ICE-closed subcats of module categories

Haruhiso Enomoto
(jt with Arashi Sakai)

§0. Intro.

Setting. k : field

Λ : a f.d k-alg.

$\mathrm{mod}\Lambda$: the cat of f.g. right Λ-modules.

$\boxed{\text{Motivation: Want to study subcat of } \mathrm{mod}\Lambda.}$

More precisely.

Want to classify$^{\text{(ii)}}$

(i) certain subcats of $\mathrm{mod}\Lambda$.

(i) subcats closed under operations

e.g. $\oplus$, submodule, quotient, Kernels, cokernels, ….

(ii) establish a bij

$$\{\text{certain subcats}\} \overset{1-1}{\longleftrightarrow} \{\text{certain good modules}\}$$

Def

- $\mathcal{T} \subseteq \mathrm{mod}\Lambda$ is a torsion class [Dickson 1966]
  $:\Longleftrightarrow$ closed under quotients and extensions.
- $\mathcal{W} \subseteq \mathrm{mod}\Lambda$ is a wide subcat [Hovey 2001]
  $:\Longleftrightarrow$ closed under kernels, cokernels and extensions.

($\mathcal{W}$: ext-closed exact abelian subcat)

Def $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is an ICE-closed subcat (ICE)

$:\Longleftrightarrow$ $\mathcal{C}$ is closed under

(i) Images, (ii) Cokernels, (iii) Extensions.

i.e.,

(i) $\forall c_1 \xrightarrow{f} c_2$ $(c_1, c_2 \in \mathcal{C})$,

$\operatorname{Im} f \in \mathcal{C}$

(ii) $\operatorname{Coker} f \in \mathcal{C}$

(iii) $0 \to C_1 \to E \to C_2 \to 0$.

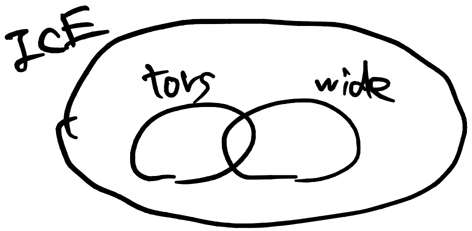
$C_1, C_2 \in \mathcal{C} \Rightarrow E \in \mathcal{C}$. $\lrcorner$

Rem

Every tors and wide is ICE.

$\exists$ many papers

new concept which I introduced



Aim

Classify ICEs!

Strategy

Use a progenerator of ICE

Def $\mathcal{E}$: ext-closed subcat of $\operatorname{mod}\Lambda$.

(1) $P \in \mathcal{E}$ is proj in $\mathcal{E}$

$\Longleftrightarrow$ $\operatorname{Ext}^1_\Lambda(P, \mathcal{E}) = 0$.

(2) $P \in \mathcal{E}$ is a progenerator

if $\circ$ P: proj in $\mathcal{E}$

$\circ$ $\forall X \in \mathcal{E}$.

$0 \to Y \to P^n \to X \to 0$

s.t. $Y \in \mathcal{E}$. $\lrcorner$

Prop If an ICE $\mathcal{C}$ has a progen P,

$$\mathcal{C} = \operatorname{cok} P := \{ X \in \operatorname{mod}\Lambda \mid \exists\ P^m \to P^n \to X \to 0 \text{ exact.} \}$$

(∵) $(\supset)$ clear.

$(\subset)$ Follows from:

$$\begin{tikzcd}
 & & & 0 & & & \\
 & 0 \arrow[r] & \overset{\mathcal{C}\,\ni}{Y} \arrow[ur] \arrow[r] & P^n \arrow[r] & \underset{\in\,\mathcal{C}}{\overset{\forall}{X}} \arrow[r] & 0 \\
 & & P^m \arrow[u] \arrow[ur] & & & \\
0 \arrow[r] & Z \arrow[ur] & & & &
\end{tikzcd}$$

$\square$

Def $M \in \operatorname{mod}\Lambda$ is rigid

$:\Longleftrightarrow \operatorname{Ext}^1_\Lambda(M,M) = 0$

We have maps

$$\{\text{ICEs with progen}\} \underset{\operatorname{cok}}{\overset{P(-)\ \text{(progen)}}{\rightleftarrows}} \{\text{rigid } \Lambda\text{-mods}\}$$

$\operatorname{cok} \circ P = \mathrm{id}.$

§1. Path alg case

Q: Dynkin quiver

e.g. $\underline{1 \leftarrow 2 \rightarrow 3}$: A_3-quiver.

kQ: path alg.

(kQ-module = representation of Q

$M: k \xleftarrow{[1\ -1]} k^2 \xrightarrow{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}} k^3$)

Thm [Gabriel]

$\exists$ bij

$$\{\text{indec } kQ\text{-mod}\} \overset{1\text{-}1}{\longleftrightarrow} \{\text{positive roots of } Q\}$$

$$M \longmapsto \underline{\dim}\, M = \sum_{i \in Q} (\dim_k M_i)\, \underbrace{\alpha_i}_{\text{simple root}}$$

(e.g.

$k \xleftarrow{1} k \to 0 \longleftrightarrow \alpha_1 + \alpha_2$

$k \xleftarrow{1} k \xrightarrow{1} k \longleftrightarrow \alpha_1 + \alpha_2 + \alpha_3$)

Fact

Every ICE of mod kQ has a progen. $\lrcorner$

Thm [Ingalls–Thomas]

$\exists$ bij

$$\operatorname{tors} kQ \underset{\operatorname{cok}(-)}{\overset{P(-)}{\rightleftarrows}} \{\text{support tilting } kQ\text{-mods}\}$$

$\{$torsion classes in mod $kQ\}$ $\lrcorner$

where. M is support tilting

$:\Leftrightarrow$ (i) M is rigid

(ii) $|M| = \#\{\text{comp. factors of } M\}$ $(\leq)$

$|M| = \#\{\text{indec summands of } M\}$

Thm [E]

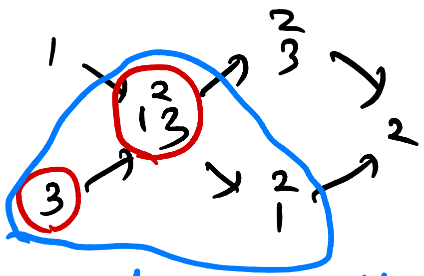
$\exists$ bij.

$$\{\text{ICEs of mod}kQ\} \underset{\operatorname{cok}(-)}{\overset{P(-)}{\rightleftarrows}} \{\text{rigid } kQ\text{-mods}\}$$

$$\begin{matrix} \cup\!\!| & & \cup\!\!| \\ \operatorname{tors} kQ & \underset{[IT]}{\longleftrightarrow} & \{\text{supp. tilt}\} \end{matrix}$$ $\lrcorner$

Ex $Q: 1 \leftarrow 2 \rightarrow 3$

mod kQ :



○ : M, : not supp tilt.

○ : cok M

Rem

$\#$ rigid kQ-mods.

$=$ $\#$ certain faces of cluster cpx of Q

$\rightsquigarrow$ $\exists$ Formula of this number, for each Dynkin e.g. A_n : $\sum_{i=0}^{1} \frac{1}{i+1}\binom{n}{i}\binom{n+i}{i}$

(Schröder number)

- E_6 : 4878

Non-Trivial Part

cok M is ICE for a rigid mod M.

:(- cok M is a torsion class in $\langle M\rangle_{\mathrm{wide}}$ (wide-closure) (by exceptional sequences)

- Tors of wide is ICE. $\square$

§2. ICE via torsion class

Except for path alg, it's difficult to characterize of propers of ICEs.

Instead, we use the poset tors Λ lattice

$(\mathcal{U} \le \mathcal{T} \Leftrightarrow \mathcal{U} \subseteq \mathcal{T})$

Def. For $\mathcal{U} \subseteq \mathcal{T}$ in torsΛ

$\mathcal{H}_{[\mathcal{U},\mathcal{T}]} := \mathcal{T} \cap \mathcal{U}^{\perp} \subseteq \mathrm{mod}\Lambda$

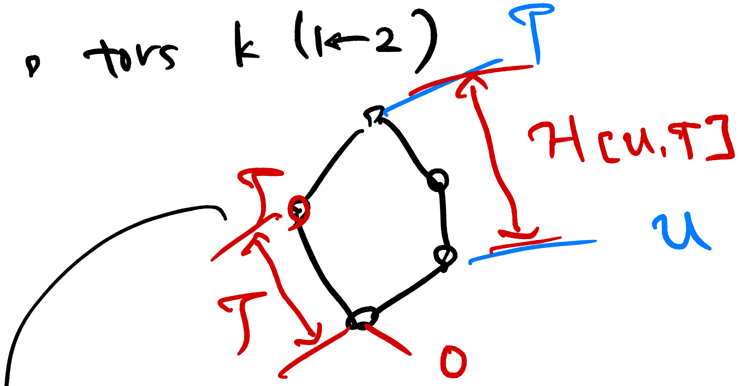
$(\mathcal{U}^{\perp} := \{X \in \mathrm{mod}\Lambda \mid \mathrm{Hom}_\Lambda(\mathcal{U}, X) = 0\}$

the heart of an interval $[\mathcal{U},\mathcal{T}]$

" $\mathcal{T} - \mathcal{U}$ "

e.g. $\mathcal{H}_{[0,\mathcal{T}]} = \mathcal{T}$ $(\mathcal{T} - 0 = \mathcal{T})$

• tors $k(1 \leftarrow 2)$



Key Prop

$\mathcal{C} \subseteq \mathrm{mod}\Lambda$: ICE

$\Longrightarrow \exists\, [\mathcal{U}, \mathcal{T}]$: interval in $\mathrm{tors}\Lambda$

s.t. $\mathcal{C} = \mathcal{H}[\mathcal{U}, \mathcal{T}]$

Def

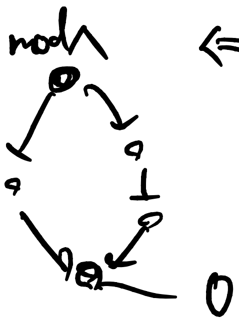
$\vec{\mathcal{H}}(\mathrm{tors}\Lambda)$: Hasse quiver.

• vtx $\longleftrightarrow$ $\mathcal{T} \in \mathrm{tors}\Lambda$

• arrow $\mathcal{T} \to \mathcal{U}$

$\Longleftrightarrow$ $\mathcal{T} \supsetneq \mathcal{U}$, $\nexists\, \mathcal{V}$

s.t. $\mathcal{T} \supsetneq \mathcal{V} \supsetneq \mathcal{U}$.



Thm [E-Sakai] TFAE

for $\mathcal{U} \leq \mathcal{T}$ in $\mathrm{tors}\Lambda$.

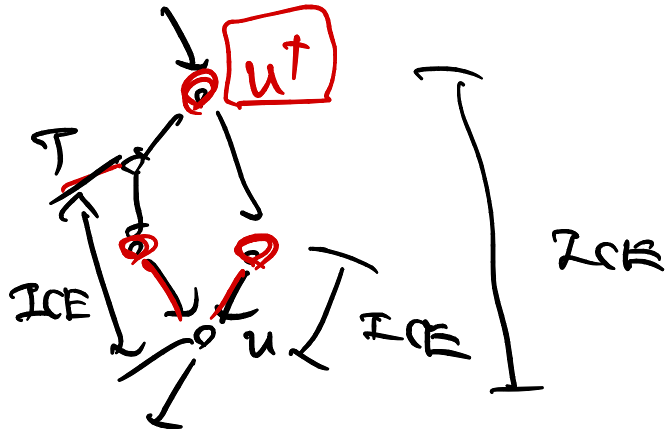
(1) $\mathcal{H}[\mathcal{U}, \mathcal{T}]$ is ICE.

(2) For

$$\mathcal{U}^+ := \bigvee \{\mathcal{U}' \mid \exists\, \mathcal{U}' \to \mathcal{U} \text{ in } \vec{\mathcal{H}}(\mathrm{tors}\Lambda)\},$$

$(\mathcal{U} \leq)\ \mathcal{T} \leq \mathcal{U}^+$

Ex



This gives a combinatorial way to construct all ICEs if $\mathrm{tors}\Lambda$ is given.

Cor For a subcat $\mathcal{C}$ of $\mathrm{mod}\Lambda$.
TFAE.

(1) $\mathcal{C}$: ICE

(2) $\mathcal{C}$ is a tors in some wide subcat of $\mathrm{mod}\Lambda$ ⌟

$$\mathcal{C} \underset{\text{tors}}{\subset} \mathcal{W} \underset{\text{wide}}{\subset} \mathrm{mod}\Lambda.$$

(pf) (2) ⇒ (1) : easy!

(1) ⇒ (2) $\mathcal{C}$: ICE

By Key Prop $\mathcal{C} = \mathcal{H}[\mathcal{U}, \mathcal{T}]$,

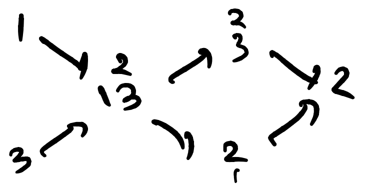
By Thm.
$$\mathcal{U} \subseteq \mathcal{T} \subseteq \mathcal{U}^+$$

$$\begin{array}{ccc}
\mathcal{U}^+ & \xrightarrow{(-)\cap\,\mathcal{U}^{\perp}} & \mathcal{H}[\mathcal{U}, \mathcal{U}^+] \\
\cup\!\!\mid & & \cup\!\!\mid \\
\mathcal{T} & \longrightarrow & \mathcal{H}[\mathcal{U}, \mathcal{T}] = \mathcal{C} \\
\cup\!\!\mid & & \cup\!\!\mid \\
\mathcal{U} & \longrightarrow & 0
\end{array}$$

$\mathrm{tors}\Lambda$

$\mathcal{H}[\mathcal{U}, \mathcal{U}^+]$ is a wide subcat and, $\mathcal{C} \in \mathrm{tors}\, \mathcal{H}[\mathcal{U}, \mathcal{U}^+]$ by Asai–Pfeifer 2019. □

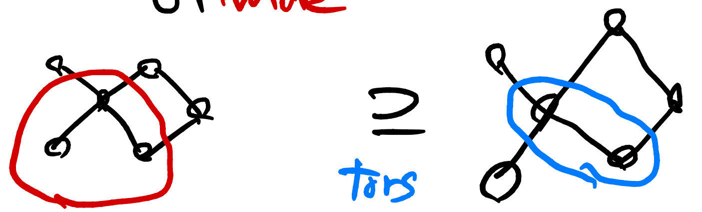
Example $Q: 1 \leftarrow 2 \rightarrow 3$



$\mathrm{mod}\, kQ$

∪ ICE.

∪I wide



What about progen of ICE?

§3. ICE via τ-tilting.

Def $M \in \operatorname{mod}\Lambda$.

(1) M is τ-rigid

$:\Leftrightarrow$ If $\exists\, M^n \twoheadrightarrow N$,

then $\operatorname{Ext}^1_\Lambda(M, N) = 0$.

($\underset{\text{if } kQ}{\overset{\Rightarrow}{\Leftarrow}}$ rigid)

(2) M is support τ-tilting

$:\Leftrightarrow$ ∘ M is τ-rigid.

∘ $|M| = \#\{\text{comp. factor of } M\}$

($\le$ in general)

($\underset{kQ}{\Leftarrow}$ supp tilt)

Thm [Adachi–Iyama–Reiten]

$$\{\text{tors with progen}\} \underset{\operatorname{cok}}{\overset{P(-)}{\rightleftarrows}} \{\text{supp } \tau\text{-tilt } \Lambda\text{-mod}\}$$

Assume.

$\#\operatorname{tors}\Lambda < \infty$ $(*)$

Fact.

∘ Every ICE has a progen.

∘ Every wide subcat $\mathcal{W}$ is equiv to $\operatorname{mod}\Gamma$ for $\exists\, \Gamma$: f.d. k-alg.

Recall

$\mathcal{E}$: ICE

$$\Leftrightarrow \mathcal{E} \overset{\text{tors}}{\subset} \mathcal{W} \overset{\text{wide}}{\subset} \operatorname{mod}\Lambda$$

$\mathcal{W} \simeq \operatorname{mod}\Gamma$

has progen, tors

Def

$M \in \operatorname{mod}\Lambda$ is wide τ-tilt

$\iff \exists\, \mathcal{W}$: wide subcat

s.t. M is supp. $\tau_{\mathcal{W}}$-tilt.

(under equiv $\mathcal{W} \simeq \operatorname{mod}\Gamma$)

Thm. [ES]

Assume $(*)$,

$$\begin{tikzcd}
\{\text{ICEs of } \operatorname{mod}\Lambda\} \arrow[r, shift left, "P(-)"] & \{\text{wide } \tau\text{-tilt}\} \arrow[l, shift left, "\operatorname{cok}"] \\
\operatorname{tors}\Lambda \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, leftrightarrow, "\text{[AIR]}"'] & \{\text{supp. } \tau\text{-tilt}\} \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}$$

Rem For kQ,

wide τ-tilt $=$ rigid kQ-mods.

In general, $\subsetneq$

Ex

$1 \leftarrow 2 \rightarrow 3$