The Jordan-Hölder property, Grothendieck monoids and Bruhat inversions

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The *Jordan-Hölder theorem* for modules says that the ways in which a module can be built up from simple modules are essentially unique. We may say that the category of modules (with finite length) satisfies *the Jordan-Hölder property*, abbreviated by (JHP). The aim of my talk is to investigate (JHP) in the setting of Quillen's *exact categories*.

As in the case of module categories, we can define simple objects, composition series and (JHP) in exact categories. Typical examples are extension-closed subcategories of $\mathsf{mod}\,\Lambda$ for an artin algebra Λ, and in this case, all objects have at least one composition series. However, it turns out that there exists many categories which does not satisfies (JHP), as well as those which does.

It is known that (JHP) implies the free-ness of the Grothendieck group, but the converse does not hold: for "nice" categories such as functorially finite torsion(-free) classes, their Grothendieck groups are free of finite rank, but (JHP) fails in some cases. Thus it is natural to consider a more sophisticated invariant than Grothendieck groups. Then I define *Grothendieck monoids*, which is a commutative monoid subject to the same universal property as the Grothendieck group. Then we have the following result:

Theorem 1. *Let $\mathcal{E}$ be an exact category. Then $\mathcal{E}$ satisfies (JHP) if and only if its Grothendieck monoid $\mathsf{M}(\mathcal{E})$ is a free monoid.*

As an application, we have the following numerical criterion.

Corollary 2. *Let $\mathcal{E}$ be a "nice" exact category. Then $\mathcal{E}$ satisfies (JHP) if and only if the number of indecomposable projective objects is equal to that of simple objects.*

We apply this to the representation theory of type A_n quiver Q by using combinatorics on the symmetric group S_{n+1}. It is known that torion-free classes of $\mathsf{mod}\,kQ$ are in bijection with c-sortable elements w of S_{n+1} ([1, 3]). Let $\mathcal{F}(w)$ be the corresponding torsion-free class. Then we obtain the following purely combinatorial description of simples and criterion for (JHP).

Theorem 3. *Simple objects in $\mathcal{F}(w)$ are in bijection with Bruhat inversions, or Bruhat lower covers, of w. In particular, $\mathcal{F}(w)$ satisfies (JHP) if and only if the number of Bruhat inversions of w is equal to that of supports of w.*

References

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