On some classes of subcategories of abelian categories

Haruhisa Enomoto
(Osaka Metropolitan Univ.)

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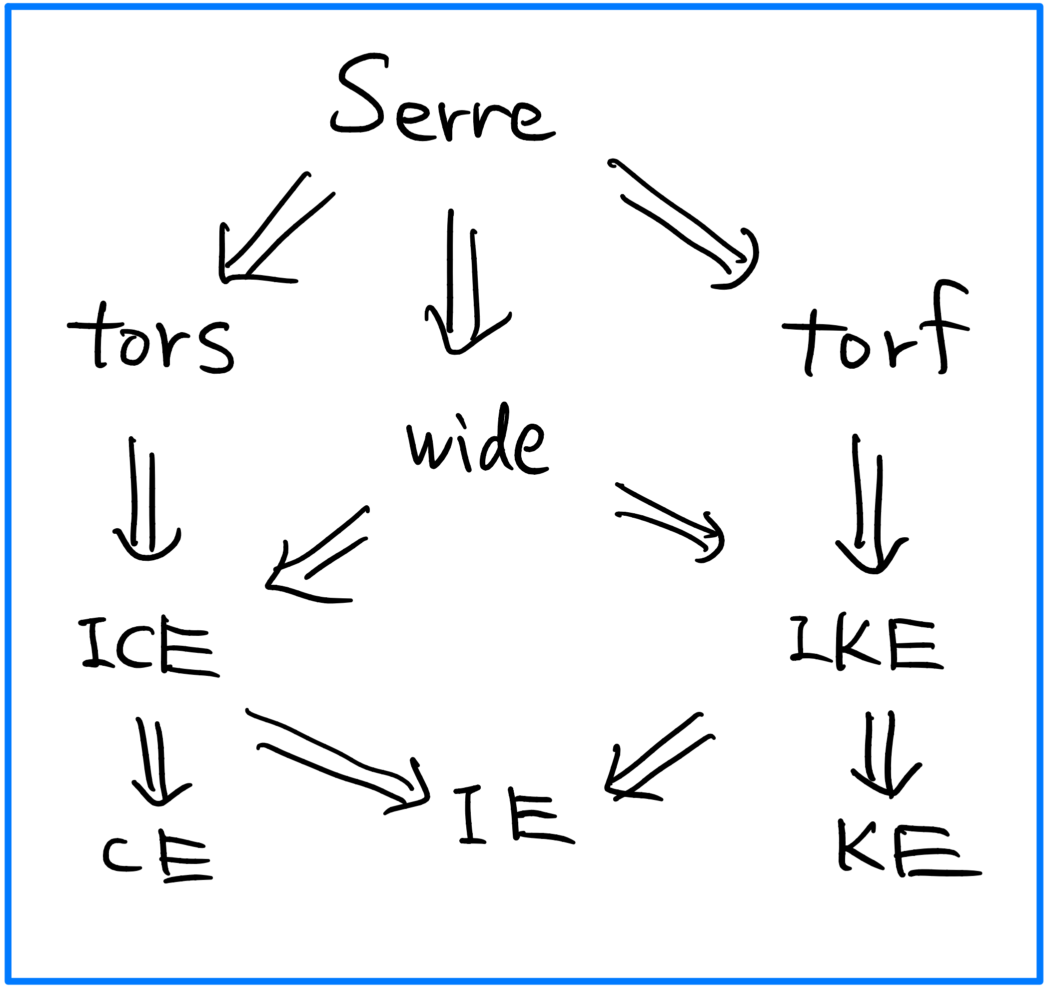
代数学特別講演

Figure of classes of subcats of abelian categories



Demo using FD Applet I developed.

Introduction

Representation theory of algebra :

$\boxed{\text{Study}}$ the abelian category mod A for

a given (non-commutative) algebra A.

(mod A := { fin. gen. A-modules })

What? How?

One direction : study subcats of an abelian cat.

- Various classes of subcategories
- Relation between them, classification, etc

Gabriel's classification ('62)

Let X be a noetherian scheme.

Then $\exists$ bij between

(1) $\boxed{\text{Serre}}$ subcats of $\mathrm{coh}\, X$ $(:= \{\text{coh. sheaves}\})$

(2) Specialization-closed subsets of X

(1) : "Categorical" structure

(2) : "Topological" structure.

Combinatorics

Thm (Krull-Schmidt)

Let A be a f.d. algebra,
Then every $M \in \operatorname{mod} A$ can be written

$$M \cong X_1 \oplus \cdots \oplus X_t$$

uniquely s.t. X_i : indecomposable

Considering a good subcat of $\operatorname{mod} A$

$\xleftrightarrow[\text{same}]{}$ a good subset of $\operatorname{ind}(\operatorname{mod} A)$

$$\operatorname{ind}(\operatorname{mod} A) := \{\text{indec. } A\text{-modules}\}/\cong.$$

combinatorical

Semisimple Example

- k : field
- $A := k \times k \times k$: 3-dim k-alg

$\rightsquigarrow$ $\operatorname{ind}(\operatorname{mod} A) = \{S_1, S_2, S_3\}$ where

$S_1 := k \times 0 \times 0$, $S_2 = 0 \times k \times 0$, $S_3 = 0 \times 0 \times k$.

$\rightsquigarrow$ $\exists$ bij between

$$\left\{ \begin{array}{l} \cdot \text{ Subcats of } \operatorname{mod} A \text{ closed under } \oplus, \text{ direct summands} \\ \cdot \text{ Subsets of } \{S_1, S_2, S_3\}. \end{array} \right.$$

(basic, but degenerate case)

Non-semisimple Example

$$A := \begin{bmatrix} k & 0 \\ k & k \end{bmatrix} \rightsquigarrow \operatorname{ind}(\operatorname{mod} A) = \{X, Y, Z\}$$

with $0 \to X \to Y \to Z \to 0$.

Subcats closed under

- $\oplus$, direct summands $\xleftrightarrow{1-1}$ subsets of $\{X, Y, Z\}$
- Some operations $\xleftrightarrow{?}$ Nice subsets?

<u>Ex</u> (i) Fac - Ext - closed (torsion class)

$$\begin{tikzcd}
& \{X,Y,Z\} \arrow[dl, dash] \arrow[dr, dash] & \\
\{X,Y\} \arrow[d, dash] & & \{Z\} \arrow[ddl, dash] \\
\{X\} \arrow[dr, dash] & & \\
& \emptyset &
\end{tikzcd}$$

(ii) Coker - Ker - Ext - closed (wide subcat)

$$\begin{tikzcd}
& \{X,Y,Z\} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\{X\} \arrow[dr, dash] & \{Y\} \arrow[d, dash] & \{Z\} \arrow[dl, dash] \\
& \emptyset &
\end{tikzcd}$$

Setting : ($\mathcal{A}$: an abelian category.
(often $\operatorname{mod} A := \{\text{f.g. } A\text{-modules}\}$ over ring A))

<u>Def</u> $\mathcal{C} \subseteq \mathcal{A}$ is closed under

(i) Extensions (E-closed)

$:\Longleftrightarrow$ $\forall\ 0 \to L \to M \to N \to 0$: ex in $\mathcal{A}$,

$$L, N \in \mathcal{C} \Longrightarrow M \in \mathcal{C}.$$

(ii) Submodules (Sub-closed)

$:\Longleftrightarrow$ $L \hookrightarrow M,\ M \in \mathcal{C} \Rightarrow L \in \mathcal{C}$

(iii) Factor modules (Fac-closed)

$:\Longleftrightarrow$ $M \twoheadrightarrow N,\ M \in \mathcal{C} \Rightarrow N \in \mathcal{C}$

(iv) Images (I - closed)

$:\Longleftrightarrow \quad \forall f : C_1 \to C_2 , \quad C_1, C_2 \in \mathcal{C} \implies \operatorname{Im} f \in \mathcal{C}$

(v) Kernels (K - closed)

$:\Longleftrightarrow$ ———————————— $\operatorname{Ker} f \in \mathcal{C}$

(vi) Cokernels (C - closed)

$:\Longleftrightarrow$ ———————————— $\operatorname{Coker} f \in \mathcal{C}$

Def

- A Serre subcat is a Sub - Fac - E - closed
- A torsion class is a Fac - E - closed tors
- A torsionfree class is a Sub - E - closed. torf

Dual

$$\begin{tikzcd} & \text{Serre} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ \text{tors} & & \text{torf} \end{tikzcd}$$

Classification of Serre subcategories

Thm

A : f.d. alg $\rightsquigarrow$ $\{S_1, \cdots, S_n\}$: all simple A-modules.

Then $\{\text{Serre subcats of } \operatorname{mod} A\}$

$$\begin{array}{c} \{\text{Serre subcats of } \operatorname{mod} A\} \\ \wr\| \\ 2^{\{S_1, \cdots, S_n\}} \end{array}$$

$\leftarrow$ power set.

($\because$ Jordan-Hölder Theorem)

If A : f.d. alg $\rightsquigarrow$ alg over noeth. ring,

then classifying Serre subcats is "more interesting".

[Gabriel '96, Kanda '12, Iyama-Kimura (preprint)]

Why torsion(-free) class?

Serre

$\swarrow \quad \searrow$

tors $\qquad$ torf

Fac-E $\qquad$ Sub-E

- Related to various things!

Thm (Happel-Reiten-Smalø '96)

Let A : f.d. alg. $\exists$ bij between

(1) Torsion class $\qquad$ $(1)^{op}$ Torsion-free classes

(2) t-structures of $D^b(\mathrm{mod}A)$

"between $\mathrm{mod}\,A$ and $(\mathrm{mod}\,A)[1]$"

Useful to study derived cat!

(e.g. Bridgeland stability)

- $A = kQ$: Dynkin path alg

$\rightsquigarrow$ tors (torf) $\overset{1-1}{\longleftrightarrow}$ "cluster-theoretic" / "Lie-theoretic" things

(Associahedra, Coxeter fan, Cambrian lattice)

- Mutation theory [Adachi-Iyama-Reiten '14]

$$\begin{array}{ccc} \text{tors} & \overset{1-1}{\longleftrightarrow} & \{\, s\tau\text{-tilting modules} \,\} \circlearrowleft \text{mutation} \\ \cup\!\!\!\mid & & \cup\!\!\!\mid \\ \text{mod}\, A & \longleftrightarrow & A \ : \text{progen}. \end{array}$$

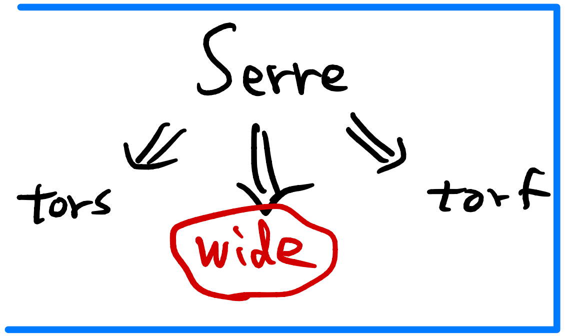
One can combinatorially obtain tors via mutation.

Demo

Wide subcategory

Def $\mathcal{W} \subseteq \mathcal{A} \leftarrow$ abelian cat

is wide $:\Leftrightarrow$ Cok - Ker - Ext-closed ($\Leftrightarrow$ E-closed abelian sub)



" abelian cat inside abelian cat "

(module —————— module →)

Thm [Ringel '76] For A : f.d. alg,

$\{$ wide in mod$A$ $\}$ $\xleftrightarrow{1-1}$ $\{$ pair-wise Hom-ortho bricks $\}$

$\mathcal{W} \longmapsto \{$ simple objs in $\mathcal{W}\}$

Schur's Lemma

Why wide?

- Relation to derived cat (wide $\overset{1-1}{\longleftrightarrow}$ {nice thick subcat})
- Relation to combinatorics

  (A: Dynkin alg ⇝ Non-crossing partition

  preproj alg ⇝ Shard intersection order on Weyl grp)

- Relation to tors

Thm [Marks–Šťovíček '17, E '23]

Suppose # tors A < ∞. Then

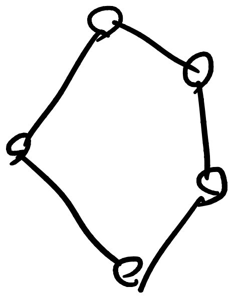
tors A $\overset{1-1}{\longleftrightarrow}$ wide A, and (wide A, ⊆) poset

can be described by (tors A, ⊆)

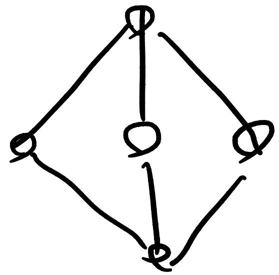
Tors v.s. Wide

Fac-E-closed | C-K-E-closed

$A = \begin{bmatrix} K & 0 \\ K & K \end{bmatrix} = K(1 \leftarrow 2)$



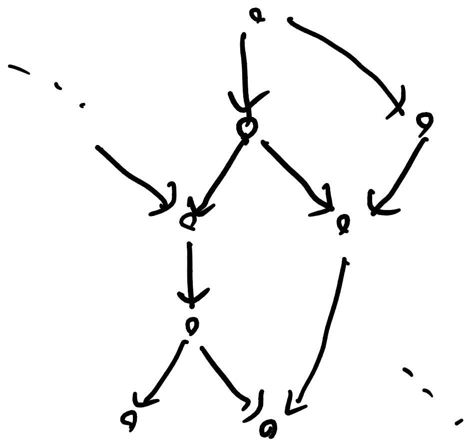
$\longleftrightarrow$

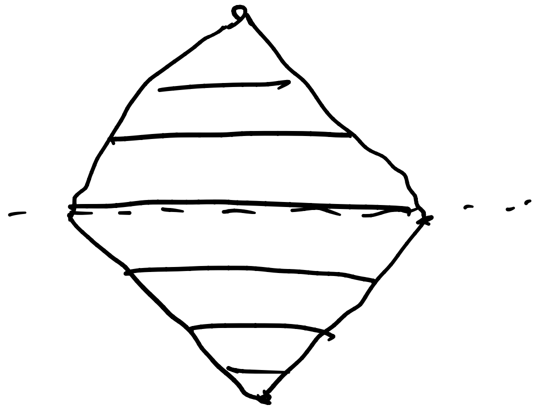


symmetric

(Hasse-regular semidistributive lattice)

(ranked poset with symmetric & unimodal rank)





Summary (for f.d. alg)

powerset of simples

$$
\begin{tikzcd}
 & \underset{\text{(Fac-Sub-E-closed)}}{\mathrm{Serre}} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] \arrow[dd, Rightarrow] & \\
\underset{\text{(Fac-E-closed)}}{\mathrm{tors}} \arrow[rr, leftrightarrow, "\text{dual}", "1-1"'] \arrow[dr, leftrightarrow, "\sim"] & & \underset{\text{(Sub-E-closed)}}{\mathrm{torf}} \arrow[dl, leftrightarrow, "\sim"'] \\
 & \underset{\text{(Cok-Ker-E-closed)}}{\mathrm{wide}} &
\end{tikzcd}
$$

Non-trivial bijection!

$$
\begin{tikzcd}
 & \text{Serre} \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{tors} \arrow[d, Rightarrow] & \text{wide} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \text{torf} \arrow[d, Rightarrow] \\
\text{ICE} & & \text{IKE}
\end{tikzcd}
$$

ICE, IKE-closed

Def [E] $\mathcal{C} \subseteq \mathcal{A}$: abelian is

ICE-closed $\iff$ Image-Coker-Ext-closed

- $\{\text{tors}\}, \{\text{wide}\} \subseteq \{\text{ICE}\}$

 : common generalization of them.

Classification

(1) [E-Sakai '21] $\mathcal{C} \subseteq \operatorname{mod} A$, A: f.d. alg.

$\rightsquigarrow$ $\mathcal{C}$: ICE $\iff$ $\mathcal{C} \underset{\text{tors}}{\subseteq} {}^{\exists}\underset{\text{wide}}{\mathcal{W}} \subseteq \operatorname{mod} A$

(2) [E '22] If $A = kQ$ for Q : Dynkin,

$$\{\text{ICE}\} \xleftrightarrow{1-1} \{ M \in \operatorname{mod} A \mid \operatorname{Ext}^1_A(M, M) = 0 \}$$

Why ICE?

- Uniform way to study both tors & wide
- Applications to derived cat

Thm [Sakai '23, preprint]

Let A be f.d. alg with $\# \operatorname{tors} A < \infty$.

Then (good) t-structures "between $\operatorname{mod} A$ and $(\operatorname{mod} A)[n]$"

can be classified using sequences

$$0 = \mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_{n+1} = \operatorname{mod} A$$

of ICE-closed subcats (satisfying some cond.)

Summary

$$\begin{tikzcd}
 & \text{Serre} \arrow[dl, Rightarrow] \arrow[dd, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{tors} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] \arrow[dd, Rightarrow] & & \text{torf} \arrow[dl, leftrightarrow] \arrow[dd, Rightarrow] \\
 & \text{wide} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{ICE} \arrow[d, equal] & & \text{IKE} \arrow[d, equal] \\
\text{tors in wide} & & \text{torf in wide}
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & \text{Serre} \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{tors} \arrow[d, Rightarrow] & \text{wide} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \text{torf} \arrow[d, Rightarrow] \\
\text{ICE} \arrow[dr, Rightarrow] & & \text{IKE} \arrow[dl, Rightarrow] \\
 & \text{IE} &
\end{tikzcd}$$

IE-closed subcat

$\Longleftrightarrow$ Image - Extension - closed.

Thm [E-Sakai '23]

$\mathcal{C} \subseteq \mathcal{A}$: IE-closed

$\Longleftrightarrow$ $\exists\, \mathcal{T} \subseteq \mathcal{A}$: tors, $\mathcal{F} \subseteq \mathcal{A}$: torf

s.t. $\mathcal{C} = \mathcal{T} \cap \mathcal{F}$

Classification [E-Sakai]

If $A = kQ$, then

$$\{\,\mathrm{IE}\,\} \longleftrightarrow \left\{ (P, I) \;\middle|\; \begin{array}{l} \mathrm{Ext}^1(P,P) = 0 = \mathrm{Ext}^1(I,I), \\ 0 \to P \to I^0 \to I^1 \to 0 \\ 0 \to P_1 \to P_0 \to I \to 0, \end{array} \right\}$$

Why IE?

- WANT to study all extension-closed subcats. But it's difficult ......
- Restring to Image-E-closed, one can use theory of tors & torf!
- But not so many properties are known, ...

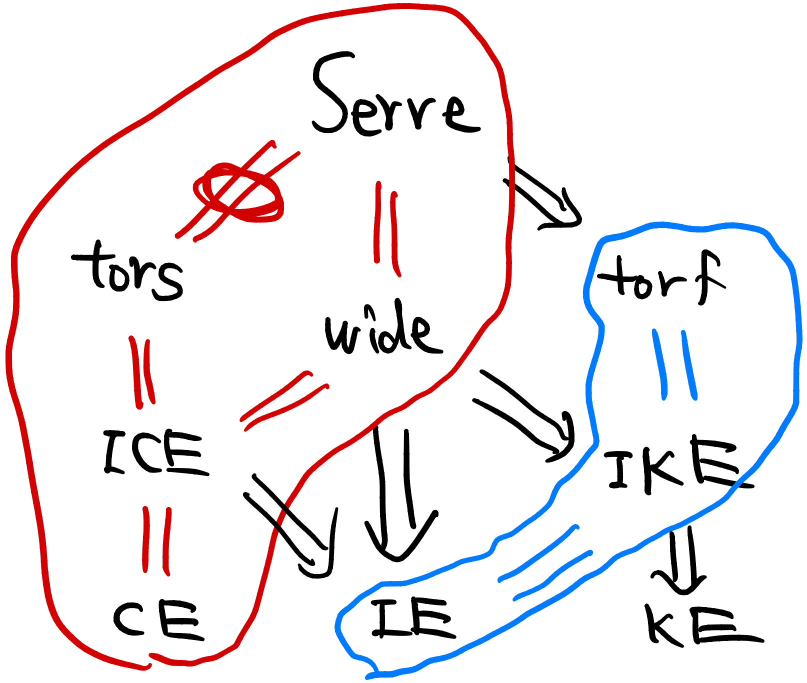
(e.g. Don't know the numbers of IEs for very basic alg!!)

Commutative case

Thm [Stanley–Wang '11, E, Kobayashi–Saito '23 (preprints)]

Let R be commutative Noetherian ring.

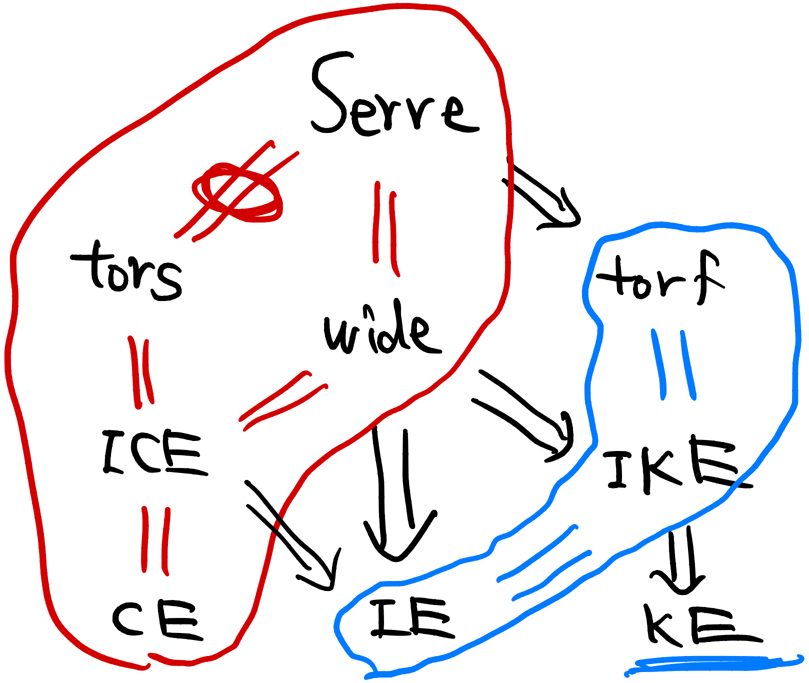
Then for $\mathcal{A} := \mathrm{mod}\, R$,



In general, "tors = Serre" implies

"CE = Serre" and "IE = torf"

Thm [Kobayashi-Saito, E, Iyama-Kimura (preprints)]



(1) Let R be nice comm. Noeth.

Then, $\dim R \leq 1$

$\iff$ KE = torf.

(2) Let A be Noetherian R-alg.

Then "Serre = tors"

($\iff$ red circle, blue circle : collapse)

$\iff \forall \mathfrak{p} \in \operatorname{Spec} R$, $A_{\mathfrak{p}}$: "essentially" local ring

(3) Let A be Noeth. (non-comm) ring.

Then ALL coincide $\iff$ A : artinian, "essentially" local ring

Future work?

[Kobayashi–Saito]
「CE = tors in tors」

- Study CE, KE!
  - Are they controllable by tilting/torsion theory?
  - Relation to derived cat?
  - How many are there (for fundamental algs)?
- Study IE!
  - How many?
    For kQ, the numbers depend on the orientation! (while tors $=$ wide NOT)
  - Classification?
- More combinatorics on ICE, IE?