ICE-closed subcategories and wide τ-tilting modules

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Overview

Today's talk

1. Study ICE-closed subcats of $\operatorname{mod}\Lambda$ using torsion classes.
2. Establish a bijection between ICE-closed subcategories and wide τ-tilting modules.
3. Hereditary & Nakayama cases

Outline

ICEs via intervals in $\operatorname{tors}\Lambda$

ICEs via wide τ-tilting

Hasse quiver of ICEs

ICEs via intervals in $\operatorname{tors} \Lambda$

Torsion classes and wide subcategories

Throughout this talk,

- Λ: f.d. algebra over a field k.
- $\operatorname{mod}\Lambda$: the cat. of f.g. right Λ-modules.

Definition

- A subcat. $\mathcal{T}$ of $\operatorname{mod}\Lambda$ is a torsion class (tors.) if it is closed under extensions and quotients.
- $\operatorname{tors}\Lambda$: the poset of torsion classes in $\operatorname{mod}\Lambda$.
- A subcat. $\mathcal{W}$ of $\operatorname{mod}\Lambda$ is wide if it is extension-closed exact abelian subcat, or equivalently, if closed under extensions, kernels and cokernels.

ICE-closed subcategories (ICEs)

Definition

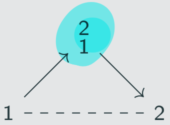
A subcategory $\mathcal{C}$ of $\operatorname{mod} \Lambda$ is ICE-closed if closed under taking Images, Cokernels and Extensions, that is,

- $\mathcal{C}$ is extension-closed.
- $f : C_1 \to C_2$ with $C_1, C_2 \in \mathcal{C}$ implies $\operatorname{Im} f, \operatorname{coker} f \in \mathcal{C}$.

Example

Easy to see chat every tors and wide is ICE.

$\operatorname{mod} k[1 \leftarrow 2]$: ICE



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Example

$$\operatorname{mod} k[1 \leftarrow 2 \to 3]$$

$$\begin{tikzcd}
3 \arrow[dr] \arrow[rr, dash, dashed] & & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[dr] & \\
& \begin{smallmatrix}2\\1\,3\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & 2 \\
1 \arrow[ur] \arrow[rr, dash, dashed] & & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] &
\end{tikzcd}$$

: ICE,
not tors,
not wide

Why ICE?

Idea

ICE: a new class of ext-closed subcat of $\operatorname{mod}\Lambda$ which is

- large enough (containing all tors & wide),
- but not so large (controlled by tors & τ-tilting)!

Previously, I gave a classification of ICEs of $\operatorname{mod}\Lambda$ for

- $\Lambda := kQ$ with Q Dynkin, and
- Λ: Nakayama.

Want to compute all ICEs!

Method

Use the heart of intervals in $\operatorname{tors}\Lambda$!

Definition

- For $\mathcal{U} \subseteq \mathcal{T}$ in $\operatorname{tors}\Lambda$, an interval is a subposet of $\operatorname{tors}\Lambda$

$$[\mathcal{U}, \mathcal{T}] := \{\mathcal{T}' \in \operatorname{tors}\Lambda \mid \mathcal{U} \subseteq \mathcal{T}' \subseteq \mathcal{T}\}.$$

- For an interval $[\mathcal{U}, \mathcal{T}]$ in $\operatorname{tors}\Lambda$, its heart is a subcat

$$\mathcal{H}_{[\mathcal{U},\mathcal{T}]} := \mathcal{T} \cap \mathcal{U}^{\perp} \quad \subseteq \operatorname{mod}\Lambda$$

About hearts of intervals

Remark

Heart construction is used in: [Jasso], [Demonet-Iyama-Reading-Reiten-Thomas], [Asai-Pfeifer], [Tattar], etc.
Terminology “hearts of twin torsion pairs" due to Tattar.

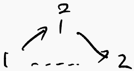
$\mathcal{H}_{[\mathcal{U},\mathcal{T}]} = \mathcal{T} \cap \mathcal{U}^{\perp}$ measures a difference “$\mathcal{T} - \mathcal{U}$.”

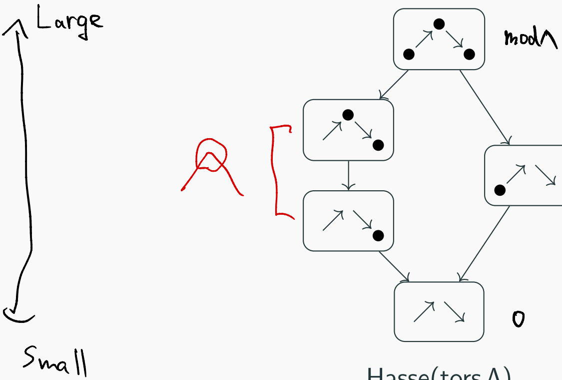
Example

- $\mathcal{H}_{[0,\mathcal{T}]} = \mathcal{T}$ “$\mathcal{T} - 0 = \mathcal{T}$”
- $\mathcal{H}_{[\mathcal{T},\operatorname{mod}\Lambda]} = \mathcal{T}^{\perp}$ “$\operatorname{mod}\Lambda - \mathcal{T} = \mathcal{T}^{\perp}$"
- $\mathcal{H}_{[\mathcal{T},\mathcal{T}]} = 0$ “$\mathcal{T} - \mathcal{T} = 0$.”
- $\mathcal{T} = \mathcal{U} * \mathcal{H}_{[\mathcal{U},\mathcal{T}]}$, “$\mathcal{T} = \mathcal{U} + (\mathcal{T} - \mathcal{U})$.”

Examples of hearts

$\Lambda := k[1 \leftarrow 2]$, mod Λ :





Hasse(tors Λ)

Every ICE is a heart

Proposition (E-Sakai)

Let $\mathcal{C}$ be an ICE-closed subcat of $\operatorname{mod}\Lambda$.
Then there is some interval $[\mathcal{U}, \mathcal{T}]$ in $\operatorname{tors}\Lambda$ satisfying

$$\mathcal{C} = \mathcal{H}_{[\mathcal{U},\mathcal{T}]} \quad (= \mathcal{T} \cap \mathcal{U}^{\perp})$$

($\mathcal{C}$ is a heart of $[{}^{\perp}\mathcal{C}, {}^{\perp}\mathcal{C} \vee \mathsf{T}(\mathcal{C})]$, but not used later)

Question

Which interval is an ICE interval, i.e. its heart is ICE?

Characterization of ICE

Theorem (E-Sakai)

The following are equivalent for an interval $[\mathcal{U}, \mathcal{T}]$ in $\operatorname{tors}\Lambda$:

1. $[\mathcal{U}, \mathcal{T}]$ is an ICE interval (i.e. $\mathcal{H}_{[\mathcal{U},\mathcal{T}]}$ is ICE).
2. There's some $\mathcal{T}' \in \operatorname{tors}\Lambda$ with $\mathcal{T} \subseteq \mathcal{T}'$ s.t. $[\mathcal{U}, \mathcal{T}']$ is wide interval (i.e. $\mathcal{H}_{[\mathcal{U},\mathcal{T}']}$ is wide).

In this case, $\mathcal{H}_{[\mathcal{U},\mathcal{T}]}$ is a torsion class in an abelian cat $\mathcal{H}_{[\mathcal{U},\mathcal{T}']}$.

$\mathcal{T}'$
$\cup\!|$
wide
$\mathcal{T}$
$\mathcal{H}_{[\mathcal{U},\mathcal{T}']}$ $\therefore$ wide.
$\cup\!|$
$\mathcal{H}_{[\mathcal{U},\mathcal{T}]}$
$\mathcal{U}$

Characterization of ICE

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Corollary (E-Sakai)

Let $\mathcal{C}$ be a subcategory of $\operatorname{mod}\Lambda$. Then TFAE:

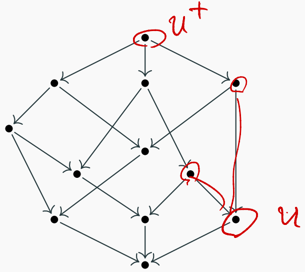
1. *$\mathcal{C}$ is an ICE-closed subcategory of $\operatorname{mod}\Lambda$.*
2. *There's some wide subcat $\mathcal{W}$ of $\operatorname{mod}\Lambda$ s.t. $\mathcal{C}$ is a torsion class in an abelian category $\mathcal{W}$.*

Combinatorial way to obtain all ICEs

Corollary (E-Sakai)

Every ICE-closed subcategory is obtained as follows:

- Choose $\mathcal{U} \in \operatorname{tors}\Lambda$
- Define $\mathcal{U}^+$ as the join of $\mathcal{U}_i$'s for all Hasse arrows $\mathcal{U} \leftarrow \mathcal{U}_i$ ending at $\mathcal{U}$.
- For each $\mathcal{T} \in [\mathcal{U}, \mathcal{U}^+]$, the heart $\mathcal{H}_{[\mathcal{U},\mathcal{T}]}$ is ICE.



tors Λ

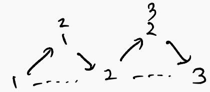
ICE as Filt of brick labels

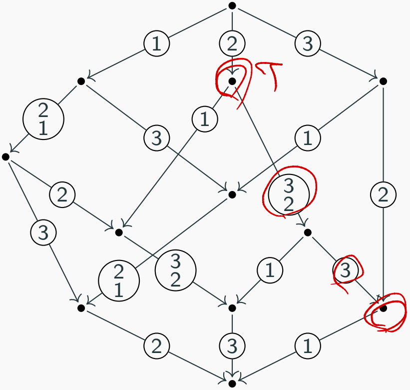
Remark

The heart of $[\mathcal{U}, \mathcal{T}]$ is Filt of brick labels in $[\mathcal{U}, \mathcal{T}]$.

$\Lambda := k[1 \leftarrow 2 \leftarrow 3]/\operatorname{rad}^2$

mod Λ :





Hasse(tors Λ) with brick labels

$[\mathcal{U}, \mathcal{T}]$: ICE int

$\|$

$\mathcal{H}_{[\mathcal{U},\mathcal{T}]}$

$\cap$

$\mathrm{Filt}(3, \begin{smallmatrix}3\\2\end{smallmatrix})$

ICEs via wide τ-tilting

Adachi-Iyama-Reiten's bijection

Theorem (Adachi-Iyama-Reiten)

There is a bijection

$$\text{f-tors}\,\Lambda \underset{\mathsf{Fac}}{\overset{P(-)}{\rightleftarrows}} \mathsf{s}\tau\text{-tilt}\,\Lambda \quad := \bigoplus \{\text{Ext-proj's}\}$$

between functorially finite tors and support τ-tilting mods.

Recall

Every ICE $\mathcal{C}$ is a torsion class in some wide subcat $\mathcal{W}$.

We can use Adachi-Iyama-Reiten's bij if

- $\mathcal{W}$ is equiv. to $\mathsf{mod}\,\Gamma$ for some Γ
  (this is equiv. to that $\mathcal{W}$ is fun. fin.)
- $\mathcal{C}$ is a fun. fin. torsion class in $\mathcal{W}$.

Doubly functorially finite ICE

Definition

An ICE-closed subcat $\mathcal{C}$ is doubly functorially finite
if there's some wide subcat $\mathcal{W}$ of $\operatorname{mod} \Lambda$ s.t.

- $\mathcal{W}$ is functorially finite.
- $\mathcal{C}$ is a funct. fin. torsion class in $\mathcal{W}$.

$\operatorname{df-ice} \Lambda$: the set of doubly fun. fin. ICEs of $\operatorname{mod} \Lambda$.

Clearly $\operatorname{f-tors} \Lambda \subseteq \operatorname{df-ice} \Lambda$ ($\mathcal{W} := \operatorname{mod} \Lambda$).

Remark

Λ is τ-tilting finite if and only if $\operatorname{ice} \Lambda = \operatorname{df-ice} \Lambda$.

Wide τ-tilting modules and a bijection

Definition

A Λ-module M is wide τ-tilting if there's some fun. fin. wide subcat $\mathcal{W}$ of $\mathrm{mod}\,\Lambda$ s.t. M is $\tau_{\mathcal{W}}$-tilting.

$\mathrm{w}\tau\text{-tilt}\,\Lambda$: the set of wide τ-tilting mods.

$$\begin{tikzcd} \mathcal{W} \arrow[r, phantom, "\simeq"] & \mathrm{mod}\,\Gamma \\ M \arrow[u, phantom, "\in" sloped] \arrow[r, mapsto] & \square \arrow[u, phantom, "\in" sloped] \end{tikzcd}$$

τ-tilt.

Support τ-tilting = wide τ-tilting with $\mathcal{W}$ being Serre.

Theorem (E-Sakai)

We have a bijection

$$\begin{tikzcd} \mathrm{df\text{-}ice}\,\Lambda \arrow[r, shift left, "P(-)"] & \mathrm{w}\tau\text{-tilt}\,\Lambda \arrow[l, shift left, "\mathrm{cok}(-)"] \\ \mathrm{f\text{-}tors}\,\Lambda \arrow[u, phantom, "\cup|" sloped] \arrow[r, leftrightarrow, "\text{[Adachi-Iyama-Reiten]}"'] & \mathrm{s}\tau\text{-tilt}\,\Lambda \arrow[u, phantom, "\cup|" sloped] \end{tikzcd}$$

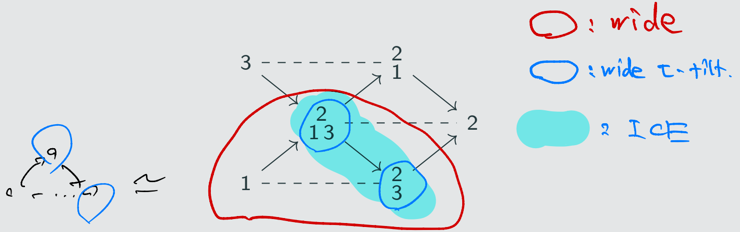
where $\mathrm{cok}\,M$ is the cat of cokernels of maps in $\mathrm{add}\,M$.

Example of bijection

$$\mathrm{w}\tau\text{-tilt}\,\Lambda \underset{P(-)}{\overset{\mathrm{cok}}{\rightleftarrows}} \text{df-ice}\,\Lambda$$

Example

$$\operatorname{mod} k[1 \leftarrow 2 \rightarrow 3]$$

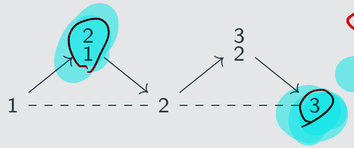


Example of bijection

$$\text{w}\tau\text{-tilt}\,\Lambda \underset{P(-)}{\overset{\text{cok}}{\rightleftarrows}} \text{df-ice}\,\Lambda$$

Example

$\operatorname{mod}\Lambda$ for $\Lambda := k[1 \leftarrow 2 \leftarrow 3]/\operatorname{rad}^2$ $\begin{smallmatrix}2\\1\end{smallmatrix} \oplus 3$



○ : wide τ-tilt

▬ : ICE

Remark

Wide τ-tilting modules is not τ-rigid in general!

Wide τ-tilting from support τ-tilting

We can obtain all wide τ-tilting modules if

- Λ is τ-tilting finite, and
- The Hasse quiver of $\mathrm{s}\tau\text{-tilt}\,\Lambda$ is given.

Proposition

Let $[\mathrm{Fac}\,U, \mathrm{Fac}\,T]$ *be an ICE interval for* $U, T \in \mathrm{s}\tau\text{-tilt}\,\Lambda$.
Then wide τ*-tilting mod. corresponding to* $\mathcal{H}_{[\mathrm{Fac}\,U, \mathrm{Fac}\,T]}$ *is:*

$$T / \mathrm{tr}_U(T)$$

"$T - U$"

T

U

U

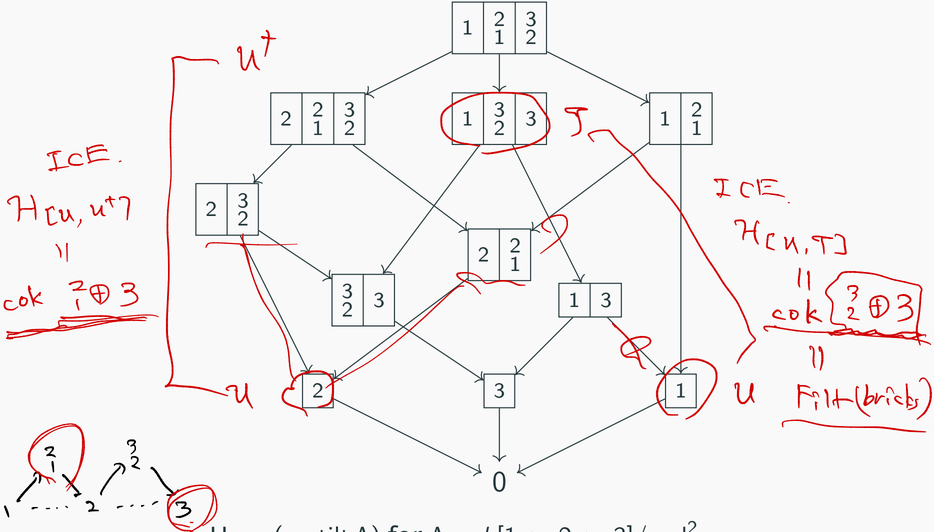
$\mathcal{H}_{[\mathrm{Fac}\,U, \mathrm{Fac}\,T]}$ ICE

where $\mathrm{tr}_U(T) := \sum\{\mathrm{Im}\,\varphi \mid \varphi\colon U \to T\}$.

$u \leq T \leq T'$

T: τ-rigid

Examples of computation of wide τ-tilting



Hasse(sτ-tilt Λ) for $\Lambda = k[1 \leftarrow 2 \leftarrow 3]/\operatorname{rad}^2$

Hasse quiver of ICEs

Hasse quiver

$|M|$: the number of indecomposable direct summands of M up to isom.

Proposition (Adachi-Iyama-Reiten)

The poset $\mathrm{s}\tau\text{-tilt}\,\Lambda$ *is Hasse* $|\Lambda|$*-regular,*
that is, for each vertex $M \in \mathrm{Hasse}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$,

$$\#\{\textit{arrows starting at } M\} + \#\{\textit{arrows ending at } M\} = |\Lambda|$$

Question

Is there any analogous combinatorial property
for wide τ-tilting (or ICE-closed subcats)?

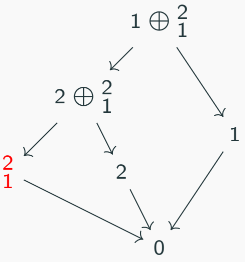
Example of Hasse(wτ-tilt Λ)

wτ-tilt Λ has the poset str. induced by
cok$(-)$: wτ-tilt $\Lambda \cong$ df-ice Λ.

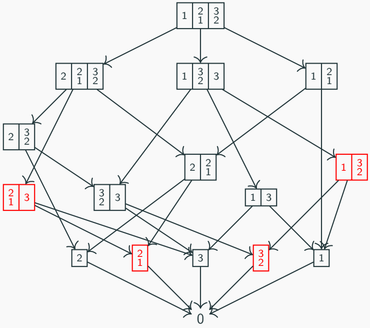
$M \leq N :\Longleftrightarrow \operatorname{cok} M \subseteq \operatorname{cok} N$

The following are Hasse(wτ-tilt Λ).

red: wide τ-tilting, not support τ-tilting.



$\Lambda = k[1 \leftarrow 2]$



$\Lambda = k[1 \leftarrow 2 \leftarrow 3]/\operatorname{rad}^2$

Hereditary case

Proposition (E-Sakai)

If Λ is hereditary, then

- *wide τ-tilting modules = rigid modules.* $\mathrm{Ext}^1_\Lambda(M,M)=0$
- *doubly fun. fin. = fun.fin. ICEs.*

Theorem (E-Sakai)

Let Λ be hereditary. Then for each $M \in \mathrm{w}\tau\text{-tilt}\,\Lambda$, there are exactly $|M|$ arrows starting at M in $\mathrm{Hasse}(\mathrm{w}\tau\text{-tilt}\,\Lambda)$.

For each indec. summand X of M, there is an Hasse arrow

$$M \to \mu_X(M)$$

with $\mu_X(M)$: generalization of left mutation of $\mathrm{s}\tau\text{-tilt}\,\Lambda$.

Nakayama case

Theorem (E, in preparation)

Let Λ be Nakayama. Then for each $M \in \mathrm{w}\tau\text{-tilt}\,\Lambda$, there are exactly $|M|$ arrows starting at M in $\mathrm{Hasse}(\mathrm{w}\tau\text{-tilt}\,\Lambda)$.

Proof uses simple objects in ICEs:

ICEs bijectively correspond to epibricks.

set of brick

s.t. $B_1 \longrightarrow B_2$

is 0 or surjection.

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