Some classes of subcategories of module categories: classifications and the relation between them

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Overview

Today's talk

- Introduce new classes of subcategories of $\mathrm{mod}\,\Lambda$.
- Give classification results of these subcategories.

Throughout this talk,

- Λ: a finite-dimensional k-algebra over a field k.
- $\mathrm{mod}\,\Lambda$: the category of finitely generated right Λ-modules.

Motivation

Slogan

Study various subcategories of $\operatorname{mod}\Lambda$!

Question

- What kinds of subcategories should we study?
- What "study" means?

(Today's) Answer

1. Subcategories controlled by torsion pairs.
2. Classify and describe poset structure.

Outline

Torsion pairs, Serre and wide subcategories

ICE-closed subcategories and torsion hearts

From $\operatorname{tors}\Lambda$ to other posets

Torsion pairs, Serre and wide subcategories

Torsion pairs

Definition (Dickson 1966)

- A subcategory $\mathcal{T}$ of $\operatorname{mod}\Lambda$ is a torsion class if it is closed under extensions and quotients.
- A subcategory $\mathcal{F}$ of $\operatorname{mod}\Lambda$ is a torsion-free class if it is closed under extensions and submodules.
- $\operatorname{tors}\Lambda$ ($\operatorname{torf}\Lambda$): the poset of torsion(-free) classes in $\operatorname{mod}\Lambda$.

- $\operatorname{tors}\Lambda$ and $\operatorname{torf}\Lambda$ are anti-isom by $(-)^{\perp}$ and ${}^{\perp}(-)$.
- If $\operatorname{tors}\Lambda$ is a finite set (τ-tilting finite), then there's a bijection between $\operatorname{tors}\Lambda$ and support τ-tilting modules [Adachi-Iyama-Reiten 2014]

Serre and wide subcategories

Definition (Serre 1953?, Hovey 2001)

- A subcategory $\mathcal{S}$ of $\operatorname{mod}\Lambda$ is a Serre subcategory if it is closed under extensions, quotients, and submodules.
- A subcategory $\mathcal{W}$ of $\operatorname{mod}\Lambda$ is a wide subcategory if it is closed under extensions, cokernels, and kernels.
- $\operatorname{Serre}\Lambda$ ($\operatorname{wide}\Lambda$): the posets of Serre (wide) subcategories of $\operatorname{mod}\Lambda$.

Clearly $\operatorname{Serre}\Lambda = \operatorname{tors}\Lambda \cap \operatorname{torf}\Lambda$ and $\operatorname{Serre}\Lambda \subseteq \operatorname{wide}\Lambda$.

Theorem (Ingalls-Thomas 2009, Marks-Šťovíček 2017)

Suppose Λ is τ-tilting finite. Then there is a bijection between $\operatorname{tors}\Lambda$ and $\operatorname{wide}\Lambda$ (but not poset-isom!).

Picture

$$
\begin{array}{ccccc}
 & & ??? & & \\
 & \rotatebox{45}{$\subset$} & & \rotatebox{-45}{$\supset$} & \\
?? & & & & ?? \\
 & \rotatebox{-45}{$\supset$} & & \rotatebox{45}{$\subset$} & \\
\cup & & \mathsf{wide}\,\Lambda & & \cup \\
 & & & & \\
\mathsf{tors}\,\Lambda & & \cup & & \mathsf{torf}\,\Lambda \\
 & \rotatebox{-45}{$\supset$} & & \rotatebox{45}{$\subset$} & \\
 & & \mathsf{Serre}\,\Lambda & &
\end{array}
$$

Picture

$$
\begin{array}{ccccc}
 & & \text{tors-heart}\,\Lambda & & \\
 & \subset & & \supset & \\
\text{ICE}\,\Lambda & & & & \text{IKE}\,\Lambda \\
 & \supset & & \subset & \\
\cup & & \text{wide}\,\Lambda & & \cup \\
 & & & & \\
\text{tors}\,\Lambda & & \cup & & \text{torf}\,\Lambda \\
 & \supset & & \subset & \\
 & & \text{Serre}\,\Lambda & &
\end{array}
$$

ICE-closed subcategories and torsion hearts

ICE-closed subcategories

Definition (E 2020)

A subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ is ICE-closed if it is closed under:

- Images ($f\colon C_1 \to C_2$ with $C_1, C_2 \in \mathcal{C} \Rightarrow \operatorname{Im} f \in \mathcal{C}$),
- Cokernels (... $\Rightarrow \operatorname{Coker} f \in \mathcal{C}$), and
- Extensions.

$\operatorname{ICE}\Lambda$: the poset of ICE-closed subcategories of $\operatorname{mod}\Lambda$.

Dually define IKE-closed subcategories (Image-Kernel-Extension-closed).

- $\operatorname{tors}\Lambda \subseteq \operatorname{ICE}\Lambda$, $\quad \operatorname{torf}\Lambda \subseteq \operatorname{IKE}\Lambda$.
- $\operatorname{ICE}\Lambda \cap \operatorname{IKE}\Lambda = \operatorname{wide}\Lambda$.

ICE-closed subcategories

Example: kQ for $Q : 1 \leftarrow 2 \rightarrow 3$

Theorem (E-Sakai 2021)

Let $\mathcal{C}$ be a subcategory of $\operatorname{mod}\Lambda$. Then TFAE.

1. $\mathcal{C}$ is an ICE-closed subcategory.
2. There is some wide subcategory $\mathcal{W}$ containing $\mathcal{C}$ such that $\mathcal{C}$ is a torsion class in $\mathcal{W}$.

Classification of ICE-closed subcategories

Corollary (E-Sakai 2021)

If Λ is τ-tilting finite, then there is a bijection between:

- *ICE-closed subcategories of* $\operatorname{mod} \Lambda$ *and*
- *wide τ-tilting modules*
  (= $\tau_{\mathcal{W}}$-tilting object in some wide subcat $\mathcal{W}$).

This generalizes Adachi-Iyama-Reiten's bijection!

Corollary (E 2020)

If Q is a Dynkin quiver, then there is a bijection between:

- *ICE-closed subcategories of* $\operatorname{mod} kQ$ *and*
- *rigid kQ-modules*
  (modules M with $\operatorname{Ext}^1_{kQ}(M, M) = 0$).

(Some questions)

- Easy characterization of wide τ-tilting modules (for non-hereditary case)?
- Interpretation of wide τ-tilting modules using the derived category? (silting complex for usual τ-tilting theory).

Torsion hearts

The proof uses the notion of torsion hearts.

Definition (Demonet-Iyama-Reading-Reiten-Thomas 2017, Tattar 2020, Asai-Pfeifer 2021, E-Sakai 2021, etc)

- To each pair $\mathcal{U} \subseteq \mathcal{T}$ in $\operatorname{tors} \Lambda$, its heart is:

$$\mathcal{H}_{[\mathcal{U},\mathcal{T}]} := \mathcal{T} \cap \mathcal{U}^{\perp} \, (= "\mathcal{T} - \mathcal{U}"),$$

- A subcategory of this form is called a torsion heart.
- $\operatorname{tors-heart} \Lambda$: the poset of torsion hearts.

The following subcategories are torsion hearts:

- Torsion(-free) classes (by $\mathcal{T} = \mathcal{H}_{[0,\mathcal{T}]}$ and its dual).
- Wide subcategories [Asai-Pfeifer 2021]
- ICE-closed subcategories (and IKE) [E-Sakai 2021].

Picture Again

$$
\begin{array}{ccccc}
 & & \text{tors-heart}\,\Lambda & & \\
 & \swarrow\!\!\!\!\subset & & \supset\!\!\!\!\searrow & \\
\text{ICE}\,\Lambda & & & & \text{IKE}\,\Lambda \\
 & \supset & & \subset & \\
\cup & & \text{wide}\,\Lambda & & \cup \\
 & & & & \\
\text{tors}\,\Lambda & & \cup & & \text{torf}\,\Lambda \\
 & \supset & & \subset & \\
 & & \text{Serre}\,\Lambda & &
\end{array}
$$

How to obtain the poset $\text{tors-heart}\,\Lambda$?

From $\operatorname{tors}\Lambda$ to other posets

From $\mathsf{tors}\,\Lambda$ to other posets

$$\mathsf{itv}(\mathsf{tors}\,\Lambda) := \{(\mathcal{U}, \mathcal{T}) \mid \mathcal{U}, \mathcal{T} \in \mathsf{tors}\,\Lambda, \quad \mathcal{U} \subseteq \mathcal{T}\}$$

"Taking hearts" gives a surj $\mathcal{H}_{(-)} \colon \mathsf{itv}(\mathsf{tors}\,\Lambda) \twoheadrightarrow \mathsf{tors\text{-}heart}\,\Lambda$.

Theorem (E, in preparation)

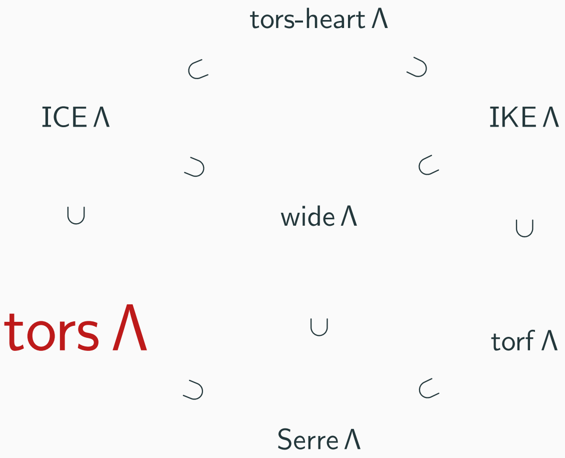
We can define a certain equivalence relation $\sim$, which depends only on the poset structure of $\mathsf{tors}\,\Lambda$, s.t.

$$\frac{\mathsf{itv}(\mathsf{tors}\,\Lambda)}{\sim} \simeq \mathsf{tors\text{-}heart}\,\Lambda.$$

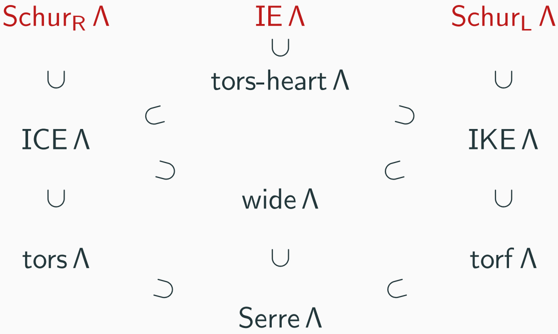
This restricts to bijections between $\{\text{certain intervals}\}/\sim$ and $\mathsf{wide}\,\Lambda$ or $\mathsf{ICE}\,\Lambda$.

The posets $\mathsf{tors\text{-}heart}\,\Lambda$, $\mathsf{ICE}\,\Lambda$, $\mathsf{IKE}\,\Lambda$, and $\mathsf{wide}\,\Lambda$ can be computed from the poset $\mathsf{tors}\,\Lambda$ (using computer)!

Picture Yet Again



Larger picture?



- $\mathrm{Schur}_R \Lambda$: right Schur subcategories [E 2020] (defined using one-sided Schur's lemma)
- $\mathrm{IE}\,\Lambda$: Image-Extension-closed subcategories [E-Sakai, in preparation] (= $\mathcal{T} \cap \mathcal{F}$ for some $\mathcal{T} \in \operatorname{tors} \Lambda$ and $\mathcal{F} \in \operatorname{torf} \Lambda$)