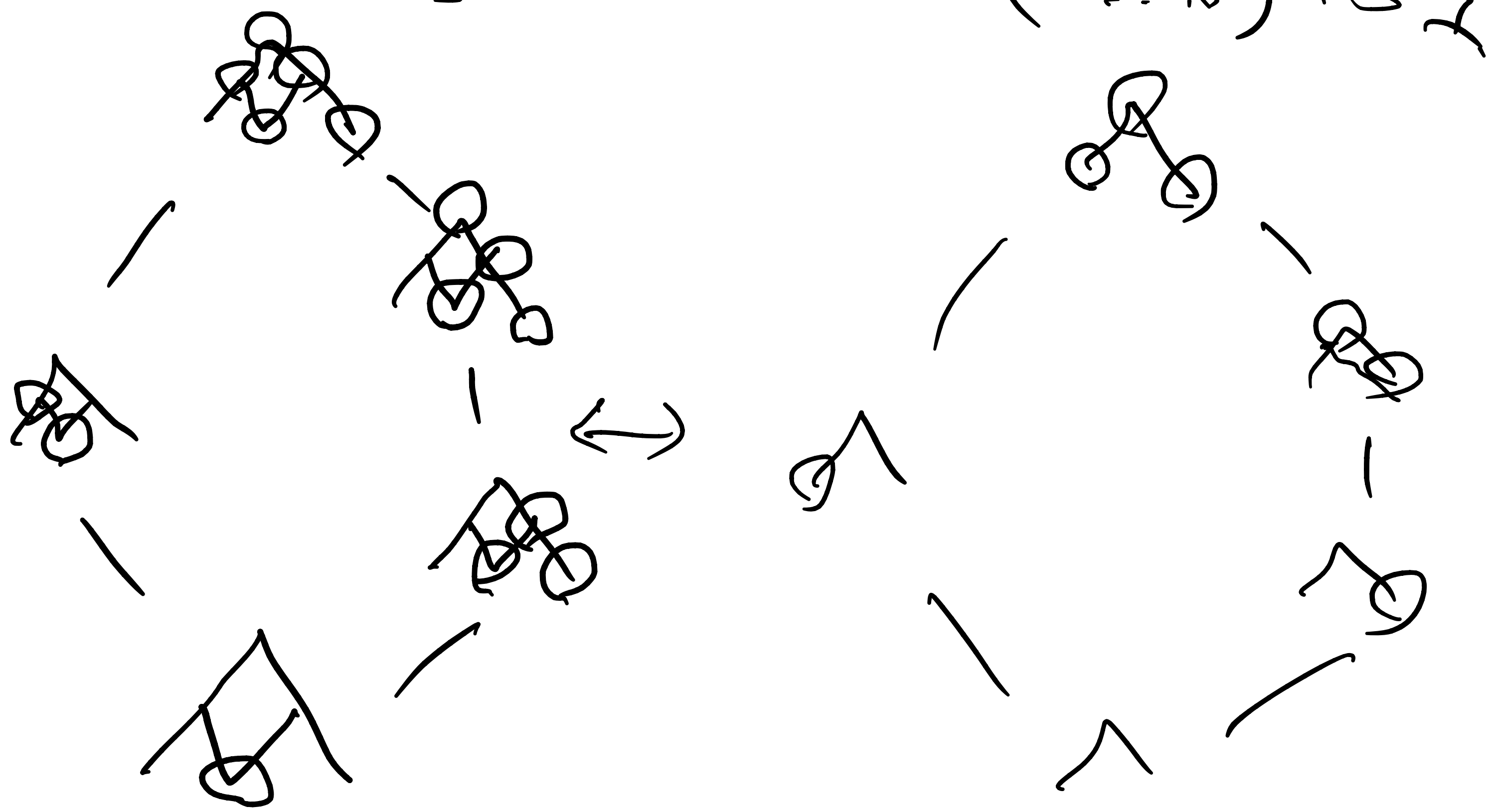
Ex



$\mathcal{U} \subsetneq \mathcal{T}$.

$\mathcal{T} \cap \mathcal{U}^{\perp}$
$\parallel$
$-$
$\therefore$ wide subcat.

wide itv. : $[\mathcal{U}, \mathcal{T}] \overset{1-1}{\longleftrightarrow}$ tors() : ≤ 2



II.2. Brick label (煉瓦)

Def $B \in \operatorname{mod} A$: brick

$:\Longleftrightarrow$ $\operatorname{End}_A(B)$: division ring (skew-field) (i.e. $f: B \to B$ は 0 or isom).

| ⇝ B : indec)

<u>Def</u> $\mathcal{C} \subseteq \mathrm{mod}A$ に

・ $\mathrm{Filt}\,\mathcal{C} := \bigcup_{n \geq 0} \overbrace{\mathcal{C} * \cdots * \mathcal{C}}^{n}$.

・ $\mathrm{brick}\,\mathcal{C} := \{ B \in \mathcal{C} \mid B : \mathrm{brick} \} / \cong$.

<u>Lem</u> [Demonet-Iyama-Reading-Reiten-Thomas]

$\forall\, 0 \neq X \in \mathrm{mod}A$, $\exists\, f: X \to X$

s.t. $\mathrm{Im} f$: brick 」

(証) $l(X)$ についての induction.

・ $l(X) = 1 \Rightarrow X$: simple.

$\Rightarrow X$: brick (Schur's lemma)

$\therefore X \xrightarrow{\mathrm{id}_X} X$ が条件みたす.

・ $l(X) > 1$ とき.

・ X : brick $\Rightarrow$ id_X で OK.

- X : not brick

$\Rightarrow \exists\ f : X \to X$

: $f \neq 0$, not isom.

$\Rightarrow$

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, two heads] & & X \\ & \mathrm{Im} f \arrow[ur, hook] & \end{tikzcd}$$

$l(\mathrm{Im} f) < l(X)$

$\rightsquigarrow$ induction. $\exists\ \mathrm{Im} f \twoheadrightarrow B \hookrightarrow \mathrm{Im} f$

B : brick.

$\rightsquigarrow X \twoheadrightarrow \mathrm{Im} f \twoheadrightarrow B \hookrightarrow \mathrm{Im} f \hookrightarrow X$

は条件みたす. □

<u>Prop</u> $[\mathcal{U}, \mathcal{T}]$: itv. heart $\mathcal{H}$

$\rightsquigarrow \mathcal{H} = \mathrm{Filt}(\mathrm{brick}\,\mathcal{H})$. ⌟

(:-/) $\mathcal{H} = \mathcal{T} \cap \mathcal{U}^{\perp}$: ext-closed.

$\therefore$ $(\supseteq)$ は OK.

$(\subseteq)$ $X \in \mathcal{H}$ について $l(X)$ の induction

$X \in \mathrm{Filt}(\mathrm{brick}\,\mathcal{H})$ をみる.

$l(X) = 0 \Rightarrow X = 0$ より OK.

$l(X) > 0$ つまり $X \neq 0$ とする。

さっきより ∃ $X \overset{\pi}{\twoheadrightarrow} \underset{\text{brick}}{B} \hookrightarrow X$.

⇝ $\underline{B \in \mathcal{H}} = \mathcal{T} \cap \mathcal{U}^{\perp}$

($\mathcal{H}$: image-closed)

$\mathcal{T} \ni X \longrightarrow Y \in \mathcal{U}^{\perp}$, $X \twoheadrightarrow \underset{\mathcal{T} \cap \mathcal{U}^{\perp}}{\underset{\cap}{I}} \hookrightarrow Y$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & tK \arrow[d] \arrow[r, phantom, "="] & tK \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & X \arrow[r, "\pi"] \arrow[d] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & fK \arrow[r] \arrow[d] & \square \arrow[r] \arrow[d] & B \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

$\mathcal{H} = \mathcal{T} \cap \mathcal{U}^{\perp} \ni tK$

$tK \in \mathcal{H}$, $\mathcal{U}^{\perp} \ni tK$

$\mathcal{U}^{\perp} \ni K$, $X \in \mathcal{H}$

$\mathcal{U}^{\perp} \ni \mathcal{T}^{\perp} \ni fK$

$\square \in \mathcal{T}$, $\square \in \mathcal{H}$

$B \in \mathcal{H} \subseteq \mathcal{U}^{\perp}$

$\mathcal{H}$ の s.e.s.

ここに induction したい！

$tK \neq 0$ とき

$\square \neq 0$ なら induction つかって OK.

$\square = 0$ なら $fK = 0$

$\Rightarrow K \in J \rightsquigarrow K \in H$

$\rightsquigarrow 0 \to K \to X \to B \xrightarrow{\neq 0} 0$

に induction できる!

$tK = 0$ とき.

$$0 \to \underset{\sim}{K} \to X \to B \to 0 \qquad K \subset T^{\perp}$$

$(B, -) \rightsquigarrow$

$$(B,K) \to (B,X) \to \underbrace{(B,B)}_{\text{1-dim in } \mathrm{mod}\,\mathrm{End}_A(B)} \to {}^{1}(B,K)$$

$(B,K) = 0$: $B \in T$ and $K \in T^{\perp}$

$(B,X) \neq 0$: $\exists\, B \hookrightarrow X$

$\therefore (B,X) \to (B,B)$: isom

$\rightsquigarrow \varepsilon$ is split $\qquad X \cong B$

$\therefore X \to K$: retr $\qquad X \in T$, $K \in T^{\perp}$ $\quad \therefore K = 0$. $\square$

Lem B: brick fun. fin と限らない.

$\Longrightarrow$ Filt B : wide subcat with unique simple obj B. ∎

∵ ker-closed のみ示す

$$\{ X \mid {}^\forall X \to B : 0 \text{ or } (\text{surj かつ } \underset{\text{Filt } B}{\underset{\cap}{\ker}}) \}$$

を考えると, B が入り,

ext-closed. HW

$\rightsquigarrow$ Filt $B \subseteq \{ \text{——} \}$.

∘ $\underset{\text{Filt }B}{\underset{\cap}{{}^\forall X}} \xrightarrow{f} \underset{\text{Filt }B}{\underset{\cap}{Y}}$, $\ker f \in$ Filt B を Y の B-Filt の長さ ℓ で induction.

∘ ($\ell = 0$), $\ell = 1$ は上のことより OK.

$\ell > 1$ とき,

$$\begin{array}{ccccccccc} & & & & X & & & & \\ & & & & \downarrow f & \searrow & & & \\ 0 & \to & \square & \to & Y & \to & B & \to & 0 \end{array}$$

0 or not zero.

smaller.. Filt B

0 なら

$$\begin{tikzcd} & X \arrow[d, "f"] \arrow[dl, dashed, "\exists \bar{f}"'] \\ \square \arrow[r, hook, "\text{inj}"'] & Y \end{tikzcd}$$

$\rightsquigarrow$ $\ker f = \ker \bar{f}$

∴ induction より Filt B に入る.

not zero

Filt B

$$\begin{tikzcd} 0 \arrow[r] & \triangle \arrow[r] \arrow[d, "\bar{f}"'] \arrow[dr, phantom, "\text{p.b}"] & X \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \square \arrow[r] & Y \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

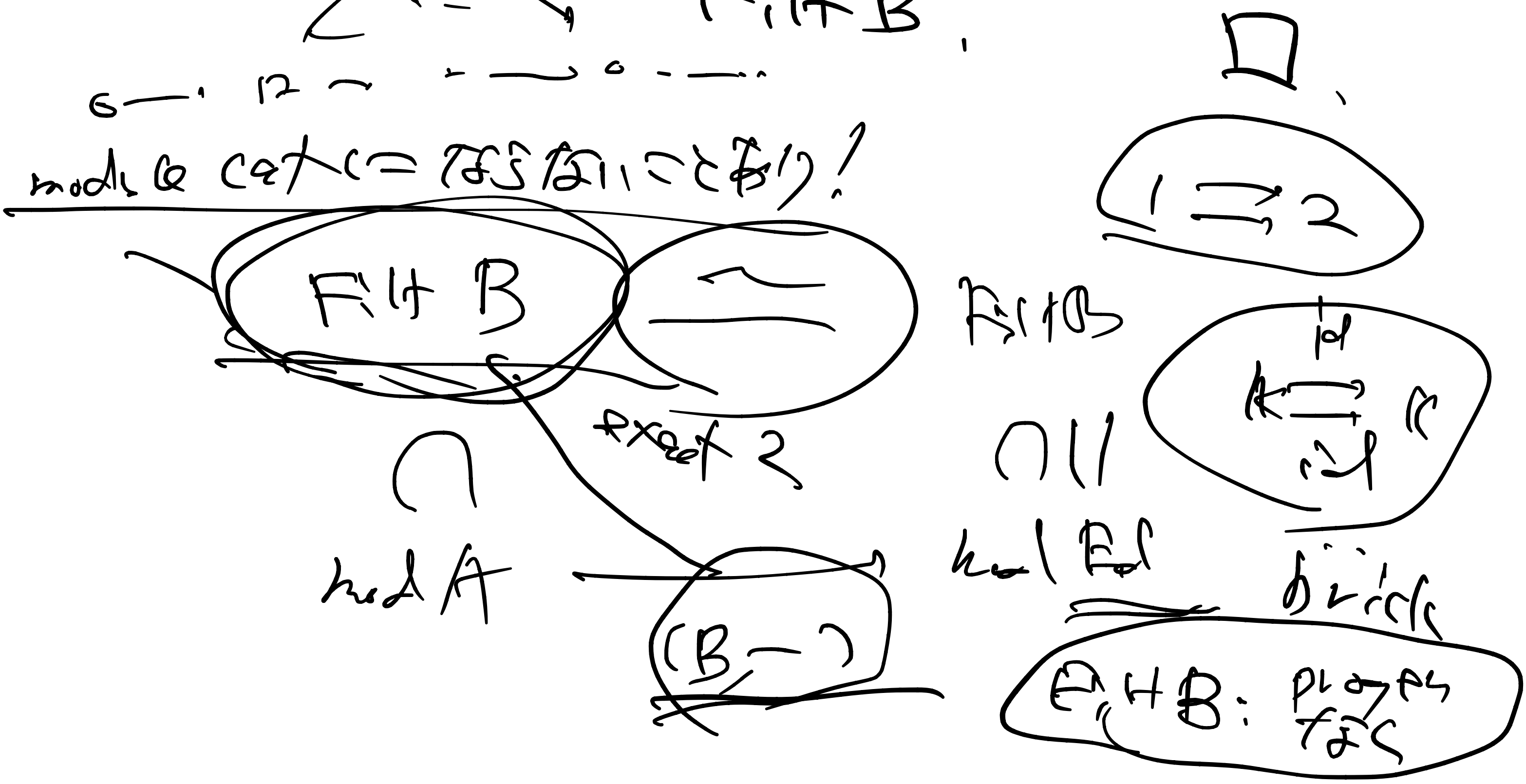
$\xrightarrow{\text{p.b}}$ $\ker f = \underbrace{\ker \bar{f}}_{}$.

① induction

k → Filt B

0 → □ → ... → 0

module cat にならないことあり!



Thm $\mathcal{U} \leq \mathcal{T}$ in tors A,

TFAE (1) $\exists\, \mathcal{T} \to \mathcal{U}$ in $\vec{\mathcal{H}}(\mathrm{tors} A)$

(2) $|\mathrm{brick}\, \mathcal{H}_{[\mathcal{U},\mathcal{T}]}| = 1$

(3) $\mathcal{H}_{[\mathcal{U},\mathcal{T}]}$: wide subcat with unique simple.

(∵) $\underline{(1) \Rightarrow (2)}$

$B_1, B_2 \in \mathcal{H}_{[\mathcal{U},\mathcal{T}]}$ を brick とする。

brick $\mathcal{H}_{[\mathcal{U},\mathcal{T}]} = \phi$
$\Downarrow$
$\mathcal{H}_{[\mathcal{U},\mathcal{T}]} = 0$
$\Downarrow$
$\mathcal{T} = \mathcal{U}$ でおかしい

$B_i \in \mathcal{T}$ $(i=1,2)$

$\cancel{\in}\, \mathcal{U}$ $\left(B_i \in \mathcal{U}^{\perp} \text{ より } B_i \in \mathcal{U} \Rightarrow (B,B) = 0 \text{ で } ムリ \right)$

∴ $\mathcal{U} \subsetneq \underline{T(\mathcal{U} \cup B_i)} \subseteq \mathcal{T}$

$\mathcal{T} \cup B_i$ を含む最小の tors

∴ $T(\mathcal{U} \cup B_1) = T(\mathcal{U} \cup B_2) = \mathcal{T}$.

$\therefore\ B_2 \in T(\mathcal{U} \cup B_1)$

一方で、

$\mathcal{C} := \{ X \mid {}^{\forall} X \to B_1 : 0 \text{ or surj} \}$

とすると $\mathcal{U} \subseteq \underset{\substack{\cup \\ B_1}}{\mathcal{C}}$. $(B_1 \in \mathcal{U}^{\perp})$

しかも $\mathcal{C}$: tors (HW)

$$\therefore\ \underset{\substack{\cup \\ B_2}}{T(\mathcal{U} \cup B_1)} \subseteq \mathcal{C}.$$

$\therefore\ B_2 \to B_1 : 0 \text{ or } \underline{\text{surj}}.$

$(B_2, B_1) = 0$ とすると、

$$\underset{\mathcal{U}}{B_2} \in \underbrace{{}^{\perp}B_1}_{\text{tors}} \quad \text{(HW)}$$

$$\therefore\ \underset{\substack{\cup \\ B_1}}{T(\mathcal{U} \cup B_2)} \subseteq {}^{\perp}B_1$$

$\rightsquigarrow (B_1, B_1) = 0$

(4 り !)

$\therefore\ \exists\ f : B_2 \to B_1$: not zero

$\Downarrow$

surj.

$\rightsquigarrow$ 入れかえて $B_1 \twoheadrightarrow B_2$

$\swarrow$

B_1 $\rightsquigarrow$ $B_1 \cong B_2$

(2) $\Rightarrow$ (3) OK. 同

($\mathcal{H} = \mathrm{Filt}(\mathrm{brick}\,\mathcal{H})$)

(3) $\Rightarrow$ (1) $[\mathcal{U}, \mathcal{T}]$: wide itv.

$\rightsquigarrow [\mathcal{U}, \mathcal{T}] \overset{1-1}{\longleftrightarrow} \mathrm{tors}\,\mathcal{H}$

$\mathcal{T}$ simple 1つ

長さ2の順序集合であるべき

$\mathcal{U} \longleftrightarrow \{0 \neq \mathcal{H}\}$: 2つのみ!

($\because 0 \neq \mathcal{X} \subseteq \mathcal{H}$: tors

$\Downarrow$

$0 \neq X \twoheadrightarrow$ simple $\in \mathcal{X}$)

$\therefore\ \exists\ \mathcal{T} \to \mathcal{U}$ in $\vec{\mathrm{H}}(\mathrm{tors}\,A)$ $\square$

2コマ目 15:20 —

Def (brick ラベル)

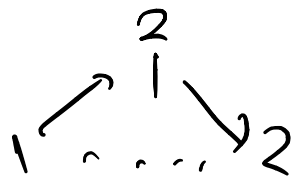
$\vec{H}(\mathrm{tors}A)$ の矢 $T \to U$ に

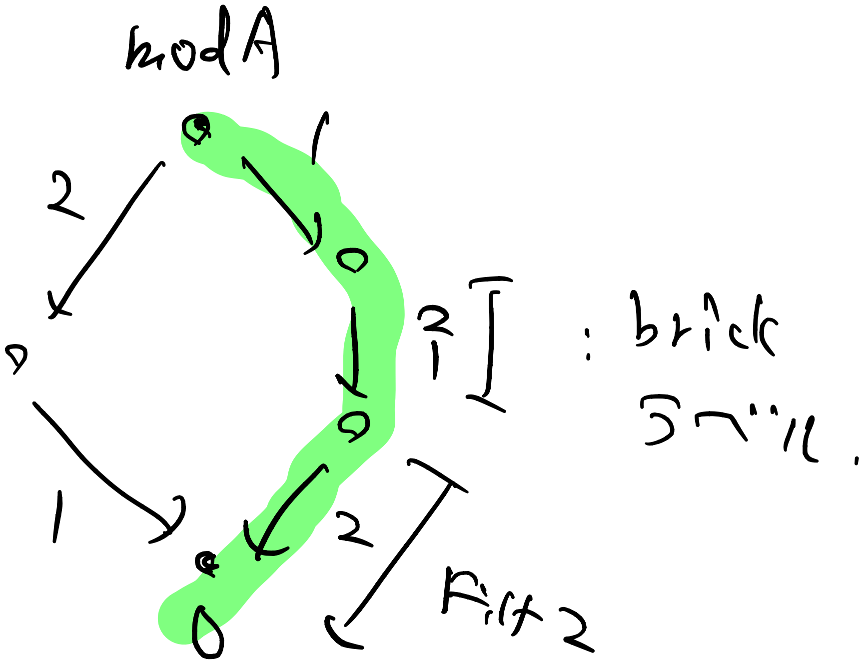
brick $H_{[U,T]}$ の unique brick

をのせたもの：brick ラベルと

よぶ！

Ex





$\mathrm{mod}A = \mathrm{Filt}\,2 * \mathrm{Filt}\begin{smallmatrix}2\\1\end{smallmatrix} * \mathrm{Filt}\,1.$

Cor

(1) $T \to U$ in $\vec{H}(\mathrm{tors}A)$ とき

[DIJ] T : fun. fin $\Longleftrightarrow$ U : fun. fin.

(2) $|\mathrm{tors}A| < \infty$ なら

$\forall$ tors は fun. fin.

Lem $\mathcal{T} \supseteq \mathcal{U}$: tors.

heart $\mathcal{H}$.

$\mathcal{T}$: fun. fin. with progen T

$\Longrightarrow$ $\mathcal{H} \ni$ progen gT ??

$$\begin{tikzcd}
(0 \arrow[r] & uT \arrow[r] & T \arrow[r] & gT \arrow[r] & 0) \\
& \mathcal{U} \arrow[u, phantom, "\cap"] & \mathcal{T} \arrow[u, phantom, "\cap"] & \mathcal{U}^{\perp} \arrow[u, phantom, "\cap"] &
\end{tikzcd}$$

$\Longrightarrow \cap\, \mathcal{T} = \mathcal{H}$

☹ $gT \in \mathcal{H}$. is OK.

gT : progen is HW

(proj? $\mathcal{U}^{\perp} \supseteq$

$(uT, \mathcal{H}) \to {}^1(gT, \mathcal{H}) \to {}^1(T, \mathcal{H})$) $\quad \mathcal{T} \supseteq$

$(uT, \mathcal{H}) = 0$, $\quad {}^1(T, \mathcal{H}) = 0$

enough? $\forall X \in \mathcal{H}$

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
& uT_0 \arrow[r, equal] \arrow[d] & uT_0 \arrow[d] & & \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & T_0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \quad : \text{in } \mathcal{T} \\
0 \arrow[r] & \square \arrow[r] \arrow[d] & gT_0 \arrow[r] \arrow[d] & X \arrow[r] & 0 \\
& 0 & 0 & &
\end{tikzcd}$$

$\mathcal{T} \ni X'$, $\quad T_0 \in \operatorname{add} T$, $\quad \mathcal{H} \ni \square$

Cor $[\mathcal{U},\mathcal{T}]$: wide itv とき

$\mathcal{T}$: fun. fin $\iff$ $\mathcal{U}$: fun. fin.

(∵) $(\Longrightarrow)$ のみ.

$\mathcal{T}$: fun. fin より

$\mathcal{H}[\mathcal{U},\mathcal{T}]$: wide subig progen

$\mathcal{H}[\mathcal{U},\mathcal{T}] \simeq \mathcal{H}$ $\Downarrow$ 前の

$\mathcal{H}$: fun. fin.

$\underset{\text{2-out-of-3}}{\Longrightarrow}$ $\mathcal{U} \in$ fun. fin. □.

Cor (1) $\mathcal{T} \to \mathcal{U}$ in $\vec{\mathcal{H}}(\operatorname{tors} A)$

$\mathcal{T}$: fun fin $\Longleftarrow$ $\mathcal{U}$: fun fin.

(2) $|\operatorname{tors} A| < \infty \Longrightarrow$

$\forall$ tors : fun. fin.

(∵) (1) は上より

(2) $\forall \mathcal{T} \supseteq 0$

$\rightsquigarrow$ $\mathcal{T} \to \cdots \to 0$ in $\vec{\mathcal{H}}(\operatorname{tors} A)$

0 : fun. fin より

$\mathcal{T}$: inductive に fun. fin. $\square$

[Demonet-Iyama-Jasso]

Thm. $\mathcal{T} \in \operatorname{tors} A$

$\mathcal{U} \subsetneq \mathcal{T}$ なる $\mathcal{U} \in \operatorname{tors} A$ について

tors もし $\mathcal{T}$: fun. fin

$\Rightarrow$ $\exists$ $\mathcal{T} \to \mathcal{T}'$ in $\vec{\mathcal{H}}(\operatorname{tors} A)$

s.t. $\mathcal{U} \subseteq \underset{\text{f-tors}A}{\underset{\cap}{\mathcal{T}'}} \subsetneq \mathcal{T}$.

$\mathcal{T}$

$\mathcal{T}'$

$\mathcal{U}$

(($|\operatorname{tors} A| < \infty$ なら 当たり前。

一般の $\mathcal{T}$ では ムリ

$\Leftarrow$ 2 $\{T_i\}_{i \in I}$: tors から 矢がのびない))

「 M : f.g. module, $N \subsetneq M$

これは $\Rightarrow$ $N \subsetneq M' \subsetneq M$ 」
maximal

の類似ver.

→ Zorn を使う!

$\underbrace{[\mathcal{U}, \mathcal{T})}_{\substack{\cup\!\!\!\mid \\ \mathcal{U}}} := \{\mathcal{C} \in \operatorname{tors} A \mid \mathcal{U} \subseteq \mathcal{C} \subsetneq \mathcal{T}\}$

non-empty poset.

$\forall$ chain が上界をもつか?

$\{\mathcal{C}_i\} : [\mathcal{U}, \mathcal{T}) : \text{chain}.$

$\Rightarrow \bigcup \mathcal{C}_i \in \underset{\sim}{\operatorname{tors} A}$ (tot. orderedより)

$\mathcal{U} \subseteq$ ↑ set. theoretic

$\bigcup \mathcal{C}_i \neq \mathcal{T}$ とすると, $\bigcup \mathcal{C}_i$ が $\{\mathcal{C}_i\}$ の上界.

Zorn より 極大元 $\mathcal{T}'$

$\rightarrow \mathcal{U} \subseteq \mathcal{T}' \lessdot \mathcal{T}$

極大性より OK.

$\bigcup \mathcal{C}_i = \mathcal{T}$ とすると,

$\mathcal{T}$: fun. fin. より $\exists M \quad \mathcal{T} = \operatorname{Fac} \underline{M}$

$M \in \mathcal{T} = \bigcup \mathcal{C}_i$

$\Rightarrow \exists i, \quad M \in \mathcal{C}_i.$

$\Rightarrow \operatorname{Fac} M \subseteq \mathcal{C}_i \subsetneq \mathcal{T} = \operatorname{Fac} M$

のでムジュン. □

Cor $\mathcal{T}, \mathcal{U} \in$ f-tors A について

$\mathcal{T} \to \mathcal{U}$ in $\vec{\mathcal{H}}$(tors A)

$\Leftrightarrow$ $\mathcal{T} \to \mathcal{U}$ in $\vec{\mathcal{H}}$(f-tors A)

(証) ($\Rightarrow$) OK

($\Leftarrow$) $\mathcal{T} \supsetneq \mathcal{U}$ より

なより $\exists\, \mathcal{T} \xrightarrow{\ \ \theta\ \ } \mathcal{T}' \supseteq \mathcal{U}$
in $\vec{\mathcal{H}}$(tors A)

$\mathcal{T}' \in$ f-tors A $\Rightarrow$ $\mathcal{T}' = \mathcal{U}$ □

IV. Hasse arrow via sp-proj.
(mutation)

メイン $\mathcal{T}$ の indec sp-proj

fun. fin.

$\updownarrow$ 1-1

$\mathcal{T}$ からのびる Hasse 矢. 」

Wide itv の rank.

Prop $\mathcal{T}$: tors with progen T.
[仁-酒井] $\cup$|
$(\mathcal{U}, \mathcal{G})$: tors pair.

$(\rightsquigarrow$ gT は $\mathcal{H}[\mathcal{U}, \mathcal{T}]$ の progen$)$

$(0 \to uT \to T \to gT \to 0)$

$$\rightsquigarrow |gT| = |\underbrace{\mathrm{ind}\,T \setminus \mathcal{U}}_{\{X \in \mathrm{ind}\,T \mid X \notin \mathcal{U}\}/\cong}|$$

:(g は functor.

: functor. HW

progen.

$$\begin{tikzcd} \mathrm{mod}\,A \arrow[r, "g"] & \mathcal{G} \\ T \in \mathcal{T} \arrow[u, phantom, "\cup"] \arrow[r, "g|_{\mathcal{T}}"] & \mathcal{T}\cap\mathcal{G} =: \mathcal{H} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$\supseteq \mathcal{U} \longmapsto 0$

は restrict.

$(\because \quad X \in \mathcal{T} \Rightarrow X \twoheadrightarrow gX, \ gX \in \mathcal{T})$

Claim

$$\frac{\mathrm{add}\,T}{[\mathcal{U}]} \xrightarrow{\sim} \mathrm{add}(gT) \ : \text{equiv}$$

を induce.

$\mathcal{U}$ を通る射全体というイデアル.

(:($g|_{\mathcal{T}}(\mathcal{U}) = 0$.

$$\rightsquigarrow \quad \frac{\mathcal{T}}{[\mathcal{U}]} \longrightarrow \mathcal{H} \text{ : induce.}$$

$$\begin{array}{ccc} \frac{\mathcal{T}}{[\mathcal{U}]} & \longrightarrow & \mathcal{H} \\ \cup & & \cup \\ \frac{\operatorname{add} T}{[\mathcal{U}]} & \longrightarrow & \operatorname{add} gT \end{array}$$

dense is OK

fully faithful?

$$\frac{\operatorname{End}_A(T)}{[\mathcal{U}](T,T)} \xrightarrow{\sim} \operatorname{End}_A(gT)$$

∴ isom or?

inj

$$\begin{tikzcd} 0 \arrow[r] & uT \arrow[r] & T \arrow[r] \arrow[d, "\varphi"] \arrow[dl, dashed, "\exists"'] & gT \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & uT \arrow[r] & T \arrow[r] & gT \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow$

$uT \in \mathcal{U}$.

∴ OK.

surj

$$\begin{tikzcd} 0 \arrow[r] & uT \arrow[r] \arrow[d] & T \arrow[r] \arrow[d, dashed, "\exists"] & gT \arrow[r] \arrow[d, "\Phi"] & 0 \\ 0 \arrow[r] & uT \arrow[r] & T \arrow[r] & gT \arrow[r] & 0 \end{tikzcd}$$

J ⊙

$T \in P(\mathcal{T}) \rightsquigarrow$ surj

Q

Claim より

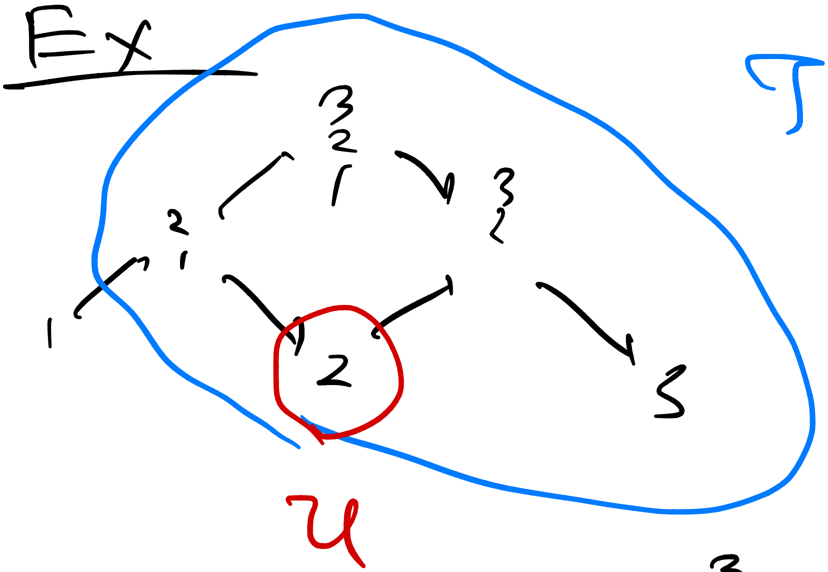
$$\left| \frac{\mathrm{add}\, T}{[\mathcal{U}]} \right| = \left| \mathrm{add}\, gT \right|$$

$$\| \;\; \text{HW} \qquad\qquad\qquad\qquad \|$$

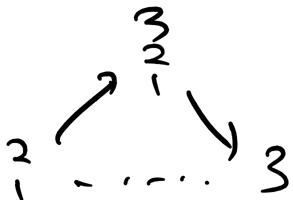
$$\left| \mathrm{ind}\, T \setminus \mathcal{U} \right| \qquad\qquad \left| gT \right|$$

$\square$

Ex



$\mathcal{H}_{[\mathcal{U},T]}$ =



: rank 2

wide subcat.

$$\left| \underline{P(T)} \setminus \mathcal{U} \right|$$

$$\|$$

$$\left| \left\{ \begin{smallmatrix}3\\2\\1\end{smallmatrix}, \begin{smallmatrix}2\\1\end{smallmatrix}, 2 \right\} \setminus \mathcal{U} \right| = \left| \begin{smallmatrix}3\\2\\1\end{smallmatrix}, \begin{smallmatrix}2\\1\end{smallmatrix} \right| = 2.$$

Key Prop $\mathcal{T} \in \text{f-tors}\, A$,

T : $\mathcal{T}$ の basic progen. (sτ-tilt)

$T = X \oplus U$ で

$X \in P_0(T)$ (X : T の sp-proj)

となる分解をとると、

$[\mathrm{Fac}\, U, \mathcal{T}]$ は wide itv で、

tors.

その heart は rank $|X|$ の

f.d. alg の module cat と equiv. ┘

スローガン

「tors の sp-proj をおとすと、

おとした rank の wide itv ができる」

$$\mathcal{H} := \mathcal{T} \cap (\mathrm{Fac}\, U)^{\perp} \overset{\text{HW}}{=} \mathcal{T} \cap U^{\perp}$$ とする。

◦ $\mathcal{H}$: wide subcat を示す

$\mathcal{H}$: image-ext-closed OK.

∴ $$\begin{tikzcd} X \arrow[r] & Y \\ Q_0 \arrow[ur, hook] & \end{tikzcd}$$

∴ ETS

$\forall\ 0 \to L \to M \to N \to 0$: exact

(i) $L, M \in \mathcal{H} \Rightarrow N \in \mathcal{H}$

(ii) $M, N \in \mathcal{H} \Rightarrow L \in \mathcal{H}$.

(i) $M \in \mathcal{T} \Rightarrow N \in \mathcal{T}$.

$N \in U^{\perp}$? $\qquad \mathcal{T} \supset U$.

$$(U, M) \to (U, N) \to {}^1(U, L)$$

$M \in U^{\perp} \Rightarrow (U,M) = 0$

$U \le_{\oplus} T \in \mathcal{P}(\mathcal{T}) \Rightarrow {}^1(U,L) = 0$

∴ $(U, N) = 0$.

(ii) $M \in U^{\perp}$: torf より

$L \in U^{\perp}$ は OK.

$L \in \mathcal{T}$?

__Claim__ $\quad T = U \oplus X$ sp-proj

$N \in \mathcal{H}$

$$\Rightarrow 0 \to \underset{\substack{\cap\\ T}}{N'} \to \underset{\substack{\cap\\ \mathrm{add}X}}{X_0} \to N \to 0$$

☹ T is progen $\quad U \oplus X$ をつかって

$$\begin{tikzcd}
 & T \ni N'' \arrow[r] & U^{\oplus} \oplus X^{\oplus} \arrow[r] & N \arrow[r] & 0 \\
0 \arrow[r] & N'' \arrow[u, phantom]
\end{tikzcd}$$

$$0 \to \underset{\substack{\rotatebox{90}{$\ni$}\\ T}}{N''} \to U^{\oplus} \oplus X^{\oplus} \to \underset{\substack{\rotatebox{90}{$\in$}\\ \mathcal{H}}}{N} \to 0 \qquad \mathcal{H} = T \cap U^{\perp}$$

$\cong$

$$\left([U^{\oplus} = U^{\oplus}] \oplus \left(0 \to \underset{\substack{\cap\\ T}}{N'} \to X^{\oplus} \to N \to 0\right) \right)$$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & T \ni & N' \arrow[r, equal] \arrow[d] & N' \arrow[d] & \in T \\
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & \square \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{pb}"] & X_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Claim

$X_0 \in \mathrm{add}X$

$T \ni \square$, $T \ni L$, $T \ni M$

$$\rightsquigarrow \quad \overset{\mathcal{T}}{\overset{\cup}{\square}} \twoheadrightarrow X_0$$

sp-proj

は retr

(by $X \in \mathcal{P}_0(\mathcal{T})$)

$$\rightsquigarrow \quad L \oplus \square \in \mathcal{T}$$

$$\therefore \quad L \in \mathcal{T}$$

以上より $\mathcal{H}$: wide subcat

$\therefore$ $[\mathrm{Fac}\, U, \underset{\mathrm{Fac}(U \oplus X)}{\overset{\parallel}{\mathcal{T}}}]$ は wide itv.

この wide の rank

は、さっきより

$$\left| \underset{T = U \oplus X}{\overset{\parallel}{\mathcal{P}(\mathcal{T})}} \setminus \mathrm{Fac}\, U \right|$$

$$= |\operatorname{ind}(U \oplus X) \setminus \operatorname{Fac} U|$$

$$= |X|$$

(∵ U の summand は $\operatorname{Fac} U$ に入る。

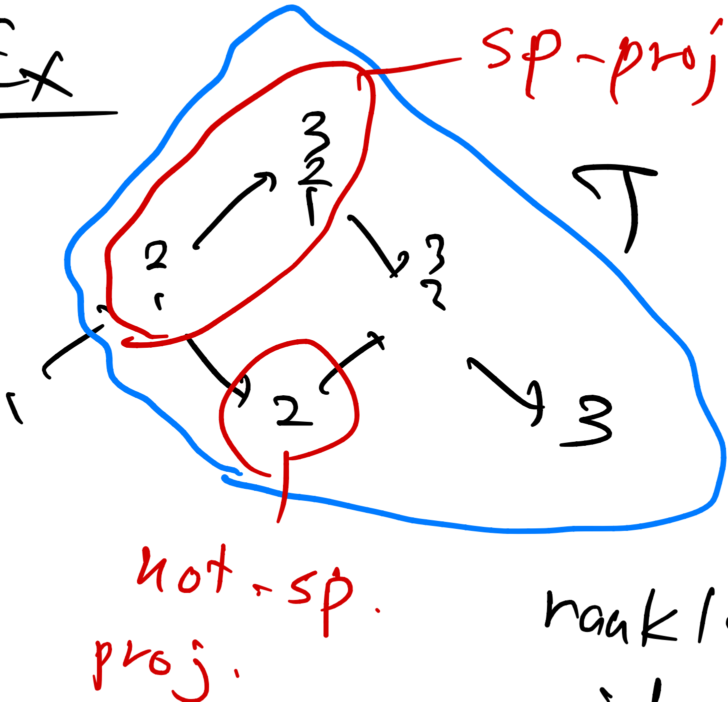
$X' \in \operatorname{ind} X$, $\operatorname{Fac} U \ni X'$ とすると、

$U^n \twoheadrightarrow X' \rightsquigarrow X' \oplus U$
∵ sp-proj

basic に反する。)

∴ X の indec smd は全て残る。 □

Ex



$X = \begin{smallmatrix}2\\1\end{smallmatrix}$: sp. proj in T

$\Downarrow$

T

rank 1 ⊕ wide

$\cup$

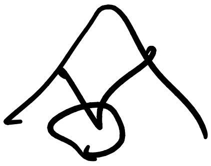
$\operatorname{Fac}\left(\begin{smallmatrix}3\\2\\1\end{smallmatrix} \oplus 2\right)$

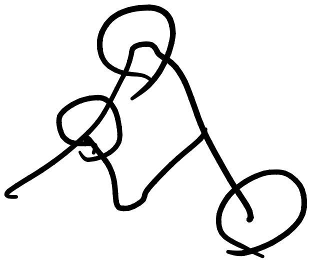
$= \operatorname{Filt} \begin{smallmatrix}2\\1\end{smallmatrix}$

- $X = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$. $(\mathcal{T} = \mathrm{Fac}\, X)$

$$\left.\begin{matrix} \mathcal{T} \\ \cup \\ \mathrm{Fac}\, 2 \end{matrix}\right] \text{ rank } 2 \text{ wide}$$

$\mathrm{Fac}\, 2 =: \mathcal{U}$





Cor $\mathcal{T} \in$ f-tors A.

T : $\mathcal{T}$ の basic progen.

(1) $X \in \mathrm{ind}\, P_0(\mathcal{T})$: indec sp-proj

$\Rightarrow \mathcal{T} \longrightarrow \mathrm{Fac}(T/X)$ in $\vec{H}(\mathrm{tors} A)$,

(2) 逆に、$\mathcal{T} \longrightarrow \mathcal{U}$ in $\vec{H}(\mathrm{tors} A)$

とすると $\exists!\, X \in \mathrm{ind}\, P_0(\mathcal{T})$ s.t.

$\mathcal{U} = \mathrm{Fac}(T/X)$]

$\rightsquigarrow$ $\mathrm{ind}\, P_0(\mathcal{T}) \xleftrightarrow{1-1} \{\mathcal{T}$ から 出る矢$\}$

(1.5) (1) $T = X \oplus U$

$\underline{\underline{\text{index}}}$ sp-proj in T.

$\overset{\text{さっき}}{\Longrightarrow}$ $[\mathrm{Fac}\,U, T]$ は

rk 1 wide itv

$\rightsquigarrow$ $T \xrightarrow{\exists} \mathrm{Fac}\,U$.

(2) $T \to \mathcal{U}$ とる. $\rightsquigarrow$ $\mathcal{U}$: fun. fin.

一方 $[\mathcal{U}, T]$: $\underbrace{\text{wide itv}}$, rank 1.

$\rightsquigarrow$ $|\mathrm{ind}\,T \setminus \mathcal{U}| = 1$

$\{X\}$ とする.

$T = X \oplus U$ とおく.

<u>Claim</u> $X \in P_0(T)$

(∵ ちがうなら, T の cover $X \oplus U$ から X をのぞける (U : T の cover)

$\rightsquigarrow$ $X \twoheadleftarrow \underline{\underline{U^{\oplus}}}$

$\rightsquigarrow$ $X \in \mathcal{U}$ に矛盾.)

$\overset{(1)}{\rightsquigarrow}$ $\mathcal{T} \longrightarrow \operatorname{Fac} U$: Hasse 矢

$\mathcal{T} \not\supsetneq \mathcal{U} \supsetneq \operatorname{Fac} U$

$\therefore\ \mathcal{U} = \operatorname{Fac} U$. $\square$

<u>まとめ</u>

$\overset{\text{progen.}}{\mathcal{T}} = \operatorname{Fac} T$ から1つ下に行くには

ind T のうち sp-proj なもの

(= 他のもののFacでかけない)

を T からとって Fac をとれば良い.

<u>Rem</u> <u>実は</u>

$\mathcal{T} \in \text{f-tors}\, A$

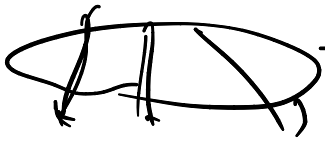
$\Rightarrow \#\{\mathcal{T} \to \cdot\} + \#\{\cdot \to \mathcal{T}\} = |A|$.

τ-rigid pair の (co)-Bongartz completion の理論.
$\mathcal{T}$

Ex

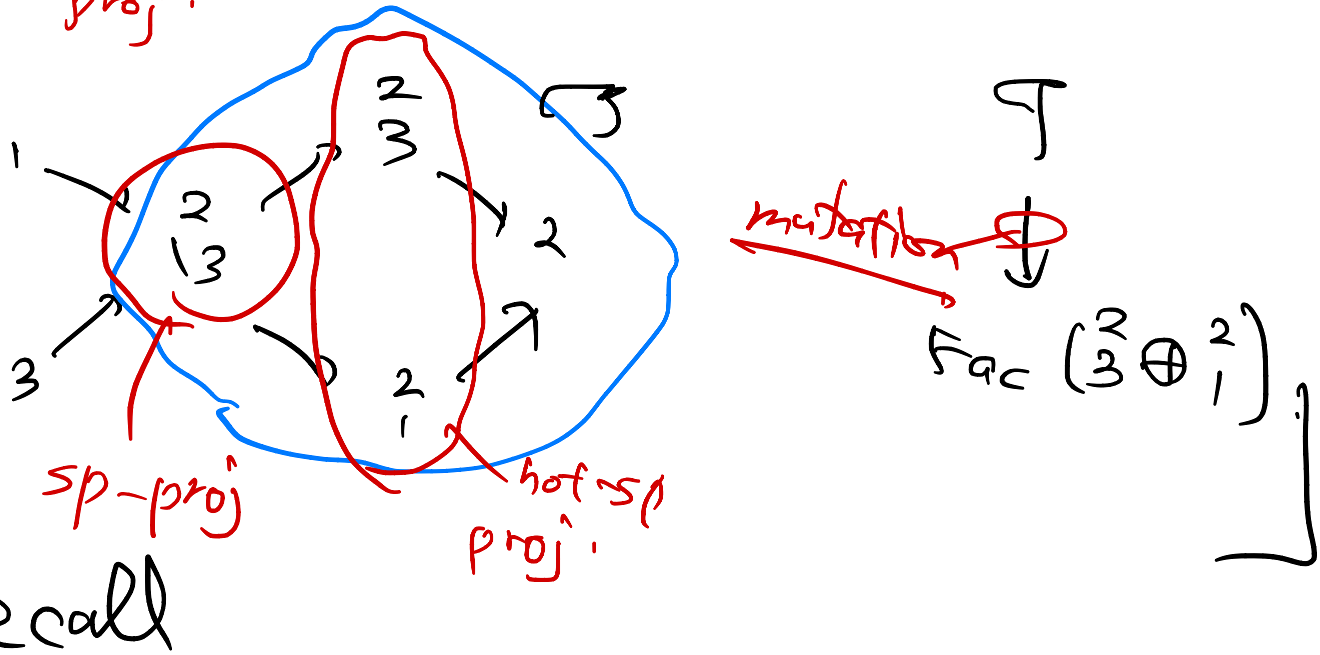
今回の話

(T, P)



left mutation

Ex



Recall

$T \in f\text{-}\mathrm{tors}\,A$

$\rightsquigarrow A \xrightarrow{f} \underbrace{T_0^{A}}_{\text{sp-proj}} \longrightarrow \underbrace{T_1^{A}}_{\text{proj, not-sp.}} \longrightarrow 0$

f: left min T-approx

$\|$

T の minimal cover

$\|$

他の元のFacでかけない

話せなかったこと (c.f. 講演原稿)

$\mathrm{Fac}(X \oplus U) \to \mathrm{Fac}\, U$

Hasse 矢

Fac U の proj は U 以外に増えうる!

どうやって出すか?

sτ-filt mod の mutation!

Fact $X \oplus U$ のとき,
indec sp proj

Fac U -approx

$\exists$

$$X \xrightarrow{\phi} U_0^X \to U_1^X \to 0$$

0 もありうる

left min U-approx

$\rightsquigarrow \quad \mathcal{P}(\mathrm{Fac}\, U) = U \oplus U_1^X$

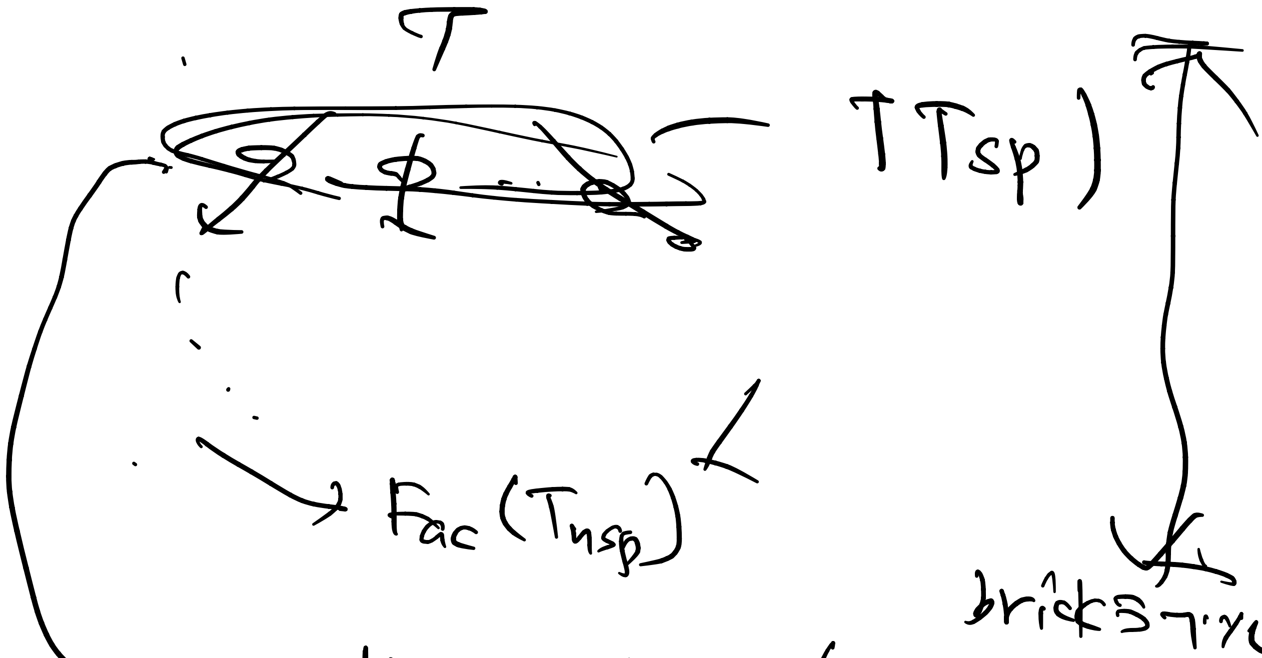
Thm [Marks–Šťovíček]

$$T = T_{sp} \oplus T_{nsp} \quad : s\tau\text{-tilt}$$

basic — T_{sp}: sp-proj — T_{nsp}: それ以外

$\rightsquigarrow$ $[\mathrm{Fac}\, T_{nsp}, \mathrm{Fac}\, T]$: wide.

その heart $\alpha T \simeq \mathcal{T}$



$\underbrace{\alpha W} = \mathrm{Filt}(\mathcal{T} \oplus)$

$\rightsquigarrow$ $\mathcal{T} = T(\alpha W)$

$\qquad$ smallest. tors

$\rightsquigarrow$ $\text{f-tors}\, A \underset{T(-)}{\overset{\alpha(-)}{\rightleftarrows}} \text{wide}\, A$

$\longleftarrow$: id.

$|\text{tors}A| < \infty$

$\text{ftors}A \underset{\tau}{\overset{\alpha}{\rightleftarrows}} \text{wide}A$: bijection

$|\text{mod}A| < \infty$

$\text{tors}A = \text{f-tors}A.$ $\not\Leftarrow$

$\Uparrow$ ok

$\Uparrow$ だいたいいる

$|\text{brick}A| < \infty$!

$\Uparrow$ ∃ wild alg

$|\text{ind}A| < \infty$ s.t $|\text{tors}A| < \infty$

[Asai-Pfeifer]

$[U, T]$: wide itv

$\iff \quad \exists \quad T$

subset

$T_1 \quad \cdots \quad T_r$

s.t. $\quad U = T_1 \cap \cdots \cap T_r$.