

Relations for Grothendieck groups and Representation-finiteness

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Outline

- 1 Introduction
 - Known Results
- 2 Main results
 - Settings and the motivating question
 - Main Results
- 3 Functorial proof
 - Effaceable functors
 - Sketch of Proof

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Representation-finiteness

Λ : artin algebra.

$\text{mod } \Lambda$: the cat. of f.g. right Λ -module.

Definition

Λ is **representation-finite** $:\Leftrightarrow$ $\text{mod } \Lambda$ has finitely many indecomposable objects (up to isom).

Classical Question

Characterize Representation-Finiteness!

Results of Butler and Auslander

Theorem (Butler 1981, Auslander 1984)

The following are equivalent:

- (1) Λ is representation-finite.
- (2) The *relations* for the Grothendieck group $K_0(\text{mod } \Lambda)$ are generated by *Auslander-Reiten sequences*.

Today, I will

Generalize this results, from $\text{mod } \Lambda$ to

- Subcategory of $\text{mod } \Lambda$, or
- Higher Krull-dimensional version.

by using **Functorial Arguments** on **Exact Categories**.

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Exact category

$0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ is a **kernel-cokernel pair** in \mathcal{E}
: $\Leftrightarrow f = \ker g$ and $g = \operatorname{coker} f$.

Definition (Quillen 1973)

An **exact category** \mathcal{E} consists of:

- an additive category \mathcal{E} , together with
- a **class of ker-coker pairs** in \mathcal{E} (called **conflations**)

satisfying some conditions.

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Example

Extension-closed subcategory of abelian categories
(e.g. torsion class, CM rep-theory, \dots)

The Grothendieck group of exact cat.

\mathcal{E} : Krull-Schmidt exact category.

Definition

- $\text{ind } \mathcal{E}$: the set of **indecomposable** obj. (up to iso) in \mathcal{E} .
- $K_0(\mathcal{E}, 0)$: the free abelian group with basis set $\text{ind } \mathcal{E}$.
 \cup
- $\text{Ex}(\mathcal{E})$: the subgroup gen. by $[X] - [Y] + [Z]$
 for all conflations $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ in \mathcal{E} .
 (we identify $[X \oplus Y] = [X] + [Y]$ in $K_0(\mathcal{E}, 0)$)
- $K_0(\mathcal{E}) := K_0(\mathcal{E}, 0) / \text{Ex}(\mathcal{E})$: the **Grothendieck group**.

Auslander-Reiten conflations

\mathcal{E} : Krull-Schmidt exact category.

Definition

A conflation $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ is an **AR conflation** if:

- g is right almost split.
- f is left almost split.

Auslander-Reiten conflations

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Definition

A conflation $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ is an **AR conflation** if:

- g is right almost split.
- f is left almost split.

Then we can define the following subgroups:

- $K_0(\mathcal{E}, 0)$: the split Grothendieck group.
 \cup
- $\text{Ex}(\mathcal{E})$: gen. by $[X] - [Y] + [Z]$ for all conflations
 $\cup \quad 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0.$
- $\text{AR}(\mathcal{E})$: — for all **AR** conflations.

Motivating Question

\mathcal{E} : Krull-Schmidt exact category.

Question

When are the following equivalent?

- (1) \mathcal{E} is **finite** ($:\Leftrightarrow \text{ind } \mathcal{E}$ is a finite set).
- (2) $\text{Ex}(\mathcal{E}) = \text{AR}(\mathcal{E})$ holds.

For this question,

- (1) \Rightarrow (2): (almost) always holds by my previous work.
- (2) \Rightarrow (1): \exists Some counter-examples.

Results for artin algebras

Λ : artin algebra.

Definition

A subcategory \mathcal{E} of $\text{mod } \Lambda$ is **resolving** if

- it is closed under extension, summands.
- contains all projectives.
- For each exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$, if Y and Z belong to \mathcal{E} , then so does X .

Theorem (E)

Let Λ be an artin algebra and \mathcal{E} a *contravariantly finite resolving* subcategory of $\text{mod } \Lambda$. Then TFAE.

- (1) \mathcal{E} is finite.
- (2) $\text{Ex}(\mathcal{E}) = \text{AR}(\mathcal{E})$.

Example

- Functorially finite torsion(free) class.
- $\mathcal{F}(\Delta)$ over standardly-stratified algebra (e.g. quasi-hereditary algebra).
- ${}^{\perp}U := \{X \in \text{mod } \Lambda \mid \text{Ext}_{\Lambda}^{>0}(X, U) = 0\}$ for a cotilting module U .
- CM Λ for Iwanaga-Gorenstein algebra Λ .

Results for orders

R : commutative regular complete local ring.

Definition

- An R -algebra Λ is **R -order**
: $\Leftrightarrow \Lambda$ is free of finite rank as an R -module.
- $\text{CM } \Lambda := \{X \in \text{mod } \Lambda \mid X \text{ is free over } R\}$
 \rightsquigarrow Krull-Schmidt exact cat. with enough proj = proj Λ .
- Λ is **Gorenstein** if $\text{CM } \Lambda$ is Frobenius exact category.
- Λ "has" an **isolated singularity** if $\text{CM } \Lambda$ has AR confluents.

Example

- $\dim R = 0 \rightsquigarrow R\text{-order} = \text{f.d. alg}, \text{CM } \Lambda = \text{mod } \Lambda$.

Result for orders

Theorem (E)

Let Λ be an R -order with isolated singularity. If either

- Λ has finite global dimension, or
- Λ is Gorenstein,

then TFAE:

- (1) $\text{CM } \Lambda$ is finite.
- (2) $\text{Ex}(\text{CM } \Lambda) = \text{AR}(\text{CM } \Lambda)$ holds.

Conjecture

(1) and (2) are equivalent for an arbitrary order Λ !
(open even for commutative case)

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Idea of Proof

\mathcal{E} : Krull-Schmidt exact category.

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\mathcal{E} : Krull-Schmidt exact category.

Question

When are the following equivalent?

- (1) \mathcal{E} is finite.
- (1.5) Every object in $\text{eff } \mathcal{E}$ has finite length.
- (2) $\text{Ex}(\mathcal{E}) = \text{AR}(\mathcal{E})$ holds.

Idea

(1.5) Categorifies (2),

Idea of Proof

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Question

When are the following equivalent?

- (1) \mathcal{E} is finite.
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Idea

(1.5) Categorifies (2), and is closer to (1)!!

Effaceable functors

\mathcal{E} : Krull-Schmidt exact category.

- \mathcal{E} -module is a contravariant additive functor
 $M : \mathcal{E}^{\text{op}} \rightarrow \mathcal{A}b. \quad \rightsquigarrow \text{Mod } \mathcal{E} : \text{abelian cat.}$
- The Yoneda emb. $X \mapsto \mathcal{E}(-, X)$ induces equivalence between \mathcal{E} and the cat of **f.g. proj. \mathcal{E} -modules**.

Effaceable functors

\mathcal{E} : Krull-Schmidt exact category.

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Definition

\mathcal{E} -module M is **effaceable**

$:\Leftrightarrow \exists$ a **conflation** $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ in \mathcal{E} with

$$0 \rightarrow \mathcal{E}(-, X) \rightarrow \mathcal{E}(-, Y) \rightarrow \mathcal{E}(-, Z) \rightarrow M \rightarrow 0$$

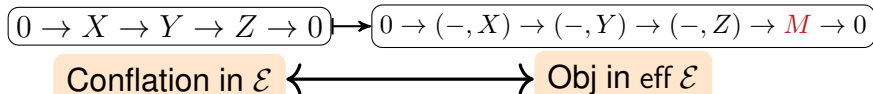
eff \mathcal{E} : the cat. of effaceable modules.

Properties of $\text{eff } \mathcal{E}$

\mathcal{E} : Krull-Schmidt exact category.

Facts

- $\text{eff } \mathcal{E}$ is **abelian** subcategory of $\text{Mod } \mathcal{E}$.
- $\text{eff } \mathcal{E}$ **reconstructs the exact structure** of \mathcal{E} .
- $\text{eff } \mathcal{E} = \text{mod } \underline{\mathcal{E}}$ if \mathcal{E} has enough projectives.



Properties of eff \mathcal{E}

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Facts

- eff \mathcal{E} is **abelian** subcategory of $\text{Mod } \mathcal{E}$.
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$$\boxed{0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0} \longrightarrow \boxed{0 \rightarrow (-, X) \rightarrow (-, Y) \rightarrow (-, Z) \rightarrow M \rightarrow 0}$$

Conflation in \mathcal{E}

Obj in eff \mathcal{E}

AR conflation

simple obj in eff \mathcal{E}

General result

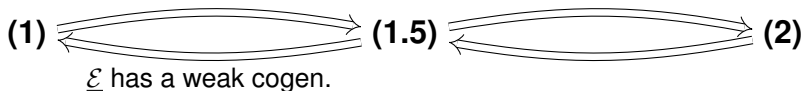
R : artinian.

\mathcal{E} : Krull-Schmidt Hom-finite exact R -cat. with a progen.

Conditions

- (1) \mathcal{E} is finite.
- (1.5) Every object in $\text{eff } \mathcal{E}$ ($= \text{mod } \underline{\mathcal{E}}$) has finite length.
- (2) $\text{Ex}(\mathcal{E}) = \text{AR}(\mathcal{E})$ holds.

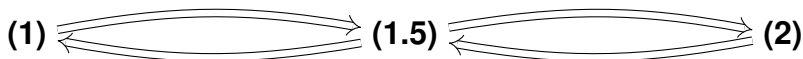
Then the following holds:



Remark

Conditions

- (1) \mathcal{E} is finite.
- (1.5) Every object in $\text{eff } \mathcal{E}$ ($= \text{mod } \underline{\mathcal{E}}$) has finite length.
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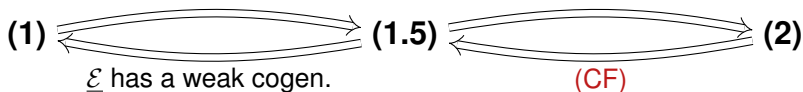
(*): $\underline{\mathcal{E}}$ has a weak cogen.

- (*) is OK for contravariantly finite resolving subcat (including CM Λ for order Λ).

Remark

Conditions

- (1) \mathcal{E} is finite.
- (1.5) Every object in $\text{eff } \mathcal{E}$ ($= \text{mod } \underline{\mathcal{E}}$) has finite length.
- (2) $\text{Ex}(\mathcal{E}) = \text{AR}(\mathcal{E})$ holds.



- (*) is OK for contravariantly finite resolving subcat (including CM Λ for order Λ).
- We need extra cond. (CF) when we drop artin-ness.

Thank you for your
attention!