Relations for Grothendieck groups and Representation-finiteness

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Outline

1. Introduction
   - Known Results
2. Main results
   - Settings and the motivating question
   - Main Results
3. Functorial proof
   - Effaceable functors
   - Sketch of Proof

Outline

Representation-finiteness

Λ: artin algebra.
$\operatorname{mod}\Lambda$: the cat. of f.g. right Λ-module.

Definition

Λ is representation-finite $:\Leftrightarrow$ $\operatorname{mod}\Lambda$ has finitely many indecomposable objects (up to isom).

Classical Question

Characterize Representation-Finiteness!

Results of Butler and Auslander

Theorem (Butler 1981, Auslander 1984)

The following are equivalent:

(1) Λ *is representation-finite.*

(2) *The relations for the Grothendieck group* $\mathrm{K}_0(\operatorname{mod} \Lambda)$ *are generated by Auslander-Reiten sequences.*

Today, I will

Generalize this results, from $\operatorname{mod} \Lambda$ to

- Subcategory of $\operatorname{mod} \Lambda$, or
- Higher Krull-dimensional version.

by using Functorial Arguments on Exact Categories.

Outline

Exact category

$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is a kernel-cokernel pair in $\mathcal{E}$
$:\Leftrightarrow f = \ker g$ and $g = \operatorname{coker} f$.

Definition (Quillen 1973)

An exact category $\mathcal{E}$ consists of:

- an additive category $\mathcal{E}$, together with
- a class of ker-coker pairs in $\mathcal{E}$ (called conflations)

satisfying some conditions.

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satisfying some conditions.

Example

Extension-closed subcategory of abelian categories
(e.g. torsion class, CM rep-theory,$\cdots$)

The Grothendieck group of exact cat.

$\mathcal{E}$: Krull-Schmidt exact category.

Definition

- ind $\mathcal{E}$: the set of indecomposable obj. (up to iso) in $\mathcal{E}$.
- $\mathsf{K}_0(\mathcal{E}, 0)$: the free abelian group with basis set ind $\mathcal{E}$.
  $\cup$
- $\mathsf{Ex}(\mathcal{E})$: the subgroup gen. by $[X] - [Y] + [Z]$
  for all conflations $0 \to X \to Y \to Z \to 0$ in $\mathcal{E}$.
  (we identify $[X \oplus Y] = [X] + [Y]$ in $\mathsf{K}_0(\mathcal{E}, 0)$)
- $\mathsf{K}_0(\mathcal{E}) := \mathsf{K}_0(\mathcal{E}, 0) / \mathsf{Ex}(\mathcal{E})$: the Grothendieck group.

Auslander-Reiten conflations

$\mathcal{E}$: Krull-Schmidt exact category.

Definition

A conflation $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is an AR conflation if:

- g is right almost split.
- f is left almost split.

Auslander-Reiten conflations

$\mathcal{E}$: Krull-Schmidt exact category.

Definition

A conflation $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is an AR conflation if:

- g is right almost split.
- f is left almost split.

Then we can define the following subgroups:

- $\mathsf{K}_0(\mathcal{E}, 0)$: the split Grothendieck group.

  $\cup$
- $\mathsf{Ex}(\mathcal{E})$: gen. by $[X] - [Y] + [Z]$ for all conflations

  $\cup$ $\quad 0 \to X \to Y \to Z \to 0.$
- $\mathsf{AR}(\mathcal{E})$: — for all AR conflations.

Motivating Question

$\mathcal{E}$: Krull-Schmidt exact category.

Question

When are the following equivalent?

(1) $\mathcal{E}$ is finite (:$\Leftrightarrow$ $\operatorname{ind}\mathcal{E}$ is a finite set).

(2) $\operatorname{Ex}(\mathcal{E}) = \operatorname{AR}(\mathcal{E})$ holds.

For this question,

(1) $\Rightarrow$ **(2)**: (almost) always holds by my previous work.
(2) $\Rightarrow$ **(1)**: $\exists$ Some counter-examples.

Results for artin algebras

Λ: artin algebra.

Definition

A subcategory $\mathcal{E}$ of $\operatorname{mod} \Lambda$ is resolving if

- it is closed under extension, summands.
- contains all projectives.
- For each exact sequence $0 \to X \to Y \to Z \to 0$, if Y and Z belong to $\mathcal{E}$, then so does X.

Theorem (E)

Let Λ be an artin algebra and $\mathcal{E}$ a contravariantly finite resolving subcategory of $\operatorname{mod}\Lambda$. Then TFAE.

(1) *$\mathcal{E}$ is finite.*

(2) $\mathrm{Ex}(\mathcal{E}) = \mathrm{AR}(\mathcal{E})$.

Example

- Functorially finite torsion(free) class.
- $\mathcal{F}(\Delta)$ over standardly-stratified algebra (e.g. quasi-hereditary algebra).
- ${}^{\perp}U := \{X \in \operatorname{mod}\Lambda \,|\, \mathrm{Ext}_{\Lambda}^{>0}(X, U) = 0\}$ for a cotilting module U.
- $\mathrm{CM}\,\Lambda$ for Iwanaga-Gorenstein algebra Λ.

Results for orders

R: commutative regular complete local ring.

Definition

- An R-algebra Λ is R-order
  $:\Leftrightarrow$ Λ is free of finite rank as an R-module.
- $\mathsf{CM}\,\Lambda := \{X \in \operatorname{mod}\Lambda \mid X$ is free over $R\}$
  $\rightsquigarrow$ Krull-Schmidt exact cat. with enough proj = $\operatorname{proj}\Lambda$.
- Λ is Gorenstein if $\mathsf{CM}\,\Lambda$ is Frobenius exact category.
- Λ "has" an isolated singularity if $\mathsf{CM}\,\Lambda$ has AR conflations.

Example

- $\dim R = 0 \rightsquigarrow R$-order = f.d. alg, $\mathsf{CM}\,\Lambda = \operatorname{mod}\Lambda$.

Result for orders

Theorem (E)

Let Λ be an R-order with isolated singularity. If either

- *Λ has finite global dimension, or*
- *Λ is Gorenstein,*

then TFAE:

(1) $\mathsf{CM}\,\Lambda$ *is finite.*

(2) $\mathsf{Ex}(\mathsf{CM}\,\Lambda) = \mathsf{AR}(\mathsf{CM}\,\Lambda)$ *holds.*

Conjecture

(1) and (2) are equivalent for an arbitrary order Λ!
(open even for commutative case)

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Idea of Proof

$\mathcal{E}$: Krull-Schmidt exact cateogory.

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Question

When are the following equivalent?

(1) $\mathcal{E}$ is finite.

(1.5) Every object in $\operatorname{eff} \mathcal{E}$ has finite length.

(2) $\operatorname{Ex}(\mathcal{E}) = \operatorname{AR}(\mathcal{E})$ holds.

Idea

(1.5) Categorifies **(2)**,

Idea of Proof

$\mathcal{E}$: Krull-Schmidt exact category.

Question

When are the following equivalent?

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(2) $\operatorname{Ex}(\mathcal{E}) = \operatorname{AR}(\mathcal{E})$ holds.

Idea

(1.5) Categorifies **(2)**, and is closer to **(1)**!!

Effaceable functors

$\mathcal{E}$: Krull-Schmidt exact category.

- $\mathcal{E}$-module is a contravariant additive functor $M : \mathcal{E}^{\mathrm{op}} \to \mathcal{A}b$. $\rightsquigarrow$ Mod $\mathcal{E}$: abelian cat.
- The Yoneda emb. $X \mapsto \mathcal{E}(-, X)$ induces equivalence between $\mathcal{E}$ and the cat of f.g. proj. $\mathcal{E}$-modules.

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Definition

$\mathcal{E}$-module M is effaceable

:$\Leftrightarrow$ $\exists$ a conflation $0 \to X \to Y \to Z \to 0$ in $\mathcal{E}$ with

$$0 \to \mathcal{E}(-, X) \to \mathcal{E}(-, Y) \to \mathcal{E}(-, Z) \to M \to 0$$

eff $\mathcal{E}$: the cat. of effaceable modules.

Properties of eff $\mathcal{E}$

$\mathcal{E}$: Krull-Schmidt exact category.

Facts

- eff $\mathcal{E}$ is abelian subcategory of Mod $\mathcal{E}$.
- eff $\mathcal{E}$ reconstructs the exact structure of $\mathcal{E}$.
- eff $\mathcal{E}$ $=$ mod $\underline{\mathcal{E}}$ if $\mathcal{E}$ has enough projectives.

$$0 \to X \to Y \to Z \to 0 \longmapsto 0 \to (-,X) \to (-,Y) \to (-,Z) \to M \to 0$$

Conflation in $\mathcal{E}$ $\longleftrightarrow$ Obj in eff $\mathcal{E}$

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Conflation in $\mathcal{E}$ $\longleftrightarrow$ Obj in eff $\mathcal{E}$

AR conflation $\longleftrightarrow$ simple obj in eff $\mathcal{E}$

General result

R: artinian.
$\mathcal{E}$: Krull-Schmidt Hom-finite exact R-cat. with a progen.

Conditions

(1) $\mathcal{E}$ is finite.

(1.5) Every object in $\mathsf{eff}\,\mathcal{E}$ ($=\mathsf{mod}\,\underline{\mathcal{E}}$) has finite length.

(2) $\mathsf{Ex}(\mathcal{E}) = \mathsf{AR}(\mathcal{E})$ holds.

Then the following holds:

(1) $\rightleftarrows$ **(1.5)** $\rightleftarrows$ **(2)**

$\underline{\mathcal{E}}$ has a weak cogen.

Remark

Conditions

- **(1)** $\mathcal{E}$ is finite.
- **(1.5)** Every object in $\mathsf{eff}\,\mathcal{E}$ ($=\mathsf{mod}\,\underline{\mathcal{E}}$) has finite length.
- **(2)** $\mathsf{Ex}(\mathcal{E}) = \mathsf{AR}(\mathcal{E})$ holds.

$$\textbf{(1)} \rightleftarrows \textbf{(1.5)} \rightleftarrows \textbf{(2)}$$

(*): $\underline{\mathcal{E}}$ has a weak cogen.

- (*) is OK for contravariantly finite resolving subcat (including $\mathsf{CM}\,\Lambda$ for order Λ).

Remark

Conditions

- **(1)** $\mathcal{E}$ is finite.
- **(1.5)** Every object in $\operatorname{eff}\mathcal{E}$ $(=\operatorname{mod}\underline{\mathcal{E}})$ has finite length.
- **(2)** $\operatorname{Ex}(\mathcal{E}) = \operatorname{AR}(\mathcal{E})$ holds.

$$\textbf{(1)} \rightleftarrows \textbf{(1.5)} \rightleftarrows \textbf{(2)}$$

(1.5) ⇒ (1): $\underline{\mathcal{E}}$ has a weak cogen.

(2) ⇒ (1.5): (CF)

- (*) is OK for contravariantly finite resolving subcat (including $\operatorname{CM}\Lambda$ for order Λ).
- We need extra cond. (CF) when we drop artin-ness.

Thank you for your attention!