Relative Auslander Correspondence via Exact Categories

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Contents

Abstract

In this thesis, we discuss the representation theory of algebras by using exact categories and give some new results. This thesis is mainly based on the author's papers [En1, En2]. The author prepared this thesis as both *Self Designed Study Report / Research report* and *Small Group class' Report* of his master thesis in Graduate School of Mathematics, Nagoya University.

This thesis consists of three chapters together with some appendices. In Chapter 1, we explain the general idea behind this thesis, including Auslander correspondence and exact categories. Then we raise two motivating questions, Problem A and B. In Chapter 2, we investigate Problem A, that is, a Morita-type characterizations of exact categories. This chapter is based on the author's published paper [En1]. In Chapter 3, we study Problem B, that is, to give an Auslander-type correspondence of exact categories. Using the result in the previous chapter, we obtain a new classification result of CM-finite algebras in terms of exact structures. This chapter is based on the author's paper [En2].

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First and foremost, I would like to thank my advisor Osamu Iyama, for his support and many helpful comments over the past three years. Without his guidance, my work would not be possible. When I was an undergraduate, I preferred viewing mathematics from a highly abstract categorical perspective, and would not be willing to know about down-to-earth objects. Since my undergraduate seminar, he has been teaching me kindly a lot of technique of actual computation, mathematical intuition and examples, along with a good deal of inspiring questions and suggestions. Thanks to his generous supervision, I recognized the importance of relating abstract things to concrete phenomena, which is the guiding principle of this thesis and my future research.

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Notation and conventions

Throughout this thesis, we adopt the following notation and conventions.

(On categories)

- All categories are *skeletally small* unless otherwise stated, that is, the isomorphism classes of objects form a set.
- All subcategories are *full* and *closed under isomorphisms*.
- Let $\mathcal{C}$ and $\mathcal{C}'$ be categories, $\mathcal{D}$ a subcategory of $\mathcal{C}$ and $F : \mathcal{C} \to \mathcal{C}'$ a functor. We denote by $F(\mathcal{D})$ the essential image of $\mathcal{D}$ under F, that is, the subcategory of $\mathcal{C}'$ consisting of all objects isomorphic to the images of objects in $\mathcal{C}$ under F. We simply call $F(\mathcal{D})$ the *image* of $\mathcal{D}$ under F.

(On additive categories)

- All functors between additive categories are additive.
- For a subcategory $\mathcal{D}$ of an additive category $\mathcal{C}$, we write $\mathsf{add}\,\mathcal{D}$ for the subcategory of $\mathcal{C}$ consisting of all objects which are summands of finite direct sums of objects in $\mathcal{D}$. For an object G in $\mathcal{C}$, we simply write $\mathsf{add}\,G$ for $\mathsf{add}\{G\}$.
- We say that an additive category $\mathcal{C}$ is *of finite type* if there exists an object G such that $\mathcal{C} = \mathsf{add}\,G$. In this case, such G is called an *additive generator* of $\mathcal{C}$.
- When we consider modules over rings or categories, we always mean right modules.

(On exact categories) We refer to [Bü, Ke] for the basics on exact categories.

- When we consider an extension-closed subcategory $\mathcal{C}$ of an exact category $\mathcal{E}$, we always regard $\mathcal{C}$ as the exact category with the exact structure inherited from those of $\mathcal{E}$.
- As for exact categories, we use the terminologies *inflations*, *deflations* and *conflations*.
- We often denote by $\rightarrowtail$ an inflation and by $\twoheadrightarrow$ a deflation.
- We say that a functor $F : \mathcal{E} \to \mathcal{E}'$ between exact categories is an *exact equivalence* if F is an equivalence of categories, F is exact and F reflects exactness.
- For an exact category $\mathcal{E}$, we denote by $\mathcal{P}(\mathcal{E})$ (resp. $\mathcal{I}(\mathcal{E})$) the subcategory of $\mathcal{E}$ consisting of all projective (resp. injective) objects. We say that $\mathcal{E}$ *has enough projectives* $\mathcal{P}$ *(resp. enough injectives* $\mathcal{I}$*)* if $\mathcal{E}$ has enough projective objects and $\mathcal{P} = \mathcal{P}(\mathcal{E})$ (resp. enough injective objects and $\mathcal{I} = \mathcal{I}(\mathcal{E})$).

CHAPTER 1

Introduction

Representation theory studies how an algebraic object acts on vector spaces. It dates back to Frobenius, who studied finite groups by using their representations. Since then, finite groups, Lie groups and Lie algebras have played leading roles in the representation theory. It was not until around 1970 that the representation theory of *algebras*, the main topic of this thesis, has attracted an attention, undoubtedly due to the influential work of Maurice Auslander. He developed the method of analyzing modules (= representation) over algebras, and it enabled us to visualize the whole structure in a quite combinatorial and explicit manner. His work consists of the beautiful combination of abstract categorical techniques and concrete description of representations.

In this chapter, we will give a rough explanation of the motivations of representation theory of algebras, with emphasis on the *Auslander correspondence*. Then we will see how the notion of *exact categories* fits in with them when considering *relative representation theory*. The general references of representation theory of algebras are [ARS, ASS]. For basics on exact categories, see [Bü, Ke].

1.1. Auslander correspondence

Let Λ be a ring. Then the purpose of the representation theory of Λ is to understand the structure of the category of Λ-modules. To get interesting results, it is common to impose some restriction on rings and modules we consider, such as noetherian and finitely generated. Then for a noetherian ring Λ, the category $\mathsf{mod}\,\Lambda$ of finitely generated right modules is a target of our investigation, which has a nice property such as being abelian. If in addition Λ is finite-dimensional over a field, then the classical Krull-Schmidt theorem holds, that is, every object in $\mathsf{mod}\,\Lambda$ uniquely decomposes into a finite direct sum of indecomposable Λ-modules. Therefore the investigation of $\mathsf{mod}\,\Lambda$ is completely reduced to that of indecomposable objects in $\mathsf{mod}\,\Lambda$.

Let us consider the simplest case, where there are only finitely many indecomposable objects in $\mathsf{mod}\,\Lambda$ up to isomorphism. Such an algebra is called *representation-finite*, and has been investigated intensively. As for this class, the following result due to Auslander [Au2] is important, which serves as a leading model for this work.

Theorem 1.1.1 (Auslander correspondence). *Let k be a field. Then there exists a bijection between the following classes (up to some equivalence).*

(1) *Representation-finite finite-dimensional k-algebras Λ.*
(2) *Abelian Hom-finite k-categories $\mathcal{E}$ with finitely many indecomposable objects.*
(3) *Finite-dimensional k-algebras Γ satisfying* $\mathrm{gl.dim}\,\Gamma \leq 2 \leq \mathrm{dom.dim}\,\Gamma$.

We omit some unexplained terminologies (see [Au2, ARS]). This relates the study of representation-finite algebras Λ to another class of algebras Γ, and the representation theory of Λ is encoded in the structural theory of Γ. The correspondence between (1) and (3) is usually referred to as an *Auslander correspondence*, however we include (2) here to help understand what is happening under this correspondence in detail. Let us divide this correspondence into two parts, (1)-(2) and (2)-(3), and explain the idea and philosophy behind them.

1.1.1. Bijection between (1) and (2): the Morita theory. The map from (1) to (2) can be easily described: it sends Λ to the category $\mathsf{mod}\,\Lambda$, which is by definition of finite type. This map is an injection since Λ can be recovered from the category $\mathsf{mod}\,\Lambda$ (this point will become more delicate when we will consider the relative version later).

The crucial point here is the surjectivity of the map. This follows from the following well-known variants of the classical *Morita theory*.

PROPOSITION 1.1.2. *The following are equivalent for a k-category $\mathcal{E}$.*

- *$\mathcal{E}$ is a Hom-finite abelian k-category with a projective generator.*
- *There is a finite-dimensional algebra Λ such that $\mathcal{E}$ is equivalent to* $\mathsf{mod}\,\Lambda$.

This gives the categorical characterizations of module categories, which is what the Morita theory is mainly aiming at. For a general account of the Morita theory, we refer the reader to [AF].

1.1.2. Bijection between (2) and (3): Auslander algebras. More mysterious and subtle is the relation between (2) and (3). Actually this part stems from a general consideration about categories with finitely many indecomposables, as we shall explain.

First recall that an additive category $\mathcal{E}$ is *of finite type* if it has an object G satisfying $\mathcal{E} = \mathsf{add}\, G$, where $\mathsf{add}\, G$ is the subcategory of $\mathcal{E}$ consisting of all direct summands of finite direct sums of G. Under mild assumption, it is equivalent to that $\mathcal{E}$ has finitely many indecomposables. For such categories, the following observation is crucial, which was essentially proved by Auslander (e.g. [Au2]).

PROPOSITION 1.1.3. *There exists a bijection between the following classes (up to some equivalence).*

(2)′ *Additive categories $\mathcal{E}$ of finite type.*
(3)′ *Rings Γ.*

We only give the construction of maps here. For an additive category $\mathcal{E}$ of finite type, choose an additive generator G of $\mathcal{E}$ and put $\Gamma := \mathrm{End}_{\mathcal{E}}(G)$. Conversely for a ring Γ, put $\mathcal{E} := \mathsf{proj}\,\Gamma$, the category of finitely generated projective right Γ-modules. This category has an additive generator Γ_Γ, hence is of finite type.

This says that considering categories of finite type is equivalent to considering algebras, and we call an algebra Γ corresponding to the category $\mathcal{E}$ is an *Auslander algebra* of $\mathcal{E}$. This is a quite interesting and suggestive phenomenon, because $\mathcal{E}$ and Γ are of a rather different nature a priori: One is a *categorical*, hence abstract, object, which we usually deal with by using conceptual categorical argument, whereas the other is kind of concrete and down-to-earth object. Here the question naturally arises: What is the relation between *categorical properties* of $\mathcal{E}$ and *homological (= representation theoretic) behavior* of Γ? Indeed the bijection of (2) and (3) claims that the ablianness of $\mathcal{E}$ can be checked by the homological condition on Γ.

Combining this with the previous observation, the general idea behind Auslander correspondence is to seek the relation between the following for a given category $\mathcal{E}$.

(1) How $\mathcal{E}$ is actually realized as a category which naturally appears in the representation theory of algebras.
(2) Categorical properties $\mathcal{E}$ has.
(3) Homological behavior of the Auslander algebra Γ of $\mathcal{E}$.

1.2. Relative representation theory

The main motivation of this project is to explore the appropriate analogue of Auslander's observation in the framework of exact categories, so that we can deal with *relative representation theory.*

1.2.1. What is the relative representation theory? As we explained earlier, the main purpose of the representation theory of an algebra Λ is to investigate the category $\mathsf{mod}\,\Lambda$. However, it is often the case that more explicit and interesting results are available if we restrict the class of modules we consider. Such a strategy is what we call the relative representation theory of algebras.

To demonstrate the validity and the origin of the relative theory, let us consider the case of commutative rings. First consider $\Lambda = \mathbb{Z}$ and the category $\mathsf{mod}\,\mathbb{Z}$ of finitely generated abelian groups. Although the whole structure of $\mathsf{mod}\,\mathbb{Z}$ is controlled by the structure theorem of finitely generated abelian groups, we obtain a simpler result if we focus on torsionfree abelian groups: they are nothing but finite direct sums of $\mathbb{Z}$ itself. More generally, for a Dedekind domain R, the structure of finitely generated torsionfree modules over R are well-understood.

Dedekind domains are one-dimensional regular rings, so let us next move to the higher-dimensional situation. For a noetherian local ring R, an R-module M is called *maximal Cohen-Macaulay* if the depth of M is equal to $\dim R$, and denote by $\mathsf{CM}\, R$ the category of such modules. This category is known to be more tractable than $\mathsf{mod}\, R$ in various aspects, and the investigation of $\mathsf{CM}\, R$ is now one of the main streams of the representation theory of commutative rings, called *Cohen-Macaulay representation theory* (see [LW, Yo]). Here rings R such that $\mathsf{CM}\, R$ is of finite type, called *CM-finite* ring, has played the central role. It is known that this theory has a close connection with algebraic geometry and singularity theory. For example, CM-finite Gorenstein rings nicely correspond to simple singularities of type ADE.

Let us consider the non-commutative analogue of this CM theory. For a nice class of finite-dimensional algebras, called *Iwanaga-Gorenstein*, there exists an appropriate analogue of $\mathsf{CM}\,\Lambda$ available. This class of algebras are ubiquitous in the representation theory, which generalize self-injective algebras and algebras with finite global dimension. Therefore, for such an algebra Λ, we are led to investigate $\mathsf{CM}\,\Lambda$ more deeply instead of $\mathsf{mod}\,\Lambda$. What we should do first is, as in the "absolute" case, the investigation of algebras such that $\mathsf{CM}\,\Lambda$ has only finitely many indecomposables, that is, *CM-finite* ones.

1.2.2. Relative Auslander correspondence fails. Taking the previous argument in account, when considering relative representation theory, it is natural to seek the analogue of the Auslander correspondence for the relative representation-finite algebras such as CM-finite ones.

If we can achieve this, then we will have a bijection between the following:

(1) CM-finite Iwanaga-Gorenstein algebras Λ.
(2) Hom-finite k-categories $\mathcal{E}$ with finitely many indecomposables *which satisfy some categorical properties.*
(3) Finite-dimensional k-algebras *satisfying some homological properties.*

The maps are defined (if possible) as follows: For Λ in (1), we define $\mathcal{E}$ in (2) to be $\mathsf{CM}\,\Lambda$, and the map between (2) and (3) is a restriction of Proposition 1.1.3. Actually this kind of strategy works for some classes of relative theory, e.g. for one-dimensional orders and 1-Iwanaga-Gorenstein algebras, see [ARo] and [Iy3] for the details. Also it is asked in [Che1, Problem D] whether this kind of bijection exists for CM-finite algebras.

However, in general, it often happens that the map from (1) to (2) is not injective. For example, there exist non-Morita-equivalent Iwanaga-Gorenstein algebras Λ and Λ' such that $\mathsf{CM}\,\Lambda$ and $\mathsf{CM}\,\Lambda'$ are equivalent. The problem is that the algebra Λ cannot be recovered from the additive structure of the representation category such as $\mathsf{CM}\,\Lambda$, unlike $\mathsf{mod}\,\Lambda$. Now we are now in position to introduce the main subject of this thesis, *exact categories*, which provide us with enough information on $\mathsf{CM}\,\Lambda$ to reconstruct Λ.

1.3. Exact categories

Quillen introduced the concept of exact categories in [Qu] with an application to higher algebraic K-theory in mind. Since its appearance, this concept has turned out to be quite useful and has been made use of in many branches of mathematics, including representation theory, algebraic topology and functional analysis. To our purpose, exact categories are very essential since they provide an appropriate framework and suitable language for relative representation theory, as we shall see.

An exact category is a generalization of abelian categories. More precisely, it is an additive category *together with* a class of "short exact sequences" in that category, hence it has more information than its additive structure. We can endow any extension-closed subcategories of abelian categories with the natural structure of exact categories. We refer the reader to Definition 3.2.9 and [Bü] for the precise definition. Since most of the relative representation categories like

$\mathsf{CM}\,\Lambda$ are extension-closed in the module category, we can (and should) regard them as exact categories, not just as additive ones.

1.3.1. Motivating problems and main results. Return to our original situation. The problem we have encountered is that Λ cannot be recovered from the additive structure of $\mathsf{CM}\,\Lambda$. As you might guess, Λ can be reconstruct from $\mathsf{CM}\,\Lambda$ together with the exact structure on it (see e.g. Proposition 3.4.2 for the detail)

Let us consider the relative representation-finite algebras in the same way as we did for the classical Auslander correspondence, but in the framework of exact categories. The bijections we would like to establish is of the following form:

(1) CM-finite Iwanaga-Gorenstein algebras Λ.
(2) Hom-finite *exact* k-categories $\mathcal{E}$ with finitely many indecomposables satisfying some categorical properties.
(3) Finite-dimensional k-algebras together with some homological information.

The first thing we have to do is the Morita-type theory concerning the bijection between (1) and (2), which can be summarized as follows:

Problem A. *Characterize when a given exact category $\mathcal{E}$ is exact-equivalent to $\mathsf{CM}\,\Lambda$ for some Iwanaga-Gorenstein algebras. More generally, describe relationship between the categorical properties of an exact category $\mathcal{E}$ and its actual representation-theoretic realization.*

Some scattered results were available on this problem, e.g. [Che2, KIWY, Ka], but no general work was available. In Chapter 2 of this thesis, this problem is tackled with in full generality, where we consider more wider case than CM category of Iwanaga-Gorenstein algebras. For the case of Iwanaga-Gorenstein algebras, we can give a quite simple answer to it as follows:

Theorem (= Corollary 2.4.13). *Let k be a field and $\mathcal{E}$ a Hom-finite exact k-category. Then the following are equivalent.*

- *There exists a finite-dimensional Iwanaga-Gorenstein k-algebra Λ such that $\mathcal{E}$ is exact equivalent to $\mathsf{CM}\,\Lambda$.*
- *$\mathcal{E}$ is idempotent complete Frobenius, and has a projective generator and higher kernels.*

Next let us consider a bijection between (2) and (3). To establish this, it seems to be necessary to give an analogue of Proposition 1.1.3 for exact categories of finite type. Since categories of finite type are the same thing as algebras by Proposition 1.1.3, we should discuss how the structure of exact categories fits in with this Auslander-algebra construction:

Problem B. *Let $\mathcal{E}$ be a category of finite type and Γ its Auslander algebra. How can we classify all the possible exact structures on $\mathcal{E}$ by means of homological properties of Γ?*

This is quite a natural question, but there was no research on this subject. We give an answer to this problem in Chapter 3 as follows.

Theorem (= Theorem 3.3.7). *Let k be a field, $\mathcal{E}$ a Hom-finite idempotent complete k-category and Γ its Auslander algebra. Then there exists a bijection between the following.*

- *Exact structures on $\mathcal{E}$.*
- *Sets of isomorphism classes of simple Γ-modules satisfying the 2-regular condition.*

Finally, by combining Problem A and B, we obtain the somewhat explicit classification of CM-finite Iwanaga-Gorenstein algebras, see Theorem H. This answered an open question raised in [Che1, Problem D] in the case of Iwanaga-Gorenstein algebras.

CHAPTER 2

Morita-type characterizations of exact categories

In this chapter, we mainly discuss about Problem A in the first chapter. Using the Morita-type embedding, we show that any exact category with enough projectives has a realization as a (pre)resolving subcategory of a module category. When the exact category has enough injectives, the image of the embedding can be described in terms of Wakamatsu tilting (=semi-dualizing) subcategories. If moreover the exact category has higher kernels, then its image coincides with the category naturally associated with a cotilting subcategory up to summands. We apply these results to the representation theory of artin algebras. In particular, we show that the ideal quotient of a module category by a functorially finite subcategory closed under submodules is a torsionfree class of some module category.

2.1. Introduction

Since Quillen introduced exact categories in [Qu], many branches of mathematics have made use of this concept. One of the most important classes of exact categories is given by a cotilting module U over a ring as the associated Ext-orthogonal category ${}^{\perp}U$, which forms an exact category with enough projectives and injectives. This class of exact categories is fundamental in the representation theory of algebras, such as the tilting theory in derived categories as well as Cohen-Macaulay representations. In this chapter, we give characterizations of such kinds of exact categories as ${}^{\perp}U$ among all exact categories by investigating the relationship between the following (= the detailed version of Problem A).

(1) Categorical properties of exact categories, e.g. enough projectives, enough injectives, Frobenius, idempotent complete, having (higher) kernels, abelian, $\cdots$.
(2) Representation theoretic realizations of exact categories, e.g. $\mathsf{mod}\,\Lambda$ for a ring Λ, Ext-orthogonal category ${}^{\perp}U$ for a cotilting module U, the exact category X_W associated with a Wakamatsu tilting module W, their resolving subcategories, $\cdots$.

As a consequence, we deduce several known results including [Che2, KIWY, Ka]. Our approach is based on the Morita-type embedding: For a skeletally small exact category $\mathcal{E}$ with enough projectives, the category $\mathcal{P}$ of projective objects in $\mathcal{E}$ gives a fully faithful exact functor (Proposition 2.2.1)

$$\mathbb{P} : \mathcal{E} \to \mathsf{Mod}\,\mathcal{P}, \qquad X \mapsto \mathcal{E}(-, X)|_{\mathcal{P}}.$$

This gives a realization of $\mathcal{E}$ as a subcategory of the module category.

First we apply this functor $\mathbb{P}$ to show the following basic observation, where the category $\mathsf{mod}\,\mathcal{C}$ consists of finitely presented $\mathcal{C}$-modules in a stronger sense, see Definition 2.2.2.

Proposition C. *(=Proposition 2.2.8) Let $\mathcal{E}$ be a skeletally small exact category. The following are equivalent.*

(1) *$\mathcal{E}$ is idempotent complete and has enough projectives.*
(2) *There exists a skeletally small additive category $\mathcal{C}$ such that $\mathcal{E}$ is exact equivalent to some resolving subcategory of $\mathsf{mod}\,\mathcal{C}$.*

When $\mathcal{E}$ has enough projectives and injectives, we show that the image of all injective objects under the above embedding $\mathbb{P}$ forms a special subcategory, which we call a *Wakamatsu tilting subcategory.* This is a categorical analogue of a Wakamatsu tilting module introduced in [Wa] as a common generalization of a tilting module and a cotilting module. It is also known as a *semi-dualizing module* [Chr, ATY], which is a certain analogue of a dualizing complex of Grothendieck

[Ha]. Any Wakamatsu tilting subcategory $\mathcal{W}$ of a module category $\mathsf{mod}\,\mathcal{C}$ gives rise to an exact subcategory $\mathsf{X}_{\mathcal{W}}$ of $\mathsf{mod}\,\mathcal{C}$ which has enough projectives and injectives, see Proposition 2.3.2. The category $\mathsf{X}_{\mathcal{W}}$ for the special case $\mathcal{W} = \mathsf{proj}\,\mathcal{C}$ is nothing but the category $\mathsf{GP}\,\mathcal{C}$ of Gorenstein projective $\mathcal{C}$-modules (see Definition 2.3.7).

Using these concepts, one can embed exact categories with enough projectives and injectives as follows.

THEOREM D. *(=Theorem 2.3.3) Let $\mathcal{E}$ be a skeletally small exact category. The following are equivalent.*

(1) *$\mathcal{E}$ is idempotent complete and has enough projectives and injectives.*
(2) *There exist a skeletally small additive category $\mathcal{C}$ and a Wakamatsu tilting subcategory $\mathcal{W}$ of $\mathsf{mod}\,\mathcal{C}$ such that $\mathcal{E}$ is exact equivalent to some resolving-coresolving subcategory of $\mathsf{X}_{\mathcal{W}}$.*

As an application, we characterize when a given exact category is exact equivalent to one of the three important cases $\mathsf{mod}\,\mathcal{C}$, $\mathsf{X}_{\mathcal{W}}$ or $\mathsf{GP}\,\mathcal{C}$ for some additive category $\mathcal{C}$ and some Wakamatsu tilting subcategory $\mathcal{W}$, see Theorem 2.2.15, 2.3.10 and 2.3.12 respectively.

Next we consider a special case when $\mathcal{W}$ is a cotilting subcategory of $\mathsf{mod}\,\mathcal{C}$. The category $\mathsf{X}_{\mathcal{W}}$ coincides with the Ext-orthogonal subcategory ${}^{\perp}\mathcal{W}$ in this situation, and we give a simple characterization when $\mathcal{E}$ and $\mathsf{X}_{\mathcal{W}}$ are exact equivalent. We extend the notion of *n-kernels* [Ja] to $n \geq -1$ (see Definition 2.4.5), and prove the following main result.

THEOREM E. *(=Theorem 2.4.11) Let $\mathcal{E}$ be a skeletally small exact category and n be an integer $n \geq 0$. The following are equivalent.*

(1) *$\mathcal{E}$ is idempotent complete, has enough projectives and injectives and has $(n-1)$-kernels.*
(2) *There exist a skeletally small additive category $\mathcal{C}$ with weak kernels and an n-cotilting subcategory $\mathcal{W}$ of $\mathsf{mod}\,\mathcal{C}$ such that $\mathcal{E}$ is exact equivalent to $\mathsf{X}_{\mathcal{W}}$.*

This theorem provides us with a concrete method to prove that a given exact category is equivalent to those associated with a cotilting module. Using this, we prove some results on artin algebras over a commutative artinian ring R.

THEOREM F. *(=Theorem 2.5.1) Let Λ be an artin R-algebra and $M \in \mathsf{mod}\,\Lambda$, and put $\mathcal{C} := \mathsf{Sub}\,M$. Consider the quotient category $\mathcal{E} = (\mathsf{mod}\,\Lambda)/[\mathcal{C}]$. Then the following hold.*

(1) *$\mathcal{E}$ admits an exact structure in which $\mathcal{E}$ has enough projectives and injectives and has 0-kernels.*
(2) *Suppose that $\mathsf{ind}(\tau^{-}\mathcal{C}) \setminus \mathsf{ind}\,\mathcal{C}$ is a finite set. Then there exist an artin R-algebra Γ and a 1-cotilting module $U \in \mathsf{mod}\,\Gamma$ such that $\mathcal{E}$ is exact equivalent to a torsionfree class ${}^{\perp}U \subset \mathsf{mod}\,\Gamma$.*

A key ingredient of the proof of Theorem F is the theory of relative homological algebra due to Auslander-Solberg [ASo1, ASo2, ASo3] and its relation to exact categories via [DRSS].

As another application of Theorem E, we deduce the following result by Auslander-Solberg.

THEOREM G. *(=Theorem 2.5.10) Let Λ be an artin R-algebra and $M \in \mathsf{mod}\,\Lambda$. Set $G := \Lambda \oplus M$, $C := D\Lambda \oplus \tau M$, $\Gamma := \mathrm{End}_{\Lambda}(G)$ and $U := \mathrm{Hom}_{\Lambda}(G, C) \in \mathsf{mod}\,\Gamma$. Then the following hold.*

(1) *U is a cotilting Γ-module with $\mathsf{id}\,U = 2$ or 0.*
(2) $\mathrm{Hom}_{\Lambda}(G, -) : \mathsf{mod}\,\Lambda \to \mathsf{mod}\,\Gamma$ *induces an equivalence* $\mathsf{mod}\,\Lambda \simeq {}^{\perp}U$.
(3) $\mathsf{mod}\,\Lambda$ *admits an exact structure such that projective objects are precisely objects in* $\mathsf{add}\,G$ *and the equivalence* $\mathsf{mod}\,\Lambda \simeq {}^{\perp}U$ *is an exact equivalence.*
(4) $\mathrm{End}_{\Lambda}(G)$ *and* $\mathrm{End}_{\Lambda}(C)$ *are derived equivalent.*

This theorem is essentially contained in [ASo2, Proposition 3.26], but we give a simple proof by using the modified exact structure on $\mathsf{mod}\,\Lambda$ and Theorem E.

Finally, let us give a brief description of the individual sections. In Section 2, we study exact categories with enough projectives. We prove Proposition C and give a characterization of exact categories of the form $\mathsf{mod}\,\mathcal{C}$ for some additive category $\mathcal{C}$. In Section 3, we study exact categories with enough projectives and injectives. To this purpose, we introduce Wakamatsu tilting subcategories and prove some properties. Then we prove Theorem D, and give a characterization of exact categories coming from Wakamatsu tilting subcategories. In Section 4, we introduce cotilting subcategories, and study their relationship to higher kernels. In Section 5, we apply these results to the representation theory of artin algebras. In Appendix A.1 in this thesis, we develop the analogue of Auslander-Buchweitz approximation theory in the context of exact categories, which we need in Section 4. In Appendix A.2, we collect some results which enable us to construct new exact structures from a given one, which we use in Section 5.

2.2. Exact categories with enough projectives

In this section, *we always assume that $\mathcal{E}$ is a skeletally small exact category with enough projectives $\mathcal{P} = \mathcal{P}(\mathcal{E})$.* We freely use basic properties of the bifunctor $\operatorname{Ext}^i_{\mathcal{E}}(-,-)$ for $i \geq 0$, which can be defined by using projective resolutions.

2.2.1. Morita-type theorem. We start with recalling the notion of modules over a category. For an additive category $\mathcal{C}$, a *right $\mathcal{C}$-module* X is a contravariant additive functor $X : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}b$ from $\mathcal{C}$ to the category of abelian groups $\mathcal{A}b$. We denote by $\mathsf{Mod}\,\mathcal{C}$ the (not skeletally small) category of right $\mathcal{C}$-modules, and morphisms are natural transformations between them. Then the category $\mathsf{Mod}\,\mathcal{C}$ is an abelian category with enough projectives, and projective objects are precisely direct summands of (possibly infinite) direct sums of representable functors $\mathcal{C}(-, C)$ for $C \in \mathcal{C}$.

For a skeletally small exact category $\mathcal{E}$ with enough projectives $\mathcal{P} := \mathcal{P}(\mathcal{E})$, we define a functor

$$\mathbb{P} : \mathcal{E} \to \mathsf{Mod}\,\mathcal{P}, \qquad X \mapsto \mathcal{E}(-, X)|_{\mathcal{P}}, \tag{2.2.1}$$

which we call the *Morita embedding* because of the following properties.

PROPOSITION 2.2.1. *Let $\mathcal{E}$ be a skeletally small exact category with enough projectives $\mathcal{P}$. Then the Morita embedding functor $\mathbb{P} : \mathcal{E} \to \mathsf{Mod}\,\mathcal{P}$ is fully faithful and exact. Moreover $\mathbb{P}$ preserves projectivity and all extension groups.*

PROOF. This is well-known and standard. For the convenience of the reader, we shall give a proof.

We first observe that $\mathbb{P}$ is an exact functor. In fact, $\mathbb{P}$ is obviously left exact, and for each projective object P and each deflation $f : Y \twoheadrightarrow Z$, the induced map $\mathcal{E}(P, Y) \to \mathcal{E}(P, Z)$ is surjective by the definition of projectivity. In addition, if $P \in \mathcal{E}$ is projective, then $\mathbb{P}P = \mathcal{P}(-, P)$ is a projective $\mathcal{P}$-module. Therefore $\mathbb{P}$ sends projective objects to projective modules.

Next we will see that $\mathbb{P}$ is fully faithful. Since $\mathcal{E}$ has enough projectives, for any object $X \in \mathcal{E}$, there exist conflations $X_2 \rightarrowtail P_1 \twoheadrightarrow X_1$ and $X_1 \rightarrowtail P_0 \twoheadrightarrow X$. By the exactness of $\mathbb{P}$, this gives an exact sequence $\mathbb{P}P_1 \to \mathbb{P}P_0 \to \mathbb{P}X \to 0$ in $\mathsf{Mod}\,\mathcal{P}$, and moreover we can check that $P_1 \to P_0 \to X$ is also a cokernel diagram in $\mathcal{E}$. Hence we have the following diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{E}(X, Y) & \longrightarrow & \mathcal{E}(P_0, Y) & \longrightarrow & \mathcal{E}(P_1, Y) \\
& \downarrow{\scriptstyle \mathbb{P}} & & \wr\downarrow{\scriptstyle \mathbb{P}} & & \wr\downarrow{\scriptstyle \mathbb{P}} \\
0 \longrightarrow & (\mathsf{Mod}\,\mathcal{P})(\mathbb{P}X, \mathbb{P}Y) & \longrightarrow & (\mathsf{Mod}\,\mathcal{P})(\mathbb{P}P_0, \mathbb{P}Y) & \longrightarrow & (\mathsf{Mod}\,\mathcal{P})(\mathbb{P}P_1, \mathbb{P}Y)
\end{array}$$

whose rows are exact and the second and third vertical morphisms are isomorphisms by the Yoneda lemma. It follows that $\mathcal{E}(X, Y) \to (\mathsf{Mod}\,\mathcal{P})(\mathbb{P}X, \mathbb{P}Y)$ is also an isomorphism, thus $\mathbb{P}$ is fully faithful.

Note that $\mathbb{P}$ sends a projective resolution of X to a projective resolution of $\mathbb{P}X$, since it preserves projectivity and exactness. Therefore $\mathbb{P}$ induces an isomorphism $\operatorname{Ext}^i_{\mathcal{E}}(X, Y) \cong \operatorname{Ext}^i_{\mathsf{Mod}\,\mathcal{P}}(X, Y)$ for all $i \geq 0$ because $\mathbb{P}$ is fully faithful. $\square$

However, $\mathbb{P}$ is far from essentially surjective because $\mathsf{Mod}\,\mathcal{P}$ is too large. This leads us to the following definition of a subcategory of $\mathsf{Mod}\,\mathcal{P}$. Suppose that $\mathcal{C}$ is an additive category. Recall that a right $\mathcal{C}$-module M is called *finitely generated* if there exists an epimorphism $\mathcal{C}(-, C) \twoheadrightarrow M$ for some $C \in \mathcal{C}$. By the Yoneda lemma, finitely generated projective $\mathcal{C}$-modules are precisely direct summands of representable functors.

Definition 2.2.2. Let $\mathcal{C}$ be a skeletally small additive category.

(1) We denote by $\mathsf{proj}\,\mathcal{C}$ the full subcategory of $\mathsf{Mod}\,\mathcal{C}$ consisting of all finitely generated projective $\mathcal{C}$-modules.

(2) We denote by $\mathsf{mod}\,\mathcal{C}$ the full subcategory of $\mathcal{C}$-modules $X \in \mathsf{Mod}\,\mathcal{C}$ such that there exists an exact sequence in $\mathsf{Mod}\,\mathcal{C}$ of the form

$$\cdots \to P_2 \to P_1 \to P_0 \to X \to 0 \tag{2.2.2}$$

where P_i is in $\mathsf{proj}\,\mathcal{C}$ for each $i \geq 0$.

In the same way, for a ring Λ we define the subcategories $\mathsf{proj}\,\Lambda$ and $\mathsf{mod}\,\Lambda$ of $\mathsf{Mod}\,\Lambda$.

Note that the notation $\mathsf{mod}\,\mathcal{C}$ often stands for weaker notions, e.g. the category of finitely presented $\mathcal{C}$-modules. In Proposition 2.2.7 below, we characterize when these notions coincide.

Let us recall the following classes of subcategories of exact categories.

Definition 2.2.3. Let $\mathcal{E}$ be an exact category with enough projectives and $\mathcal{C}$ a full subcategory of $\mathcal{E}$ which is closed under isomorphisms.

(1) We say that $\mathcal{C}$ is *resolving* if $\mathcal{C}$ satisfies the following conditions.
   (a) $\mathcal{C}$ contains all projective objects in $\mathcal{E}$.
   (b) $\mathcal{C}$ is closed under extensions, that is, for each conflation $L \rightarrowtail M \twoheadrightarrow N$, if both L and N are in $\mathcal{C}$, then so is M.
   (c) $\mathcal{C}$ is closed under kernels of deflations, that is, for each conflation $L \rightarrowtail M \twoheadrightarrow N$, if both M and N are in $\mathcal{C}$, then so is L.
   (d) $\mathcal{C}$ is closed under summands.

   Dually we define *coresolving* subcategories for an exact category with enough injectives.

(2) $\mathcal{C}$ is called *thick* if it is closed under extensions, kernels of deflations, cokernels of inflations and summands.

In addition to these classical concepts, we need the following weaker notion when we later investigate exact categories which are not idempotent complete.

Definition 2.2.4. Let $\mathcal{E}$ be an exact category with enough projectives and $\mathcal{C}$ a full subcategory of $\mathcal{E}$ closed under isomorphisms. We call $\mathcal{C}$ *preresolving* if $\mathcal{C}$ satisfies the following conditions.

(a) $\mathsf{add}\,\mathcal{C}$ contains all projective objects in $\mathcal{E}$.
(b) $\mathcal{C}$ is closed under extensions.
(c) For each X in $\mathcal{C}$, there exists a conflation $\Omega X \rightarrowtail P \twoheadrightarrow X$ in $\mathcal{E}$ such that all terms are in $\mathcal{C}$ and P is projective in $\mathcal{E}$.

Dually we define *precoresolving* subcategories for an exact category with enough injectives.

We point out that these subcategories are actually *exact* subcategories of $\mathcal{E}$, because they are closed under extensions.

The following lemma gives the relationship between resolving and preresolving subcategories.

Lemma 2.2.5. *Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$. Then a subcategory $\mathcal{C}$ of $\mathcal{E}$ is resolving if and only if it is preresolving and closed under summands.*

Proof. The "only if" part is clear. Now we suppose that $\mathcal{C}$ is preresolving and closed under summands. Since $\mathsf{add}\,\mathcal{C} = \mathcal{C}$ in this case, it suffices to show that $\mathcal{C}$ is closed under kernels of deflations. Let $X \rightarrowtail Y \twoheadrightarrow Z$ be a conflation in $\mathcal{E}$ in which Y and Z are in $\mathcal{C}$. By our assumption for $\mathcal{C}$, there exists a conflation $\Omega Z \rightarrowtail P \twoheadrightarrow Z$ with ΩZ in $\mathcal{C}$ and P in $\mathcal{C} \cap \mathcal{P}$. We then have the following pullback diagram.

$$\begin{array}{ccccc}
 & & \Omega Z & = & \Omega Z \\
 & & \downarrow & & \downarrow \\
X & \rightarrowtail & E & \twoheadrightarrow & P \\
\| & & \downarrow & & \downarrow \\
X & \rightarrowtail & Y & \twoheadrightarrow & Z
\end{array}$$

Because $\mathcal{C}$ is closed under extensions, the middle column shows that E is in $\mathcal{C}$. Furthermore the middle row splits because P is projective in $\mathcal{E}$. From this X is a summand of $E \in \mathcal{C}$, which shows that X is actually in $\mathcal{C}$ since $\mathcal{C}$ is closed under summands. $\square$

We give basic properties of $\mathsf{mod}\,\mathcal{C}$.

Proposition 2.2.6. *Let $\mathcal{C}$ be a skeletally small additive category. Then $\mathsf{mod}\,\mathcal{C}$ is a thick subcategory of $\mathsf{Mod}\,\mathcal{C}$. In addition, $\mathsf{mod}\,\mathcal{C}$ has enough projectives, and its projective objects are precisely objects in $\mathsf{proj}\,\mathcal{C}$.*

Proof. It follows from the horseshoe lemma that $\mathsf{mod}\,\mathcal{C}$ is closed under extensions. First we will show that $\mathsf{mod}\,\mathcal{C}$ is closed under cokernels of monomorphisms. Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\mathsf{Mod}\,\mathcal{C}$ and suppose that Y is in $\mathsf{mod}\,\mathcal{C}$. By the definition of $\mathsf{mod}\,\mathcal{C}$, there is an exact sequence $0 \to \Omega Y \to P \to Y \to 0$ such that P is in $\mathsf{proj}\,\mathcal{C}$ and ΩY is in $\mathsf{mod}\,\mathcal{C}$. We then have the following commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega Y & = & \Omega Y & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \Omega Z & \to & P & \to & Z & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & X & \to & Y & \to & Z & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{2.2.3}$$

whose rows and columns are exact. If X is also in $\mathsf{mod}\,\mathcal{C}$, we have that ΩZ is also in $\mathsf{mod}\,\mathcal{C}$ since $\mathsf{mod}\,\mathcal{C}$ is closed under extensions. Then it follows that Z is in $\mathsf{mod}\,\mathcal{C}$ by the middle row. Hence $\mathsf{mod}\,\mathcal{C}$ is closed under cokernels of monomorphisms.

Next we show that $\mathsf{mod}\,\mathcal{C}$ is closed under summands. Let $Z \in \mathsf{Mod}\,\mathcal{C}$ be a summand of $Y \in \mathsf{mod}\,\mathcal{C}$. We claim that there exists an exact sequence $0 \to \Omega Z \to P \to Z \to 0$ such that P is in $\mathsf{proj}\,\mathcal{C}$ and ΩZ is a summand of some object in $\mathsf{mod}\,\mathcal{C}$, which inductively implies that $\mathsf{mod}\,\mathcal{C}$ is closed under summands. We have a split exact sequence $0 \to X \to Y \to Z \to 0$ with Y in $\mathsf{mod}\,\mathcal{C}$, so we can use the diagram (2.2.3) again and obtain the exact sequence $0 \to \Omega Z \to P \to Z \to 0$. By taking the direct sum of $0 \to Z = Z$ and the first column of (2.2.3), we obtain an exact sequence $0 \to \Omega Y \to \Omega Z \oplus Z \to X \oplus Z \to 0$. Since $Y \cong X \oplus Z$ and ΩY are in $\mathsf{mod}\,\mathcal{C}$ and $\mathsf{mod}\,\mathcal{C}$ is closed under extensions, it follows that $\Omega Z \oplus Z$ is in $\mathsf{mod}\,\mathcal{C}$. Thus ΩZ is a summand of some object in $\mathsf{mod}\,\mathcal{C}$.

By the same argument as in the proof of Lemma 2.2.5, one can show that $\mathsf{mod}\,\mathcal{C}$ is closed under kernels of epimorphisms, which completes the proof that $\mathsf{mod}\,\mathcal{C}$ is thick in $\mathsf{Mod}\,\mathcal{C}$. The last statement of the proposition is obvious by the construction of $\mathsf{mod}\,\mathcal{C}$. $\square$

As a corollary, we obtain the following relation between $\mathsf{mod}\,\mathcal{C}$ and the category of finitely presented $\mathcal{C}$-modules (cf. [Au1, Proposition 2.1]). Recall that $\mathcal{C}$ is said to have *weak kernels* if for any morphism $f : C_1 \to C_0$ in $\mathcal{C}$, there exists a morphism $g : C_2 \to C_1$ such that $\mathcal{C}(-, C_2) \xrightarrow{\mathcal{C}(-,g)} \mathcal{C}(-, C_1) \xrightarrow{\mathcal{C}(-,f)} \mathcal{C}(-, C_0)$ is exact in $\mathsf{Mod}\,\mathcal{C}$.

Proposition 2.2.7. *Let $\mathcal{C}$ be a skeletally small additive category. Then the category $\mathsf{mod}\,\mathcal{C}$ coincides with the category of finitely presented $\mathcal{C}$-modules if and only if $\mathcal{C}$ has weak kernels. In this case $\mathsf{mod}\,\mathcal{C}$ is an abelian category.*

Proof. Suppose that $\mathcal{C}$ has weak kernels and X is a finitely presented $\mathcal{C}$-module. Then we have an exact sequence $\mathcal{C}(-, C_1) \to \mathcal{C}(-, C_0) \to X \to 0$ for some objects C_0 and C_1 in $\mathcal{C}$. By the Yoneda lemma, the morphism $\mathcal{C}(-, C_1) \to \mathcal{C}(-, C_0)$ is induced from a morphism $C_1 \to C_0$. Since $\mathcal{C}$ has weak kernels, it follows that this morphism has a series of weak kernels $\cdots \to C_2 \to C_1 \to C_0$, which yields an exact sequence $\cdots \to \mathcal{C}(-, C_2) \to \mathcal{C}(-, C_1) \to \mathcal{C}(-, C_0) \to X \to 0$. Therefore X is in $\mathsf{mod}\,\mathcal{C}$.

Conversely, suppose that all finitely presented $\mathcal{C}$-modules are in $\mathsf{mod}\,\mathcal{C}$. Let $C_1 \to C_0$ be a morphism in $\mathcal{C}$. Consider the exact sequence $0 \to X \to \mathcal{C}(-, C_1) \to \mathcal{C}(-, C_0) \to Z \to 0$ in $\mathsf{Mod}\,\mathcal{C}$. Then Z is in $\mathsf{mod}\,\mathcal{C}$ by our assumption. Since $\mathsf{mod}\,\mathcal{C}$ is thick in $\mathsf{Mod}\,\mathcal{C}$ by Proposition 2.2.6, we have $X \in \mathsf{mod}\,\mathcal{C}$. In particular we have an exact sequence $\mathcal{C}(-, C_2) \to \mathcal{C}(-, C_1) \to \mathcal{C}(-, C_0)$ since X is finitely generated, which gives a weak kernel of $C_1 \to C_0$.

Finally we show that $\mathsf{mod}\,\mathcal{C}$ is abelian if these conditions are satisfied. It is easy to see that the category of finitely presented $\mathcal{C}$-modules is closed under cokernels in $\mathsf{Mod}\,\mathcal{C}$, thus we only have to show that $\mathsf{mod}\,\mathcal{C}$ is closed under kernels. However this is clear since $\mathsf{mod}\,\mathcal{C}$ is thick in $\mathsf{Mod}\,\mathcal{C}$ by Proposition 2.2.6. $\square$

Now we can prove that $\mathcal{E}$ can be embedded into $\mathsf{mod}\,\mathcal{P}$ as a (pre)resolving subcategory.

PROPOSITION 2.2.8. *Let $\mathcal{E}$ be a skeletally small exact category with enough projectives $\mathcal{P}$ and $\mathbb{P} : \mathcal{E} \to \mathsf{mod}\,\mathcal{P}$ the Morita embedding* (2.2.1). *Then $\mathbb{P}(\mathcal{E})$ is a preresolving subcategory of $\mathsf{mod}\,\mathcal{P}$ and $\mathcal{E}$ is exact equivalent to $\mathbb{P}(\mathcal{E})$. It is resolving in $\mathsf{mod}\,\mathcal{P}$ if and only if $\mathcal{E}$ is idempotent complete.*

PROOF. Since $\mathcal{E}$ has enough projectives and $\mathbb{P}$ is an exact functor by Proposition 2.2.1, one can easily check that $\mathbb{P}(\mathcal{E})$ is contained in $\mathsf{mod}\,\mathcal{P}$. To show that $\mathbb{P}(\mathcal{E})$ is preresolving in $\mathsf{mod}\,\mathcal{P}$, we check the conditions of Definition 2.2.4.

(a): Clearly $\mathsf{add}\,\mathbb{P}(\mathcal{E})$ contains all projective objects in $\mathsf{mod}\,\mathcal{P}$, since projective objects in $\mathsf{mod}\,\mathcal{P}$ are direct summands of representable functors of the form $\mathcal{P}(-, P) = \mathbb{P}P$ for $P \in \mathcal{P}$.

(b): We show that $\mathbb{P}(\mathcal{E})$ is closed under extensions. Suppose that $0 \to \mathbb{P}L \to X \to \mathbb{P}M \to 0$ is an exact sequence in $\mathsf{mod}\,\mathcal{P}$. Then there exists a conflation $\Omega M \rightarrowtail P \twoheadrightarrow M$ in $\mathcal{E}$, and sending this by $\mathbb{P}$ gives us the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{P}\Omega M & \longrightarrow & \mathbb{P}P & \longrightarrow & \mathbb{P}M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathbb{P}L & \longrightarrow & X & \longrightarrow & \mathbb{P}M & \longrightarrow & 0 \end{array} \tag{2.2.4}$$

with exact rows. The vertical morphisms exist by the projectivity of $\mathbb{P}P$. Then we obtain the corresponding morphism $\Omega M \to L$ in $\mathcal{E}$ because $\mathbb{P}$ is fully faithful. By taking pushout, we get the following commutative diagram

$$\begin{array}{ccccc} \Omega M & \rightarrowtail & P & \twoheadrightarrow & M \\ \downarrow & & \downarrow & & \| \\ L & \rightarrowtail & N & \twoheadrightarrow & M \end{array}$$

in $\mathcal{E}$. Because $\mathbb{P}$ is exact, $\mathbb{P}$ sends this diagrams to the diagram isomorphic to (2.2.4). This proves that $X \cong \mathbb{P}N$.

(c): Let M be any object in $\mathcal{E}$. Since there exists a conflation $\Omega M \rightarrowtail P \twoheadrightarrow M$ with P in $\mathcal{P}$, we have an exact sequence $0 \to \mathbb{P}\Omega M \to \mathbb{P}P \to \mathbb{P}M \to 0$ in $\mathsf{mod}\,\mathcal{P}$. Since $\mathbb{P}P$ is projective, the condition (c) holds.

Next we show that $\mathcal{E}$ is exact equivalent to $\mathbb{P}(\mathcal{E})$. We have to confirm that for morphisms $L \to M \to N$ in $\mathcal{E}$, if $0 \to \mathbb{P}L \to \mathbb{P}M \to \mathbb{P}N \to 0$ is exact, then $L \to M \to N$ is a conflation. First note that $L \to M \to N$ is a kernel and cokernel pair because $\mathbb{P}$ is fully faithful and $0 \to \mathbb{P}L \to \mathbb{P}M \to \mathbb{P}N \to 0$ is exact. Take a deflation $P \twoheadrightarrow N$ in $\mathcal{E}$ with P projective. Since $\mathbb{P}P$ is projective, $\mathbb{P}P \to \mathbb{P}N$ factors through $\mathbb{P}M \twoheadrightarrow \mathbb{P}N$. Because $\mathbb{P}$ is fully faithful, this implies that $P \twoheadrightarrow N$ also factors through $M \to N$. This shows that the composition $P \to M \to N$ is a deflation in $\mathcal{E}$, and $M \to N$ has a kernel $L \to M$. By [Ke, Proposition A.1.(c)], we have that $M \to N$ is a deflation, which shows that $L \to M \to N$ is a conflation in $\mathcal{E}$.

Finally we check the last statement. If $\mathcal{E}$ is idempotent complete, then clearly so is $\mathbb{P}(\mathcal{E})$. This shows that $\mathbb{P}(\mathcal{E})$ is closed under summands in $\mathsf{mod}\,\mathcal{P}$. Thus $\mathcal{E}$ is resolving by Lemma 2.2.5. Conversely, suppose that $\mathbb{P}(\mathcal{E})$ is resolving. Since $\mathsf{mod}\,\mathcal{P}$ is idempotent complete by Proposition 2.2.6, the assumption that $\mathbb{P}(\mathcal{E}) \subset \mathsf{mod}\,\mathcal{P}$ is closed under summands clearly implies that $\mathbb{P}(\mathcal{E})$ is idempotent complete, so is $\mathcal{E}$. $\square$

This proposition gives a simple criterion for a morphism in an exact category with enough projectives to be a deflation.

Corollary 2.2.9. *Let $\mathcal{E}$ be a skeletally small idempotent complete exact category with enough projectives, and let $f : X \to Y$ be a morphism in $\mathcal{E}$. Then f is a deflation in $\mathcal{E}$ if and only if for any projective object $P \in \mathcal{E}$, the induced map $\mathcal{E}(P, f) : \mathcal{E}(P, X) \to \mathcal{E}(P, Y)$ is surjective.*

Proof. The "only if" part is clear. Suppose that $\mathcal{E}(P, X) \to \mathcal{E}(P, Y)$ is surjective for any projective object P. This is clearly equivalent to that $\mathbb{P}X \to \mathbb{P}Y$ is surjective. Since both $\mathbb{P}(\mathcal{E}) \subset \mathsf{mod}\,\mathcal{P}$ and $\mathsf{mod}\,\mathcal{P} \subset \mathsf{Mod}\,\mathcal{P}$ are subcategories closed under kernels of epimorphisms, so is $\mathbb{P}(\mathcal{E}) \subset \mathsf{Mod}\,\mathcal{P}$. Thus we obtain an exact sequence $0 \to \mathbb{P}Z \to \mathbb{P}X \to \mathbb{P}Y \to 0$ in $\mathbb{P}(\mathcal{E})$. Since $\mathbb{P}$ reflects exactness and is fully faithful by Proposition 2.2.8, we have a conflation $Z \rightarrowtail X \twoheadrightarrow Y$, which shows that f is a deflation. □

As a consequence, we give a correspondence between (pre)resolving subcategories of module categories and exact categories with enough projectives. To state it accurately, we need some preparation.

Let us recall the following classical Morita theorem.

Definition-Proposition 2.2.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be skeletally small additive categories. The following are equivalent.*

(1) *There exists an equivalence* $\mathsf{Mod}\,\mathcal{C} \simeq \mathsf{Mod}\,\mathcal{D}$.
(2) *There exists an exact equivalence* $\mathsf{mod}\,\mathcal{C} \simeq \mathsf{mod}\,\mathcal{D}$.
(3) *There exists an equivalence* $\mathsf{proj}\,\mathcal{C} \simeq \mathsf{proj}\,\mathcal{D}$.

In this case, we say that $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent*. For example, $\mathcal{C}$ and* $\mathsf{proj}\,\mathcal{C}$ *are always Morita equivalent. Moreover, if $\mathcal{C}$ and $\mathcal{D}$ are both idempotent complete, then these are Morita equivalent if and only if $\mathcal{C} \simeq \mathcal{D}$.*

Proof. The last assertion and the equivalence of (1) and (3) are classical, and we refer the reader to [Au3, Proposition 2.6] for the proof.

(1) ⇒ (2): The equivalence $\mathsf{Mod}\,\mathcal{C} \simeq \mathsf{Mod}\,\mathcal{D}$ induces an equivalence $\mathsf{proj}\,\mathcal{C} \simeq \mathsf{proj}\,\mathcal{D}$ since *finitely generated projective* is a categorical notion. It follows from the definition of $\mathsf{mod}\,\mathcal{C}$ that it also induces an exact equivalence $\mathsf{mod}\,\mathcal{C} \simeq \mathsf{mod}\,\mathcal{D}$.

(2) ⇒ (3): Projective objects in the exact category $\mathsf{mod}\,\mathcal{C}$ are precisely objects in $\mathsf{proj}\,\mathcal{C}$ by Proposition 2.2.6. Then the assertion follows immediately. □

Note that $\mathcal{C}$ is idempotent complete if and only if $\mathsf{proj}\,\mathcal{C} \simeq \mathcal{C}$. Also observe that any additive functor $F : \mathcal{C} \to \mathcal{D}$ induces an additive functor $\mathsf{proj}\,F : \mathsf{proj}\,\mathcal{C} \to \mathsf{proj}\,\mathcal{D}$.

Our aim is to classify exact categories with enough projectives such that the subcategories of projective objects are (Morita) equivalent to a fixed category $\mathcal{C}$. To this purpose, we have to introduce the appropriate notion of equivalence between these categories.

Definition 2.2.11. Suppose that $\mathcal{E}$ and $\mathcal{E}'$ are skeletally small exact categories with enough projectives. Assume that $\mathcal{P}(\mathcal{E})$ and $\mathcal{P}(\mathcal{E}')$ are Morita equivalent to $\mathcal{C}$ via equivalences $F : \mathsf{proj}\,\mathcal{C} \simeq \mathsf{proj}\,\mathcal{P}(\mathcal{E})$ and $F' : \mathsf{proj}\,\mathcal{C} \simeq \mathsf{proj}\,\mathcal{P}(\mathcal{E}')$. We say that pairs $(\mathcal{E}, F)$ and $(\mathcal{E}', F')$ are *$\mathcal{C}$-equivalent* if there exists an exact equivalence $G : \mathcal{E} \simeq \mathcal{E}'$ such that the following diagram commutes up to natural isomorphism

$$\begin{array}{ccccc}
\mathsf{proj}\,\mathcal{P}(\mathcal{E}) & \xleftarrow[\simeq]{F} & \mathsf{proj}\,\mathcal{C} & \xrightarrow[\simeq]{F'} & \mathsf{proj}\,\mathcal{P}(\mathcal{E}') \\
{\scriptstyle \mathsf{proj}\,\iota}\downarrow & & & & \downarrow{\scriptstyle \mathsf{proj}\,\iota'} \\
\mathsf{proj}\,\mathcal{E} & & \xrightarrow[\mathsf{proj}\,G]{\simeq} & & \mathsf{proj}\,\mathcal{E}'
\end{array}$$

where ι and ι' are the inclusions.

Note that if $\mathcal{C}$, $\mathcal{E}$ and $\mathcal{E}'$ are idempotent complete, then the above diagram can be replaced by the following one.

$$\begin{array}{ccccc} \mathcal{P}(\mathcal{E}) & \xleftarrow[\simeq]{F} & \mathcal{C} & \xrightarrow[\simeq]{F'} & \mathcal{P}(\mathcal{E}') \\ \iota \downarrow & & & & \downarrow \iota' \\ \mathcal{E} & & \xrightarrow[G]{\simeq} & & \mathcal{E}' \end{array}$$

Now we are in position to state the following Morita-type theorem. This theorem says that exact categories with enough projectives with a fixed category $\mathcal{C}$ of projective objects are completely classified by resolving subcategories of $\mathsf{mod}\,\mathcal{C}$.

THEOREM 2.2.12. *Let $\mathcal{C}$ be a skeletally small additive category.*

(1) *There exists a bijection between the following two classes.*
   (a) *$\mathcal{C}$-equivalence classes of pairs $(\mathcal{E}, F)$ where $\mathcal{E}$ is a skeletally small exact category with enough projectives $\mathcal{P}$ such that $\mathcal{P}$ is Morita equivalent to $\mathcal{C}$.*
   (b) *Preresolving subcategories of $\mathsf{mod}\,\mathcal{C}$.*

   It sends $(\mathcal{E}, F)$ in (a) *to the image of $\mathcal{E} \to \mathsf{mod}\,\mathcal{P} \simeq \mathsf{mod}\,\mathcal{C}$, and the inverse map sends $\mathcal{E}$ in* (b) *to $(\mathcal{E}, \mathrm{id})$ for the identity functor $\mathrm{id} : \mathsf{proj}\,\mathcal{C} = \mathsf{proj}\,\mathcal{C}$.*

(2) *If $\mathcal{C}$ is idempotent complete, the bijection of* (1) *restricts to a bijection between the following.*
   (a) *$\mathcal{C}$-equivalence classes of pairs $(\mathcal{E}, F)$ where $\mathcal{E}$ is a skeletally small idempotent complete exact categories with enough projectives $\mathcal{P}$ such that $\mathcal{P}$ is equivalent to $\mathcal{C}$.*
   (b) *Resolving subcategories of $\mathsf{mod}\,\mathcal{C}$.*

To prove this, we need the following preparation.

LEMMA 2.2.13. *Pairs $(\mathcal{E}, F)$ and $(\mathcal{E}', F')$ are $\mathcal{C}$-equivalent if and only if the following diagram commutes up to natural isomorphism*

$$\begin{array}{ccccc} \mathsf{mod}\,\mathcal{P}(\mathcal{E}) & \xrightarrow[\simeq]{(-)\circ F} & \mathsf{mod}\,\mathcal{C} & \xleftarrow[\simeq]{(-)\circ F'} & \mathsf{mod}\,\mathcal{P}(\mathcal{E}') \\ \mathbb{P} \uparrow & & & & \uparrow \mathbb{P} \\ \mathcal{E} & & \xrightarrow[G]{\simeq} & & \mathcal{E}' \end{array} \tag{2.2.5}$$

where we identify $\mathsf{mod}\,\mathcal{P}$ (resp. $\mathsf{mod}\,\mathcal{P}'$) with $\mathsf{mod}(\mathsf{proj}\,\mathcal{P})$ (resp. $\mathsf{mod}(\mathsf{proj}\,\mathcal{P}')$).

PROOF. The assertion follows immediately from the following general fact. Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories and $K, L : \mathcal{A} \rightrightarrows \mathcal{B}$ fully faithful functors. Consider the composition $K', L' : \mathcal{B} \to \mathsf{Mod}\,\mathcal{B} \rightrightarrows \mathsf{Mod}\,\mathcal{A}$, where $\mathcal{B} \to \mathsf{Mod}\,\mathcal{B}$ is the Yoneda embedding and $\mathsf{Mod}\,\mathcal{B} \rightrightarrows \mathsf{Mod}\,\mathcal{A}$ are $(-) \circ K$ and $(-) \circ L$. Then K and L are isomorphic if and only if K' and L' are isomorphic. The details are left to the reader. $\square$

PROOF OF THEOREM 2.2.12. (1): The map from (a) to (b) is well-defined by Proposition 2.2.8 and Lemma 2.2.13.

To see that the map from (b) to (a) is well-defined, it suffices to show that the subcategory $\mathcal{P}$ of projective objects in $\mathcal{E}$ is Morita equivalent to $\mathcal{C}$. Since $\mathcal{E}$ is a preresolving subcategory of $\mathsf{mod}\,\mathcal{C}$, it easily follows that $\mathcal{P}$ is Morita equivalent to $\mathcal{P}(\mathsf{mod}\,\mathcal{C}) = \mathsf{proj}\,\mathcal{C}$, which is Morita equivalent to $\mathcal{C}$.

These maps are easily seen to be inverse to each other.

(2): Suppose that $\mathcal{C}$ is idempotent complete. If $\mathcal{E}$ is idempotent complete exact category with enough projectives, then $\mathcal{P}(\mathcal{E})$ is also idempotent complete. By Definition-Proposition 2.2.10, one can see that the bijection in (1) restricts to the one in (2). $\square$

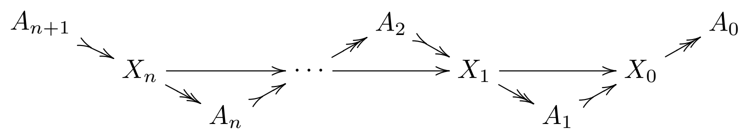
Restring this theorem to the case of rings, one can obtain the following result, whose details are left to the reader. When $\mathcal{E}$ has enough projectives $\mathcal{P}$ and $\mathcal{P} = \mathsf{add}\,P$ for an object P in $\mathcal{E}$, we call P a *projective generator*.

COROLLARY 2.2.14. *Let Λ be a ring.*

(1) *There exists a bijection between the following two classes.*
  (a) $(\mathsf{proj}\,\Lambda)$*-equivalence classes of pairs* $(\mathcal{E}, F)$ *where* $\mathcal{E}$ *is a skeletally small exact category with a projective generator* P *such that* $\mathrm{End}_{\mathcal{E}}(P)$ *is Morita equivalent to* Λ*.*
  (b) *Preresolving subcategories of* $\mathsf{mod}\,\Lambda$*.*
(2) *The bijection of* (1) *restricts to a bijection between the following.*
  (a) $(\mathsf{proj}\,\Lambda)$*-equivalence classes of pairs* $(\mathcal{E}, F)$ *where* $\mathcal{E}$ *is a skeletally small idempotent complete exact category with a projective generator* P *such that* $\mathrm{End}_{\mathcal{E}}(P)$ *is isomorphic to* Λ*.*
  (b) *Resolving subcategories of* $\mathsf{mod}\,\Lambda$*.*

2.2.2. A characterization of $\mathsf{mod}\,\mathcal{P}$. In general $\mathsf{mod}\,\mathcal{P}$ has a lot of resolving subcategories. In this subsection, we shall characterize when $\mathbb{P}(\mathcal{E})$ and $\mathsf{mod}\,\mathcal{P}$ coincide. As an application, we will give a criterion for a given exact category to be equivalent to $\mathsf{mod}\,\mathcal{C}$ for some additive category $\mathcal{C}$.

Let us introduce some terminologies. We say that a complex $A \xrightarrow{f} B \xrightarrow{g} C$ in an additive category $\mathcal{C}$ is $\mathcal{C}(\mathcal{D}, -)$-*exact* for a subcategory $\mathcal{D}$ of $\mathcal{C}$ if $\mathcal{C}(D, A) \xrightarrow{f\circ(-)} \mathcal{C}(D, B) \xrightarrow{g\circ(-)} \mathcal{C}(D, C)$ is exact for all $D \in \mathcal{D}$. Dually we define the $\mathcal{C}(-, \mathcal{D})$-*exactness* in the obvious way. We say that a complex $X_n \to \cdots \to X_1 \to X_0$ in an exact category $\mathcal{E}$ *decomposes into conflations* if there exist a commutative diagram in $\mathcal{E}$



where $A_{n+1} \rightarrowtail X_n \twoheadrightarrow A_n$, $\cdots$, $A_1 \rightarrowtail X_0 \twoheadrightarrow A_0$ are conflations.

Theorem 2.2.15. *Let* $\mathcal{E}$ *be a skeletally small exact category.*
(1) *The following conditions are equivalent.*
  (a) *There exists a skeletally small additive category* $\mathcal{C}$ *such that* $\mathcal{E}$ *is exact equivalent to* $\mathsf{mod}\,\mathcal{C}$*.*
  (b) $\mathcal{E}$ *has enough projectives* $\mathcal{P}$*, idempotent complete, and for any* $\mathcal{E}(\mathcal{P}, -)$*-exact complex*
$$\cdots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0$$
  with all terms in $\mathcal{P}$*, the morphism* $f_1 : P_1 \to P_0$ *can be factored as a deflation followed by an inflation.*
(2) *The following conditions are equivalent.*
  (a′) *There exists a skeletally small additive category* $\mathcal{C}$ *with weak kernels such that* $\mathcal{E}$ *is exact equivalent to* $\mathsf{mod}\,\mathcal{C}$*.*
  (b′) $\mathcal{E}$ *has enough projectives, idempotent complete, and any morphism in* $\mathcal{P}$ *can be factored as a deflation followed by an inflation.*

Proof. (a) ⇒ (b): We may assume that $\mathcal{E} = \mathsf{mod}\,\mathcal{C}$. Then $\mathsf{mod}\,\mathcal{C}$ is idempotent complete and has enough projectives by Proposition 2.2.6. Note that in $\mathsf{mod}\,\mathcal{C}$, the notion of $(\mathsf{mod}\,\mathcal{C})(\mathsf{proj}\,\mathcal{C}, -)$-exactness is equivalent to exactness in $\mathsf{Mod}\,\mathcal{C}$ by the Yoneda Lemma. Suppose that there exists an exact sequence
$$\cdots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0$$
in $\mathsf{Mod}\,\mathcal{C}$ whose all terms are in $\mathsf{proj}\,\mathcal{C}$. Then the cokernel of f_1 is clearly in $\mathsf{mod}\,\mathcal{C}$. Since $\mathsf{mod}\,\mathcal{C} \subset \mathsf{Mod}\,\mathcal{C}$ is resolving and $\mathrm{Coker}\, f_1$ is in $\mathsf{mod}\,\mathcal{C}$, we must have $\mathrm{Im}\, f_i$ is in $\mathsf{mod}\,\mathcal{C}$ for all $i \geq 1$. This clearly implies that the above complex decomposes into conflations in $\mathsf{mod}\,\mathcal{C}$. Especially, f_1 is decomposed into $P_1 \twoheadrightarrow \mathrm{Im}\, f_1 \rightarrowtail P_0$.

(a)′ ⇒ (b)′: Let $f : P_1 \to P_0$ be a morphism in $\mathsf{proj}\,\mathcal{C}$. Then $\mathrm{Coker}\, f$ is finitely presented, hence is in $\mathsf{mod}\,\mathcal{C}$ by Proposition 2.2.7. Since $\mathsf{mod}\,\mathcal{C}$ is resolving in $\mathsf{Mod}\,\mathcal{C}$, the same argument as above implies that $P_1 \twoheadrightarrow \mathrm{Im}\, f \rightarrowtail P_0$ is the desired factorization in $\mathsf{mod}\,\mathcal{C}$.

(b) ⇒ (a): Recall that the Morita embedding $\mathbb{P}$ realizes $\mathcal{E}$ as a resolving subcategory of $\mathsf{mod}\,\mathcal{P}$ by Proposition 2.2.8. We must check that $\mathbb{P} : \mathcal{E} \to \mathsf{mod}\,\mathcal{P}$ is essentially surjective. Let

X be an arbitrary object in $\mathsf{mod}\,\mathcal{P}$. By the definition of $\mathsf{mod}\,\mathcal{P}$ we have an exact sequence $\cdots \to \mathbb{P}P_2 \to \mathbb{P}P_1 \to \mathbb{P}P_0 \to X \to 0$ with P_i in $\mathcal{P}$ for all $i \geq 0$. By (b), the corresponding $P_1 \to P_0$ in $\mathcal{P}$ factorizes into a deflation followed by an inflation in $\mathcal{E}$. Then it immediately follows that the cokernel of $\mathbb{P}P_1 \to \mathbb{P}P_0$ is in $\mathbb{P}(\mathcal{E})$, which proves that $\mathbb{P}$ is essentially surjective.

(b)′ ⇒ (a)′: We only have to check that $\mathcal{P}$ has weak kernels. By (b)′, for any $P_1 \to P_0$ in $\mathcal{P}$ there exist conflations $X_1 \rightarrowtail P_1 \twoheadrightarrow X$ and $X \rightarrowtail P_0 \twoheadrightarrow X'$. Moreover since $\mathcal{E}$ has enough projectives, there exists a conflation $X_2 \rightarrowtail P_2 \twoheadrightarrow X_1$. It is clear that the composition $P_2 \twoheadrightarrow X_1 \rightarrowtail P_1$ is a weak kernel of $P_1 \to P_0$. □

2.3. Exact categories with enough projectives and injectives

In this section, we study exact categories with both enough projectives and enough injectives. We will show that an arbitrary exact category with enough projectives and injectives can be realized as a preresolving-precoresolving subcategory of the exact category $\mathsf{X}_{\mathcal{W}}$, the exact category associated to a Wakamatsu tilting subcategory $\mathcal{W}$ of a module category. We freely use results in the previous section.

2.3.1. Wakamatsu tilting subcategories. Wakamatsu introduced a generalization of the classical concept of tilting modules, called *Wakamatsu tilting modules*, which have possibly infinite projective dimensions [Wa, MR]. They are also called *semi-dualizing modules* by some authors [Chr, ATY]. With a Wakamatsu tilting module W, we can associate an exact category X_W with enough projectives and injectives. Typical examples are the category of Gorenstein projective Λ-modules $\mathsf{GP}\,\Lambda$ and the Ext-orthogonal category ${}^{\perp}U$ for a cotilting module U. In what follows, we introduce a categorical analogue of Wakamatsu tilting modules, called *Wakamatsu tilting subcategories*.

In this subsection, *we denote by $\mathcal{E}$ an exact category with enough projectives.* Let ${}^{\perp}\mathcal{W}$ denote the category consisting of objects X in $\mathcal{E}$ satisfying $\mathrm{Ext}^{>0}_{\mathcal{E}}(X, \mathcal{W}) = 0$.

For a subcategory $\mathcal{W}$ of $\mathcal{E}$, we first introduce the following subcategory $\mathsf{X}_{\mathcal{W}}$ of $\mathcal{E}$ in which objects in $\mathcal{W}$ behaves like an injective cogenerator, see Proposition 2.3.2. We say that a subcategory $\mathcal{W}$ of an exact category $\mathcal{E}$ is *self-orthogonal* if $\mathrm{Ext}^{>0}_{\mathcal{E}}(\mathcal{W}, \mathcal{W}) = 0$.

Definition 2.3.1. Let $\mathcal{E}$ be an exact category with enough projectives and $\mathcal{W}$ an additive subcategory of $\mathcal{E}$.

(1) We denote by $\mathsf{X}_{\mathcal{W}}$ the full subcategory of $\mathcal{E}$ consisting of all objects $X^0 \in {}^{\perp}\mathcal{W}$ such that there exist conflations $X^0 \rightarrowtail W^0 \twoheadrightarrow X^1$, $X^1 \rightarrowtail W^1 \twoheadrightarrow X^2$, $\cdots$ with $W^i \in \mathcal{W}$ and $X^i \in {}^{\perp}\mathcal{W}$ for $i \geq 0$.

(2) We say that $\mathcal{W}$ is a *Wakamatsu tilting subcategory* if it satisfies the following conditions.
   (a) $\mathcal{W}$ is self-orthogonal.
   (b) $\mathsf{X}_{\mathcal{W}}$ contains all projective objects in $\mathcal{E}$.

Note that the subcategory $\mathcal{P}(\mathcal{E})$ of $\mathcal{E}$ consisting of all projective objects is always Wakamatsu tilting in $\mathcal{E}$. A Λ-module W for a ring Λ is said to be a *Wakamatsu tilting module* or a *semi-dualizing module* if $\mathsf{add}\,W$ is a Wakamatsu tilting subcategory of $\mathsf{mod}\,\Lambda$. Our definition coincides with the usual one.

Basic properties of the category $\mathsf{X}_{\mathcal{W}}$ are as follows.

Proposition 2.3.2. *Let $\mathcal{E}$ be an exact category with enough projectives and $\mathcal{W}$ an additive self-orthogonal subcategory of $\mathcal{E}$.*

(1) $\mathsf{X}_{\mathcal{W}}$ *is closed under extensions, kernels of deflations and summands in $\mathcal{E}$ and it is thick in ${}^{\perp}\mathcal{W}$.*

If moreover $\mathcal{W}$ is Wakamatsu tilting, then the following hold.

(2) $\mathsf{X}_{\mathcal{W}}$ *is a resolving subcategory of $\mathcal{E}$.*
(3) $\mathsf{X}_{\mathcal{W}}$ *has enough projectives and injectives.*
(4) *P in $\mathsf{X}_{\mathcal{W}}$ is projective in $\mathsf{X}_{\mathcal{W}}$ if and only if P is projective in $\mathcal{E}$.*
(5) *I in $\mathsf{X}_{\mathcal{W}}$ is injective in $\mathsf{X}_{\mathcal{W}}$ if and only if I is in $\mathsf{add}\,\mathcal{W}$.*

PROOF. The same proof of Proposition 5.1 in [AR3] applies. For the convenience of the reader, we shall give a proof.

(1): We first show that $\mathsf{X}_{\mathcal{W}}$ is closed under extensions. Suppose that $A \rightarrowtail B \twoheadrightarrow C$ is a conflation in $\mathcal{E}$ with A and C in $\mathsf{X}_{\mathcal{W}}$. It is easy to see that ${}^{\perp}\mathcal{W}$ is closed under extensions, thus we have B is in ${}^{\perp}\mathcal{W}$. By the definition of $\mathsf{X}_{\mathcal{W}}$, there exist conflations $A \rightarrowtail W \twoheadrightarrow A^1$ and $C \rightarrowtail W' \twoheadrightarrow C^1$ such that W and W' are in $\mathcal{W}$ and A^1 and C^1 are in $\mathsf{X}_{\mathcal{W}}$,

Consider the pushout diagram

$$\begin{array}{ccccc} A & \rightarrowtail & B & \twoheadrightarrow & C \\ \downarrow & & \downarrow & & \| \\ W & \rightarrowtail & U & \twoheadrightarrow & C \\ \downarrow & & \downarrow & & \\ A^1 & = & A^1 & & \end{array}$$

in $\mathcal{E}$. Since C is in $\mathsf{X}_{\mathcal{W}} \subset {}^{\perp}\mathcal{W}$, the middle row splits. Thus we may assume that $U = W \oplus C$. We then have the commutative diagram.

$$\begin{array}{ccccc} B & \rightarrowtail & W \oplus C & \twoheadrightarrow & A^1 \\ \| & & \downarrow & & \downarrow \\ B & \rightarrowtail & W \oplus W' & \twoheadrightarrow & D \\ & & \downarrow & & \downarrow \\ & & C^1 & = & C^1 \end{array}$$

where the middle columns is a direct sum of $W = W \to 0$ and $C \rightarrowtail W' \twoheadrightarrow C^1$. Since A^1 and C^1 are in $\mathsf{X}_{\mathcal{W}}$, if follows that D is an extension of objects in $\mathsf{X}_{\mathcal{W}}$. By considering the middle row, one may proceed this process to see that B is in $\mathsf{X}_{\mathcal{W}}$.

We can prove that $\mathsf{X}_{\mathcal{W}}$ is closed under kernels of deflations and summands by the same argument as the proof in Proposition 2.2.6, so we leave it to the reader.

Next we show that $\mathsf{X}_{\mathcal{W}}$ is a thick subcategory of ${}^{\perp}\mathcal{W}$. Obviously it suffices to see that $\mathsf{X}_{\mathcal{W}}$ is closed under cokernels of inflations in ${}^{\perp}\mathcal{W}$. Let $A \rightarrowtail B \twoheadrightarrow C$ be a conflation with all terms in ${}^{\perp}\mathcal{W}$ and assume that A and B are in $\mathsf{X}_{\mathcal{W}}$. We have a conflation $A \rightarrowtail W \twoheadrightarrow A'$ with W in $\mathcal{W}$ and A' in $\mathsf{X}_{\mathcal{W}}$. Then consider the following pushout diagram.

$$\begin{array}{ccccc} A & \rightarrowtail & B & \twoheadrightarrow & C \\ \downarrow & & \downarrow & & \| \\ W & \rightarrowtail & D & \twoheadrightarrow & C \\ \downarrow & & \downarrow & & \\ A' & = & A' & & \end{array}$$

Because A' and B are in $\mathsf{X}_{\mathcal{W}}$ and $\mathsf{X}_{\mathcal{W}}$ is closed under extensions, D must be in $\mathsf{X}_{\mathcal{W}}$. On the other hand, the middle row splits because C is in ${}^{\perp}\mathcal{W}$, which implies that C is a summand of D. Since $\mathsf{X}_{\mathcal{W}}$ is closed under summands, it follows that C is in $\mathsf{X}_{\mathcal{W}}$, which shows that $\mathsf{X}_{\mathcal{W}} \subset {}^{\perp}\mathcal{W}$ is thick.

(2)-(5): If $\mathcal{W}$ is in addition Wakamatsu tilting, then by definition all projective objects are in $\mathsf{X}_{\mathcal{W}}$, which clearly implies that $\mathsf{X}_{\mathcal{W}}$ is resolving in $\mathcal{E}$. The remaining assertions are obvious from the definition. □

2.3.2. Morita-type theorem. We assume that *$\mathcal{E}$ is a skeletally small exact category with enough projectives $\mathcal{P} = \mathcal{P}(\mathcal{E})$ and enough injectives $\mathcal{I} = \mathcal{I}(\mathcal{E})$.* The following result gives an explicit description of the image of the Morita embedding functor $\mathbb{P} : \mathcal{E} \to \mathsf{mod}\,\mathcal{P}$ (2.2.1) in terms of a Wakamatsu tilting subcategory.

THEOREM 2.3.3. *Let $\mathcal{E}$ be a skeletally small exact category with enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$. For the Morita embedding $\mathbb{P} : \mathcal{E} \to \mathsf{mod}\,\mathcal{P}$, we set $\mathcal{W} = \mathbb{P}(\mathcal{I})$. Then $\mathcal{W}$ is a Wakamatsu tilting subcategory of* $\mathsf{mod}\,\mathcal{P}$*, and $\mathbb{P}(\mathcal{E})$ is a preresolving-precoresolving subcategory of $\mathsf{X}_{\mathcal{W}}$ (Definition 2.2.4). Moreover it is resolving-coresolving if and only if $\mathcal{E}$ is idempotent complete.*

Proof. First we see that $\mathbb{P}(\mathcal{E})$ is contained in $\mathsf{X}_{\mathcal{W}}$. Since $\mathcal{E}$ has enough injectives $\mathcal{I}$, for any object $X \in \mathcal{E}$, there exist conflations $X \rightarrowtail I^0 \twoheadrightarrow X^1$, $X^1 \rightarrowtail I^1 \twoheadrightarrow X^2$, $\cdots$ with I^i injective for all i. Applying $\mathbb{P}$, we obtain conflations $\mathbb{P}X \rightarrowtail \mathbb{P}I^0 \twoheadrightarrow \mathbb{P}X^1$, $\mathbb{P}X^1 \rightarrowtail \mathbb{P}I^1 \twoheadrightarrow \mathbb{P}X^2$, $\cdots$ with $\mathbb{P}I^i$ in $\mathcal{W}$ for all i. Recall that $\mathbb{P}$ preserves all extension groups by Proposition 2.2.1, thus $\mathbb{P}(\mathcal{E}) \subset {}^{\perp}\mathcal{W}$ holds. Therefore, the conflations show that $\mathbb{P}X$ is in $\mathsf{X}_{\mathcal{W}}$.

Next we show that $\mathcal{W}$ is a Wakamatsu tilting subcategory of $\mathsf{mod}\,\mathcal{P}$. Since $\mathbb{P}(\mathcal{E}) \subset \mathsf{X}_{\mathcal{W}} \subset {}^{\perp}\mathcal{W}$ holds, $\mathcal{W}$ is clearly self-orthogonal and $\mathbb{P}(\mathcal{P})$ is contained in $\mathsf{X}_{\mathcal{W}}$. On the other hand, $\mathsf{X}_{\mathcal{W}}$ is closed under direct sums and summands by Proposition 2.3.2. Thus it follows that $\mathsf{proj}\,\mathcal{P} = \mathsf{add}\,\mathbb{P}(\mathcal{P}) \subset \mathsf{X}_{\mathcal{W}}$, which implies that $\mathcal{W}$ is Wakamatsu tilting.

It remains to prove that $\mathbb{P}(\mathcal{E})$ is a preresolving-precoresolving subcategory of $\mathsf{X}_{\mathcal{W}}$. Since $\mathbb{P}(\mathcal{E}) \subset \mathsf{mod}\,\mathcal{P}$ and $\mathsf{X}_{\mathcal{W}} \subset \mathsf{mod}\,\mathcal{P}$ are extension-closed subcategories, it follows that $\mathbb{P}(\mathcal{E})$ is closed under extensions in $\mathsf{X}_{\mathcal{W}}$. By Proposition 2.3.2(4)(5), projective (resp. injective) objects in $\mathsf{X}_{\mathcal{W}}$ are precisely objects in $\mathsf{proj}\,\mathcal{P}$ (resp. $\mathsf{add}\,\mathcal{W}$). Thus $\mathsf{add}\,\mathbb{P}(\mathcal{E})$ contains both projective and injective objects in $\mathsf{X}_{\mathcal{W}}$. Moreover the images of projective and injective resolutions in $\mathcal{E}$ under $\mathbb{P} : \mathcal{E} \to \mathsf{X}_{\mathcal{W}}$ yield desired conflations in Definition 2.2.4(c). Thus $\mathbb{P}(\mathcal{E})$ is preresolving-precoresolving in $\mathsf{X}_{\mathcal{W}}$.

If $\mathcal{E}$ is idempotent complete, then $\mathbb{P}(\mathcal{E})$ is closed under summands in $\mathsf{mod}\,\mathcal{P}$, which implies that $\mathbb{P}(\mathcal{E})$ is resolving and coresolving in $\mathsf{X}_{\mathcal{W}}$ by Lemma 2.2.5. Conversely, suppose that $\mathbb{P}(\mathcal{E})$ is resolving or coresolving in $\mathsf{X}_{\mathcal{W}}$. Since $\mathsf{mod}\,\mathcal{P}$ is idempotent complete and $\mathsf{X}_{\mathcal{W}}$ is closed under summands, $\mathbb{P}(\mathcal{E})$ is clearly idempotent complete, hence so is $\mathcal{E}$. □

Note that in the case of Frobenius categories, Theorem 2.3.3 was shown in [Che2, Theorem 4.2]

Conversely, any preresolving-precoresolving subcategories of $\mathsf{X}_{\mathcal{W}}$ clearly have enough projectives and injectives. Hence one obtains the following classification of exact categories with enough projectives and injectives. First we modify the notion of the $\mathcal{C}$-equivalence defined in Definition 2.2.11.

Definition 2.3.4. Suppose that $\mathcal{E}$ and $\mathcal{E}'$ are skeletally small exact categories with enough projectives and injectives. Assume that $\mathcal{P}(\mathcal{E})$ and $\mathcal{P}(\mathcal{E}')$ are Morita equivalent to $\mathcal{C}$ via equivalences $F : \mathsf{proj}\,\mathcal{C} \simeq \mathsf{proj}\,\mathcal{P}(\mathcal{E})$ and $F' : \mathsf{proj}\,\mathcal{C} \simeq \mathsf{proj}\,\mathcal{P}(\mathcal{E}')$, and that $\mathcal{I}(\mathcal{E})$ and $\mathcal{I}(\mathcal{E}')$ are Morita equivalent to $\mathcal{W}$ via equivalences $H : \mathsf{proj}\,\mathcal{W} \simeq \mathsf{proj}\,\mathcal{I}(\mathcal{E})$ and $H' : \mathsf{proj}\,\mathcal{W} \simeq \mathsf{proj}\,\mathcal{I}(\mathcal{E}')$ We say that pairs $(\mathcal{E}, F, H)$ and $(\mathcal{E}', F', H')$ are $(\mathcal{C}, \mathcal{W})$-*equivalent* if there exists an exact equivalence $G : \mathcal{E} \simeq \mathcal{E}'$ which makes the following diagrams commute up to natural isomorphism

$$\begin{array}{ccccc} \mathsf{proj}\,\mathcal{P}(\mathcal{E}) & \xleftarrow[\simeq]{F} & \mathsf{proj}\,\mathcal{C} & \xrightarrow[\simeq]{F'} & \mathsf{proj}\,\mathcal{P}(\mathcal{E}') \\ \downarrow{\scriptstyle \mathsf{proj}\,\iota} & & & & \downarrow{\scriptstyle \mathsf{proj}\,\iota'} \\ \mathsf{proj}\,\mathcal{E} & & \xrightarrow[\mathsf{proj}\,G]{\simeq} & & \mathsf{proj}\,\mathcal{E}' \end{array} \qquad \begin{array}{ccccc} \mathsf{proj}\,\mathcal{I}(\mathcal{E}) & \xleftarrow[\simeq]{H} & \mathsf{proj}\,\mathcal{W} & \xrightarrow[\simeq]{H'} & \mathsf{proj}\,\mathcal{I}(\mathcal{E}') \\ \downarrow{\scriptstyle \mathsf{proj}\,\iota} & & & & \downarrow{\scriptstyle \mathsf{proj}\,\iota'} \\ \mathsf{proj}\,\mathcal{E} & & \xrightarrow[\mathsf{proj}\,G]{\simeq} & & \mathsf{proj}\,\mathcal{E}' \end{array}$$

where ι and ι' are the inclusions.

Theorem 2.3.5. *Let $\mathcal{C}$ be an additive category and $\mathcal{W}$ a Wakamatsu tilting subcategory in* $\mathsf{mod}\,\mathcal{C}$.

(1) *There exists a bijection between the following two classes.*
   (a) *$(\mathcal{C}, \mathcal{W})$-equivalence classes of pairs $(\mathcal{E}, F, H)$ where $\mathcal{E}$ is a skeletally small exact category with enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$ such that $\mathcal{P}$ is Morita equivalent to $\mathcal{C}$ and $\mathcal{I}$ is Morita equivalent to $\mathcal{W}$.*
   (b) *Preresolving-precoresolving subcategories of* $\mathsf{X}_{\mathcal{W}}$.

   Moreover, any exact categories with enough projectives and injectives occur in this way.
(2) *Suppose that $\mathcal{C}$ and $\mathcal{W}$ are idempotent complete. Then the bijection of* (1) *restricts to a bijection between the following.*
   (a) *$(\mathcal{C}, \mathcal{W})$-equivalence classes of pairs $(\mathcal{E}, F, H)$ where $\mathcal{E}$ is a skeletally small idempotent complete exact category with enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$ such that $\mathcal{P}$ is equivalent to $\mathcal{C}$ and $\mathcal{I}$ is equivalent to $\mathcal{W}$.*
   (b) *Resolving-coresolving subcategories of* $\mathsf{X}_{\mathcal{W}}$

PROOF. The proof of Theorem 2.2.12 applies. □

One can conclude the following results on a ring immediately.

COROLLARY 2.3.6. *Let Λ be a ring and W a Wakamatsu tilting Λ-module.*

(1) *There exists a bijection between the following two classes.*
   (a) $(\mathsf{proj}\,\Lambda, \mathsf{add}\,W)$*-equivalence classes of pairs $(\mathcal{E}, F, H)$ where $\mathcal{E}$ is a skeletally small exact category with a projective generator P and an injective cogenerator I such that $\mathrm{End}_{\mathcal{E}}(P)$ is Morita equivalent to Λ and $\mathrm{End}_{\mathcal{E}}(I)$ is Morita equivalent to $\mathrm{End}_{\Lambda}(W)$.*
   (b) *Preresolving-precoresolving subcategories of X_W.*

   Moreover, any exact categories with projective generators and injective cogenerators occur in this way.
(2) *The bijection of* (1) *restricts to a bijection between the following.*
   (a) $(\mathsf{proj}\,\Lambda, \mathsf{add}\,W)$*-equivalence classes of pairs $(\mathcal{E}, F, H)$ where $\mathcal{E}$ is a skeletally small idempotent complete exact category with a projective generator P and an injective cogenerator I such that $\mathrm{End}_{\mathcal{E}}(P)$ is isomorphic to Λ and $\mathrm{End}_{\mathcal{E}}(I)$ is isomorphic to $\mathrm{End}_{\Lambda}(W)$.*
   (b) *Resolving-coresolving subcategories of X_W.*

Next we consider the case of Frobenius categories. Recall that an exact category $\mathcal{E}$ is called *Frobenius* if it has enough projectives and injectives, and projective objects and injective objects coincide. A typical example of a Frobenius category is given by the following.

DEFINITION 2.3.7. For a skeletally small additive category $\mathcal{C}$, we denote by $\mathsf{GP}\,\mathcal{C}$ the exact subcategory $\mathsf{X}_{\mathcal{W}}$ of $\mathsf{mod}\,\mathcal{C}$ for the Wakamatsu tilting subcategory $\mathcal{W} = \mathsf{proj}\,\mathcal{C}$ of $\mathsf{mod}\,\mathcal{C}$. Modules in $\mathsf{GP}\,\mathcal{C}$ are called *Gorenstein projective*. Similarly for a ring Λ, we denote by $\mathsf{GP}\,\Lambda$ the exact subcategory X_{Λ} of $\mathsf{mod}\,\Lambda$.

It follows from Proposition 2.3.2 that $\mathsf{GP}\,\mathcal{C}$ is a Frobenius category. We remark that different names such as Cohen-Macaulay or totally reflexive are used for $\mathsf{GP}\,\mathcal{C}$.

Using the bijection of Theorem 2.3.5, one can easily show the following results about Frobenius categories, where we use the notion of $\mathcal{C}$-equivalence defined in Definition 2.2.11.

THEOREM 2.3.8. *Let $\mathcal{C}$ be a skeletally small additive category.*

(1) *There exists a bijection between the following two classes.*
   (a) *$\mathcal{C}$-equivalence classes of pairs $(\mathcal{E}, F)$ where $\mathcal{E}$ is a skeletally small Frobenius category such that $\mathcal{P}(\mathcal{E})$ is Morita equivalent to $\mathcal{C}$.*
   (b) *Preresolving-precoresolving subcategories of $\mathsf{GP}\,\mathcal{C}$.*
(2) *Suppose that $\mathcal{C}$ is idempotent complete. Then the bijection of* (1) *restricts to a bijection between the following.*
   (a) *$\mathcal{C}$-equivalence classes of pairs $(\mathcal{E}, F)$ where $\mathcal{E}$ is a skeletally small idempotent complete Frobenius category such that $\mathcal{P}(\mathcal{E})$ is equivalent to $\mathcal{C}$.*
   (b) *Resolving-coresolving subcategories of $\mathsf{GP}\,\mathcal{C}$.*

COROLLARY 2.3.9. *Let Λ be a ring.*

(1) *There exists a bijection between the following two classes.*
   (a) $(\mathsf{proj}\,\Lambda)$*-equivalence classes of pairs $(\mathcal{E}, F)$ where $\mathcal{E}$ is a skeletally small Frobenius category $\mathcal{E}$ with a projective generator P such that $\mathrm{End}_{\mathcal{E}}(P)$ is Morita equivalent to Λ.*
   (b) *Preresolving-precoresolving subcategories of $\mathsf{GP}\,\Lambda$.*
(2) *The bijection of* (1) *restricts to a bijection between the following.*
   (a) $(\mathsf{proj}\,\Lambda)$*-equivalence classes of pairs $(\mathcal{E}, F)$ where $\mathcal{E}$ is a skeletally small idempotent complete Frobenius category $\mathcal{E}$ with a projective generator P such that $\mathrm{End}_{\mathcal{E}}(P)$ is isomorphic to Λ.*
   (b) *Resolving-coresolving subcategories of $\mathsf{GP}\,\Lambda$.*

2.3.3. A characterization of $\mathsf{X}_{\mathcal{W}}$. Let $\mathcal{E}$ be a skeletally small exact category with enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$. We have the Morita embedding $\mathbb{P} : \mathcal{E} \to \mathsf{X}_{\mathcal{W}}$ for $\mathcal{W} = \mathbb{P}(\mathcal{I})$ by Theorem 2.3.3. The following theorem characterizes when $\mathbb{P}(\mathcal{E})$ and $\mathsf{X}_{\mathcal{W}}$ (or $\mathsf{GP}\,\mathcal{P}$) actually coincide.

THEOREM 2.3.10. *Let $\mathcal{E}$ be a skeletally small exact category. Then the following are equivalent.*

(1) *There exist a skeletally small additive category $\mathcal{C}$ and a Wakamatsu tilting subcategory $\mathcal{W}$ of $\mathsf{mod}\,\mathcal{C}$ such that $\mathcal{E}$ is exact equivalent to $\mathsf{X}_{\mathcal{W}}$.*

(2) *$\mathcal{E}$ is idempotent complete and has enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$, and any $\mathcal{E}(\mathcal{P}, -)$-exact and $\mathcal{E}(-, \mathcal{I})$-exact complex*
$$\cdots \to P_2 \to P_1 \to P_0 \to I^0 \to I^1 \to I^2 \to \cdots$$
with P_i in $\mathcal{P}$ and I^i in $\mathcal{I}$ for $i \geq 0$ decomposes into conflations.

To prove this, we need the following technical lemma.

LEMMA 2.3.11. *Let $\mathcal{C}$ be a skeletally small additive category and $\mathcal{W}$ an additive self-orthogonal subcategory of $\mathsf{mod}\,\mathcal{C}$. Suppose that $\cdots \to M_2 \to M_1 \to M_0 \to M_{-1} \to M_{-2} \to \cdots$ is a complex in ${}^{\perp}\mathcal{W}$ which decomposes into conflations $X_{i+1} \rightarrowtail M_i \twoheadrightarrow X_i$ in $\mathsf{mod}\,\mathcal{C}$ for all $i \in \mathbb{Z}$. If this complex is $(\mathsf{mod}\,\mathcal{C})(-, \mathcal{W})$-exact, then X_i is in ${}^{\perp}\mathcal{W}$ for all $i \in \mathbb{Z}$.*

PROOF. Put $\mathcal{F} := \mathsf{mod}\,\mathcal{C}$ for simplicity. Since M_i's are in ${}^{\perp}\mathcal{W}$, we have an exact sequence
$$\mathcal{F}(M_i, \mathcal{W}) \to \mathcal{F}(X_{i+1}, \mathcal{W}) \to \mathrm{Ext}^1_{\mathcal{F}}(X_i, \mathcal{W}) \to 0.$$
by the long exact sequence of Ext. We first show that $\mathrm{Ext}^1_{\mathcal{F}}(X_i, \mathcal{W})$ vanishes. Consider the following commutative diagram.
$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{F}(X_{i+1}, \mathcal{W}) & \longrightarrow & \mathcal{F}(M_{i+1}, \mathcal{W}) & \longrightarrow & \mathcal{F}(M_{i+2}, \mathcal{W}) \\ & \uparrow & & \| & & \| \\ & \mathcal{F}(M_i, \mathcal{W}) & \longrightarrow & \mathcal{F}(M_{i+1}, \mathcal{W}) & \longrightarrow & \mathcal{F}(M_{i+2}, \mathcal{W}) \end{array}$$
Since given complex is $\mathcal{F}(-, \mathcal{W})$-exact, the bottom row is exact. One can easily show that $M_{i+2} \to M_{i+1} \to X_{i+1}$ is a cokernel diagram, which implies that the top row is also exact. Thus we see that $\mathcal{F}(M_i, \mathcal{W}) \to \mathcal{F}(X_{i+1}, \mathcal{W})$ is surjective, which shows that $\mathrm{Ext}^1_{\mathcal{F}}(X_i, \mathcal{W}) = 0$. Since $\mathrm{Ext}^j_{\mathcal{F}}(X_i, \mathcal{W}) = \mathrm{Ext}^1_{\mathcal{F}}(X_{i+j-1}, \mathcal{W}) = 0$ for $j \geq 1$ by the dimension shift, it follows that X_i is in ${}^{\perp}\mathcal{W}$. □

PROOF OF THEOREM 2.3.10. (1) ⇒ (2): We may assume that $\mathcal{E} = \mathsf{X}_{\mathcal{W}}$, and in this case $\mathcal{I}$ coincides with $\mathsf{add}\,\mathcal{W}$ by Proposition 2.3.2(5). Suppose that
$$\cdots \to P_2 \to P_1 \to P_0 \to I^0 \to I^1 \to I^2 \to \cdots$$
is a complex in $\mathsf{X}_{\mathcal{W}}$ satisfying the condition. Since $\mathcal{E}(\mathcal{P}, -)$-exactness is equivalent to exactness in $\mathsf{Mod}\,\mathcal{P}$, we can decompose it into short exact sequences $X_{i+1} \rightarrowtail P_i \twoheadrightarrow X_i$ and $X^i \rightarrowtail I^i \twoheadrightarrow X^{i+1}$ for $i \geq 0$ with $X_0 = X^0$. Since the X_0 is obviously in $\mathsf{mod}\,\mathcal{C}$ and $\mathsf{mod}\,\mathcal{C} \subset \mathsf{Mod}\,\mathcal{C}$ is thick, all X_i and X^i are in $\mathsf{mod}\,\mathcal{C}$ for $i \geq 0$.

By Lemma 2.3.11, X^i is in ${}^{\perp}\mathcal{W}$ for all $i \geq 0$. Thus the conflations $X^i \rightarrowtail I^i \twoheadrightarrow X^{i+1}$, $X^{i+1} \rightarrowtail I^{i+1} \twoheadrightarrow X^{i+2}$, $\cdots$ imply that X^i is in $\mathsf{X}_{\mathcal{W}}$ for $i \geq 0$. Since $\mathsf{X}_{\mathcal{W}} \subset \mathsf{mod}\,\mathcal{C}$ is resolving, X_i is also in $\mathsf{X}_{\mathcal{W}}$ for $i \geq 0$, which proves that this complex decomposes into conflations in $\mathsf{X}_{\mathcal{W}}$.

(2) ⇒ (1): Suppose that (2) holds. Since the Morita embedding $\mathbb{P} : \mathcal{E} \to \mathsf{X}_{\mathcal{W}}$ in Theorem 2.3.3 is fully faithful for $\mathcal{W} = \mathbb{P}(\mathcal{I})$, it suffices to show that it is essentially surjective. Let X be an object in $\mathsf{X}_{\mathcal{W}}$. Since X is in $\mathsf{mod}\,\mathcal{P}$ and in $\mathsf{X}_{\mathcal{W}}$, there exists a complex
$$\cdots \to P_2 \to P_1 \to P_0 \to I^0 \to I^1 \to I^2 \to \cdots \tag{2.3.1}$$
with P_i in $\mathcal{P}$ and I^i in $\mathcal{I}$ for $i \geq 0$ satisfying the following properties:
$$\cdots \to \mathbb{P}P_2 \to \mathbb{P}P_1 \to \mathbb{P}P_0 \to \mathbb{P}I^0 \to \mathbb{P}I^1 \to \mathbb{P}I^2 \to \cdots \tag{2.3.2}$$
is exact, X is the cokernel of $\mathbb{P}P_1 \to \mathbb{P}P_0$ and all the kernels of $\mathbb{P}I^i \to \mathbb{P}I^{i+1}$ are in ${}^{\perp}\mathcal{W}$ for $i \geq 0$. Decompose (2.3.2) into short exact sequences $X_{i+1} \rightarrowtail \mathbb{P}P_i \twoheadrightarrow X_i$ and $X^i \rightarrowtail \mathbb{P}I^i \twoheadrightarrow X^{i+1}$ in $\mathsf{Mod}\,\mathcal{P}$

for $i \geq 0$ with $X_0 = X = X^0$. Since X is in $\mathsf{X}_{\mathcal{W}} \subset \mathsf{mod}\,\mathcal{P}$ and $\mathsf{mod}\,\mathcal{P}$ is thick in $\mathsf{Mod}\,\mathcal{P}$, it follows that X_i and X^i are in $\mathsf{mod}\,\mathcal{P}$ for all $i \geq 0$. Moreover, X^i is clearly in $\mathsf{X}_{\mathcal{W}}$ for $i \geq 0$. We have that X_i is in $\mathsf{X}_{\mathcal{W}}$ for $i \geq 0$ since $\mathsf{X}_{\mathcal{W}}$ is a resolving subcategory of $\mathsf{mod}\,\mathcal{P}$. It follows that (2.3.2) is $(\mathsf{mod}\,\mathcal{P})(\mathbb{P}(\mathcal{P}), -)$-exact and $(\mathsf{mod}\,\mathcal{P})(-, \mathcal{W})$-exact, which shows that (2.3.1) is $\mathcal{E}(\mathcal{P}, -)$-exact and $\mathcal{E}(-, \mathcal{I})$-exact. Hence by (2) the complex (2.3.1) can be decomposed into conflations in $\mathcal{E}$. Since $\mathbb{P}$ is an exact functor, it immediately follows that X is in $\mathbb{P}(\mathcal{E})$. $\square$

In the case of Frobenius categories, this gives an internal characterization of categories of Gorenstein projective modules $\mathsf{GP}\,\mathcal{C}$.

COROLLARY 2.3.12. *Let $\mathcal{E}$ be a skeletally small exact category. The following are equivalent.*

(1) *There exists a skeletally small additive category $\mathcal{C}$ such that $\mathcal{E}$ is exact equivalent to $\mathsf{GP}\,\mathcal{C}$.*
(2) *$\mathcal{E}$ is an idempotent complete Frobenius category, and any $\mathcal{E}(\mathcal{P}, -)$-exact and $\mathcal{E}(-, \mathcal{P})$-exact complexes*

$$\cdots \to P_2 \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$$

in $\mathcal{P}$ decomposes into conflations in $\mathcal{E}$.

Note that this class of Frobenius categories $\mathcal{E}$ such that the image of the Morita embedding $\mathbb{P} : \mathcal{E} \to \mathsf{mod}\,\mathcal{P}$ coincides with $\mathsf{GP}\,\mathcal{P}$ are called *standard* in [Che2]. The above corollary gives an intrinsic characterization of standard Frobenius categories. Also the category X_W for a Wakamatsu tilting Λ-module W and the category $\mathsf{GP}\,\Lambda$ for a ring Λ can be characterized by similar conditions to Theorem 2.3.10(2) and Corollary 2.3.12(2), which we leave to the reader.

2.4. Exact categories with enough projectives and injectives and higher kernels

In this section we study the special class of Wakamatsu tilting subcategories, *cotilting subcategories*, and study its relationship with *higher kernels*. More precisely, we show that an exact category $\mathcal{E}$ with enough projectives and injectives is equivalent to $\mathsf{X}_{\mathcal{W}}$ for a cotilting subcategory $\mathcal{W}$ of a module category if and only if $\mathcal{E}$ has higher kernels.

2.4.1. Cotilting subcategories. First we introduce the notion of n-cotilting subcategories, which is a generalization of cotilting modules, as we will see in Proposition 2.4.3 below.

DEFINITION 2.4.1. Let $\mathcal{E}$ be an exact category with enough projectives and $\mathcal{W}$ an additive subcategory of $\mathcal{E}$. We say that $\mathcal{W}$ is an *n-cotilting subcategory* if it satisfies the following conditions.

(1) For every $W \in \mathcal{W}$, we have $\mathsf{id}\,W \leq n$, that is, $\mathrm{Ext}_{\mathcal{E}}^{>n}(-, W) = 0$.
(2) $\mathcal{W}$ is self-orthogonal, that is, $\mathrm{Ext}_{\mathcal{E}}^{>0}(\mathcal{W}, \mathcal{W}) = 0$.
(3) The categories ${}^{\perp}\mathcal{W}$ and $\mathsf{X}_{\mathcal{W}}$ coincide (see Definition 2.3.1 for the category $\mathsf{X}_{\mathcal{W}}$).

Note that an n-cotilting subcategory is always Wakamatsu tilting.

Next we study basic properties of cotilting subcategories. For a subcategory $\mathcal{X}$ of an exact category $\mathcal{E}$, we denote by $\widehat{\mathcal{X}}^n$ the subcategory of $\mathcal{E}$ consisting of all objects M such that there exists a complex which decomposes into conflations in $\mathcal{E}$

$$0 \to X_n \to X_{n-1} \to \cdots \to X_0 \to M \to 0,$$

where X_i is in $\mathcal{X}$ for $0 \leq i \leq n$. We write $\widehat{\mathcal{X}}$ for the subcategory of $\mathcal{E}$ whose objects are those in $\widehat{\mathcal{X}}^n$ for some n. We set $\widehat{\mathcal{X}}^0 = \mathcal{X}$ and $\widehat{\mathcal{X}}^n = 0$ for $n < 0$.

PROPOSITION 2.4.2. *Let $\mathcal{E}$ be an exact category with enough projectives and $\mathcal{W}$ an additive self-orthogonal subcategory of $\mathcal{E}$. For an integer $n \geq 0$, the following are equivalent.*

(1) *$\mathcal{W}$ is an n-cotilting subcategory of $\mathcal{E}$.*
(2) $\widehat{\mathsf{X}_{\mathcal{W}}}^n = \mathcal{E}$.
(3) *$\mathsf{X}_{\mathcal{W}}$ contains all projective objects, and for any complex which decomposes into conflations*

$$0 \to X_n \to X_{n-1} \to \cdots \to X_0 \to M \to 0,$$

if X_i is in ${}^{\perp}\mathcal{W}$ for $0 \leq i < n$, then X_n is in $\mathsf{X}_{\mathcal{W}}$.

PROOF. (1) ⇒ (3): All projective objects are clearly contained in ${}^{\perp}\mathcal{W}$, thus in $\mathsf{X}_{\mathcal{W}}$. Let

$$0 \to X_n \to X_{n-1} \to \cdots \to X_0 \to M \to 0$$

be a complex which decomposes into conflations with X_i is in ${}^{\perp}\mathcal{W}$ for $0 \le i < n$. The dimension shift argument shows that $\mathrm{Ext}_{\mathcal{E}}^{>0}(X_n, \mathcal{W}) = \mathrm{Ext}_{\mathcal{E}}^{>n}(M, \mathcal{W}) = 0$ since $\mathsf{id}\,\mathcal{W} \le n$. Hence X_n is in ${}^{\perp}\mathcal{W} = \mathsf{X}_{\mathcal{W}}$, which shows (3).

(3) ⇒ (2): This is clear since $\mathcal{E}$ has enough projectives and all projectives are in $\mathsf{X}_{\mathcal{W}}$.

(2) ⇒ (1): For any M in $\mathcal{E} = \widehat{\mathsf{X}_{\mathcal{W}}}^{n}$, there exist conflations

$$X_n \rightarrowtail X_{n-1} \twoheadrightarrow M_{n-1}, \quad M_{n-1} \rightarrowtail X_{n-2} \twoheadrightarrow M_{n-2}, \quad \cdots, \quad M_1 \rightarrowtail X_0 \twoheadrightarrow M \tag{2.4.1}$$

where X_i is in $\mathsf{X}_{\mathcal{W}}$ for $0 \le i \le n$. The dimension shift argument shows that $\mathrm{Ext}_{\mathcal{E}}^{>n}(M, \mathcal{W}) = \mathrm{Ext}_{\mathcal{E}}^{>0}(X_n, \mathcal{W}) = 0$, which proves that $\mathsf{id}\,\mathcal{W} \le n$.

It remains to check that ${}^{\perp}\mathcal{W} = \mathsf{X}_{\mathcal{W}}$. If M is in ${}^{\perp}\mathcal{W}$, then all terms in (2.4.1) are in ${}^{\perp}\mathcal{W}$ because ${}^{\perp}\mathcal{W}$ is resolving in $\mathcal{E}$. On the other hand, according to Proposition 2.3.2, the category $\mathsf{X}_{\mathcal{W}}$ is closed under cokernels of inflations in ${}^{\perp}\mathcal{W}$. This clearly implies that $M_{n-1}, \cdots, M_1, M$ are in $\mathsf{X}_{\mathcal{W}}$, thus we have ${}^{\perp}\mathcal{W} = \mathsf{X}_{\mathcal{W}}$. □

Let us describe the relation between our cotilting subcategories and classical cotilting modules. The notion of cotilting modules over artin R-algebras is the dual notion of *tilting* modules, and both are widely studied in the representation theory of algebras. Let Λ be an artin R-algebra and U a finitely generated Λ-module. Then U is called a *cotilting module* if it satisfies the following conditions.

(1) $\mathsf{id}\, U_{\Lambda}$ is finite.
(2) U is self-orthogonal.
(3) $D\Lambda$ belongs to $\widehat{\mathsf{add}\, U}$.

The following statement illustrates the relation between our definition and the classical one.

PROPOSITION 2.4.3. *Let Λ be an artin R-algebra and $\mathcal{W}$ a subcategory of $\mathsf{mod}\,\Lambda$ satisfying $\mathsf{add}\,\mathcal{W} = \mathcal{W}$. For an integer $n \ge 0$, the following are equivalent.*

(1) *$\mathcal{W}$ is an n-cotilting subcategory of $\mathsf{mod}\,\Lambda$.*
(2) *There exists a cotilting Λ-module U with $\mathsf{id}\, U \le n$ such that $\mathcal{W} = \mathsf{add}\, U$.*

PROOF. We refer the reader to [AR3, Theorem 5.4] for the proof. □

2.4.2. Higher kernels and the main result. Our aim is to characterize the exact category of the form ${}^{\perp}\mathcal{W} = \mathsf{X}_{\mathcal{W}}$ for some n-cotilting subcategory $\mathcal{W}$ of a module category $\mathsf{mod}\,\mathcal{C}$. In this subsection *we assume that $\mathcal{E}$ is a skeletally small exact category with enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$.* Recall that we have the Morita embedding (2.2.1) $\mathbb{P} : \mathcal{E} \to \mathsf{X}_{\mathcal{W}}$ with $\mathcal{W} = \mathbb{P}(\mathcal{I})$ by Theorem 2.3.3.

The following result gives a criterion for this embedding $\mathbb{P} : \mathcal{E} \to \mathsf{X}_{\mathcal{W}}$ to be dense up to summands. A similar method was used in the proof of [KIWY, Theorem 2.7]. We simplify the proof by using the analogue of Auslander-Buchweitz approximation for exact categories, which we refer to Appendix A.1.

PROPOSITION 2.4.4. *Let $\mathcal{E}$ be a skeletally small exact category with enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$. Consider the Morita embedding $\mathbb{P} : \mathcal{E} \to \mathsf{mod}\,\mathcal{P}$ and put $\mathcal{W} := \mathbb{P}(\mathcal{I})$. If $\widehat{\mathbb{P}(\mathcal{E})}^{n} = \mathsf{mod}\,\mathcal{P}$ holds for some $n \ge 0$, then $\mathcal{W}$ is an n-cotilting subcategory of $\mathsf{mod}\,\mathcal{P}$ and we have $\mathsf{X}_{\mathcal{W}} = \mathsf{add}\,\mathbb{P}(\mathcal{E})$. In this case, $\mathsf{X}_{\mathcal{W}} = \mathbb{P}(\mathcal{E})$ holds if and only if $\mathcal{E}$ is idempotent complete.*

PROOF. Since we have $\mathbb{P}(\mathcal{E}) \subset \mathsf{X}_{\mathcal{W}}$ by Theorem 2.3.3, the equality $\widehat{\mathbb{P}(\mathcal{E})}^{n} = \mathsf{mod}\,\mathcal{P}$ implies $\widehat{\mathsf{X}_{\mathcal{W}}}^{n} = \mathsf{mod}\,\mathcal{P}$. Thus $\mathcal{W}$ is n-cotilting by Proposition 2.4.2. On the other hand, recall that $\mathbb{P}(\mathcal{E})$ is preresolving in $\mathsf{mod}\,\mathcal{P}$ and has enough injectives $\mathcal{W}$. By Proposition 2.3.2, $\mathsf{X}_{\mathcal{W}}$ is resolving in $\mathsf{mod}\,\mathcal{P}$ and has enough injectives $\mathsf{add}\,\mathcal{W}$. Applying Corollary A.1.4 to $\mathcal{X} = \mathsf{X}_{\mathcal{W}}$ and $\mathcal{X}' = \mathbb{P}(\mathcal{E})$, we have $\mathsf{X}_{\mathcal{W}} = \mathsf{add}\,\mathbb{P}(\mathcal{E})$. The last assertion is clear. □

To investigate further properties of cotilting subcategories, we extend the notion of *n-kernels* [Ja] to $n \geq -1$, which simplify our results below.

DEFINITION 2.4.5. Let $\mathcal{C}$ be an additive category.

(1) Let n be an integer $n \geq 1$. We say that $\mathcal{C}$ *has n-kernels* if for any morphism $f : X \to Y$ in $\mathcal{C}$, there exists a complex
$$0 \to X_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X \xrightarrow{f} Y$$
in $\mathcal{C}$ such that the following diagram is exact.
$$0 \to \mathcal{C}(-, X_n) \xrightarrow{\mathcal{C}(-,f_n)} \cdots \xrightarrow{\mathcal{C}(-,f_2)} \mathcal{C}(-, X_1) \xrightarrow{\mathcal{C}(-,f_1)} \mathcal{C}(-, X) \xrightarrow{\mathcal{C}(-,f)} \mathcal{C}(-, Y)$$

(2) Suppose that $\mathcal{C}$ is in addition an exact category.
   (a) We say that $\mathcal{C}$ has *0-kernels* if every morphism f in $\mathcal{C}$ can be factored as a deflation followed by a monomorphism, that is, $f = ig$ holds for a deflation g and a monomorphism i.
   (b) We say that $\mathcal{C}$ has *(-1)-kernels* if every morphism can be factored as a deflation followed by an inflation.

Dually we define the notion of *n-cokernels* for $n \geq -1$.

Note that having n-kernels implies having m-kernels for $m \geq n \geq -1$. Also we point out that an exact category $\mathcal{C}$ has (-1)-kernels if and only if $\mathcal{C}$ is abelian and its exact structure is the usual exact structure on abelian categories.

We remark that having higher kernels is almost equivalent to the finiteness of the global dimension, as the following classical proposition shows.

PROPOSITION 2.4.6. *Let $\mathcal{C}$ be a skeletally small idempotent complete additive category with weak kernels and n an integer $n \geq 1$. Then the following hold.*

(1) *$\mathcal{C}$ has n-kernels if and only if the global dimension of $\mathsf{mod}\,\mathcal{C}$ is at most $n + 1$.*
(2) *Suppose that $\mathrm{End}_{\mathcal{C}}(M)$ is right coherent for an object $M \in \mathcal{C}$. Then $\mathsf{add}\,M$ has n-kernels if and only if the right global dimension of $\mathrm{End}_{\mathcal{C}}(M)$ is at most $n + 1$.*

The following proposition gives a necessary condition for $\mathcal{E}$ to be equivalent to $\mathsf{X}_{\mathcal{W}}$ for an n-cotilting subcategory $\mathcal{W}$ of a module category $\mathsf{mod}\,\mathcal{C}$.

PROPOSITION 2.4.7. *Let $\mathcal{C}$ be a skeletally small additive category with weak kernels and $\mathcal{W}$ an n-cotilting subcategory of $\mathsf{mod}\,\mathcal{C}$ for $n \geq 0$. Then $\mathsf{X}_{\mathcal{W}}$ has $(n-1)$-kernels.*

PROOF. Recall that $\mathsf{X}_{\mathcal{W}} = {}^{\perp}\mathcal{W}$ since $\mathcal{W}$ is n-cotilting. Also note that $\mathsf{mod}\,\mathcal{C}$ is abelian by Proposition 2.2.7 since $\mathcal{C}$ has weak kernels. If $n = 0$, that is, $\mathsf{X}_{\mathcal{W}} = \mathsf{mod}\,\mathcal{C}$, then $\mathsf{X}_{\mathcal{W}}$ clearly has (-1)-kernels since $\mathsf{mod}\,\mathcal{C}$ is abelian.

Let $f : X \to Y$ be a morphism in $\mathsf{X}_{\mathcal{W}}$. Then we have conflations $M_1 \rightarrowtail X \twoheadrightarrow M_0$ and $M_0 \rightarrowtail Y \twoheadrightarrow M_{-1}$ in $\mathsf{mod}\,\mathcal{C}$ such that f is the composition $X \twoheadrightarrow M_0 \rightarrowtail Y$. Suppose that $n = 1$. It follows that $\mathrm{Ext}^{>0}_{\mathsf{mod}\,\mathcal{C}}(M_0, \mathcal{W}) = \mathrm{Ext}^{>1}_{\mathsf{mod}\,\mathcal{C}}(M_{-1}, \mathcal{W}) = 0$ since $\mathsf{id}\,\mathcal{W} \leq 1$. Thus M_0 is in ${}^{\perp}\mathcal{W} = \mathsf{X}_{\mathcal{W}}$. Since ${}^{\perp}\mathcal{W}$ is resolving in $\mathsf{mod}\,\mathcal{C}$, it follows that M_1 is also in ${}^{\perp}\mathcal{W}$. Consequently $X \twoheadrightarrow M_0$ is a deflation in ${}^{\perp}\mathcal{W}$, which shows that $\mathsf{X}_{\mathcal{W}}$ has 0-kernels.

Next let us consider the case $n \geq 2$. Note that $\mathsf{X}_{\mathcal{W}}$ is contravariantly finite in $\mathsf{mod}\,\mathcal{C}$ by Corollary A.1.5. This gives a right $\mathsf{X}_{\mathcal{W}}$-approximation $X_1 \to M_1$. Since $\mathsf{X}_{\mathcal{W}}$ contains all projective objects, this morphism is an epimorphism. Hence we obtain a conflation $M_2 \rightarrowtail X_1 \twoheadrightarrow M_1$ in $\mathsf{mod}\,\mathcal{C}$. Repeat this construction until we get $M_{n-1} \rightarrowtail X_{n-2} \twoheadrightarrow M_{n-2}$. (If $n = 2$, we interpret $X_0 = X$.) By the dimension shift argument, we have $\mathrm{Ext}^{>0}_{\mathsf{mod}\,\mathcal{C}}(M_{n-1}, \mathcal{W}) = \cdots = \mathrm{Ext}^{>n}_{\mathsf{mod}\,\mathcal{C}}(M_{-1}, \mathcal{W}) = 0$, which implies $M_{n-1} \in \mathsf{X}_{\mathcal{W}}$. Consider the complex
$$0 \to M_{n-1} \to X_{n-2} \to \cdots \to X_1 \to X \to Y$$
in $\mathsf{X}_{\mathcal{W}}$. Since the morphism $X_i \twoheadrightarrow M_i$ in each conflation $M_{i+1} \rightarrowtail X_i \twoheadrightarrow M_i$ is a right $\mathsf{X}_{\mathcal{W}}$-approximation for $i \geq 1$, it is easy to see that
$$0 \to \mathsf{X}_{\mathcal{W}}(-, M_{n-1}) \to \cdots \to \mathsf{X}_{\mathcal{W}}(-, X_1) \to \mathsf{X}_{\mathcal{W}}(-, X) \to \mathsf{X}_{\mathcal{W}}(-, Y)$$
is exact. Hence $\mathsf{X}_{\mathcal{W}}$ has $(n-1)$-kernels. $\square$

Surprisingly, having n-kernels is not only necessary but also sufficient for an exact category to be equivalent to $\mathsf{X}_{\mathcal{W}}$ for an n-cotilting subcategory $\mathcal{W}$ of $\mathsf{mod}\,\mathcal{C}$, as the following proposition shows. This was proved in [KIWY, Theorem 2.7] for the case of Frobenius categories.

Proposition 2.4.8. *Let $\mathcal{E}$ be a skeletally small exact category with enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$. Consider the Morita embedding (2.2.1) $\mathbb{P} : \mathcal{E} \to \mathsf{mod}\,\mathcal{P}$. Suppose that there exists a subcategory $\mathcal{M}$ of $\mathcal{E}$ which contains $\mathcal{P}$ and has $(n-1)$-kernels for $n \geq 0$. Then the following hold.*

(1) *$\mathcal{W} := \mathbb{P}(\mathcal{I})$ is an n-cotilting subcategory of $\mathsf{mod}\,\mathcal{P}$.*
(2) *The equality $\mathsf{add}\,\mathbb{P}(\mathcal{E}) = \mathsf{X}_{\mathcal{W}}$ holds. Moreover $\mathbb{P}(\mathcal{E}) = \mathsf{X}_{\mathcal{W}}$ if and only if $\mathcal{E}$ is idempotent complete.*
(3) *$\mathcal{P}$ has weak kernels.*

Proof. Since $\mathcal{M}$ has weak kernels and every object in $\mathcal{M}$ has a deflation from some object in $\mathcal{P}$, it is easy to see that $\mathcal{P}$ has weak kernels. Thus (3) holds.

By Proposition 2.4.4, we only have to show $\widehat{\mathbb{P}(\mathcal{E})}^{n} = \mathsf{mod}\,\mathcal{P}$ to prove (1) and (2). Let X be any object in $\mathsf{mod}\,\mathcal{P}$. Fix a morphism $P_1 \to P_0$ in $\mathcal{P}$ such that $\mathbb{P}P_1 \to \mathbb{P}P_0 \to X \to 0$ is exact.

First suppose that $n \geq 2$. Then by taking the $(n-1)$-kernel of $P_1 \to P_0$ inside $\mathcal{M}$, we get a complex

$$0 \to M_n \to \cdots \to M_2 \to P_1 \to P_0$$

in $\mathcal{M}$. Since $\mathcal{M}$ contains $\mathcal{P}$, it easily follows that

$$0 \to \mathbb{P}M_n \to \cdots \to \mathbb{P}P_1 \to \mathbb{P}P_0 \to X \to 0$$

is a complex which decomposes into conflations in $\mathsf{mod}\,\mathcal{P}$. This gives $\widehat{\mathbb{P}(\mathcal{E})}^{n} = \mathsf{mod}\,\mathcal{P}$.

The case $n = 0$ and 1 are quite similar and left to the reader. □

This immediately gives the following characterization of cotilting subcategories of a module category $\mathsf{mod}\,\mathcal{C}$ amongst Wakamatsu tilting subcategories.

Corollary 2.4.9. *Let $\mathcal{C}$ be a skeletally small additive category and $\mathcal{W}$ a Wakamatsu tilting subcategory of $\mathsf{mod}\,\mathcal{C}$. Then the following are equivalent for $n \geq 0$.*

(1) *$\mathcal{C}$ has weak kernels and $\mathcal{W}$ is n-cotilting.*
(2) *$\mathsf{X}_{\mathcal{W}}$ has $(n-1)$-kernels.*
(3) *$\mathsf{X}_{\mathcal{W}}$ has a subcategory $\mathcal{M}$ which contains all projective objects and has $(n-1)$-kernels.*

Proof. (1) ⇒ (2): This is Proposition 2.4.7.

(2) ⇒ (3): This is trivial.

(3) ⇒ (1): This is obvious from Proposition 2.4.8, since the Morita embedding $\mathbb{P}$ can be identified with the natural inclusion $\mathsf{X}_{\mathcal{W}} \to \mathsf{mod}\,\mathcal{C}$ in this case. □

As an application, we immediately get the following information about global dimensions.

Corollary 2.4.10. *Let Λ be an artin R-algebra and W a Wakamatsu tilting Λ-module which is* not *a cotilting module. Then for any $M \in \mathsf{X}_{\mathcal{W}}$ such that $\Lambda \in \mathsf{add}\,M$, the global dimension of $\mathrm{End}_{\Lambda}(M)$ is infinite.*

Proof. Let W be a Wakamatsu tilting Λ-module and suppose that there exists a Λ-module $M \in \mathsf{X}_{\mathcal{W}}$ such that $\Lambda \in \mathsf{add}\,M$ and the global dimension of $\mathrm{End}_{\Lambda}(M)$ is finite. Then by Proposition 2.4.6, $\mathsf{add}\,M$ has n-kernels for some $n \geq 0$. Applying Corollary 2.4.9 to $\mathcal{M} = \mathsf{add}\,M$, we have that M must be a cotilting module. □

Summarizing these results, we obtain the internal characterizations of exact categories associated with cotilting subcategories.

Theorem 2.4.11. *Let $\mathcal{E}$ be a skeletally small exact category. For an integer $n \geq 0$, the following are equivalent.*

(1) *There exist a skeletally small additive category $\mathcal{C}$ with weak kernels and an n-cotilting subcategory $\mathcal{W}$ of $\mathsf{mod}\,\mathcal{C}$ such that $\mathcal{E}$ is exact equivalent to $\mathsf{X}_{\mathcal{W}}$.*

(2) *$\mathcal{E}$ is idempotent complete, has enough projectives and injectives and has $(n-1)$-kernels.*
(3) *$\mathcal{E}$ is idempotent complete, has enough projectives and injectives and $\mathcal{E}$ has a subcategory $\mathcal{M}$ such that $\mathcal{M}$ contains $\mathcal{P}(\mathcal{E})$ and $\mathcal{M}$ has $(n-1)$-kernels.*

By restricting our attention to artin R-algebras, we have the following criterion.

Corollary 2.4.12. *Let $\mathcal{E}$ be a skeletally small Hom-finite exact R-category. For an integer $n \geq 0$, the following are equivalent.*

(1) *There exist an artin R-algebra Λ and a cotilting Λ-module U with $\mathsf{id}\, U \leq n$ such that $\mathcal{E}$ is exact equivalent to ${}^{\perp}U$.*
(2) *$\mathcal{E}$ is idempotent complete, has a projective generator P and has enough injectives, and has $(n-1)$-kernels.*
(3) *$\mathcal{E}$ is idempotent complete, has a projective generator P and has enough injectives, and $\mathcal{E}$ has a subcategory $\mathcal{M}$ such that $\mathcal{M}$ contains P and $\mathcal{M}$ has $(n-1)$-kernels.*
(4) *$\mathcal{E}^{\mathrm{op}}$ satisfies one of the conditions* (1)-(3).

Proof. The conditions (1)-(3) are equivalent by Theorem 2.4.11 since the notion of n-cotilting subcategories in $\mathsf{mod}\,\Lambda$ coincides with usual cotilting modules by Proposition 2.4.3 for an artin R-algebra Λ. It remains to show (4). It suffices to check that (1) is self-dual. Suppose that $\mathcal{E} = {}^{\perp}U$ for a cotilting Λ-module with $\mathsf{id}\, U \leq n$. The Brenner-Butler theorem implies that ${}_{\Gamma}U$ is a cotilting Γ-module for $\Gamma := \mathrm{End}_{\Lambda}(U)$ and $\mathrm{Hom}_{\Lambda}(-, U)$ gives an exact duality $\mathcal{E} = {}^{\perp}(U_{\Lambda}) \to {}^{\perp}({}_{\Gamma}U)$. Thus $\mathcal{E}^{\mathrm{op}}$ is exact equivalent to ${}^{\perp}({}_{\Gamma}U)$, so $\mathcal{E}^{\mathrm{op}}$ satisfies (1). □

Next we specialize these results to Frobenius categories. Recall that a two-sided noetherian ring Λ is called *Iwanaga-Gorenstein* if the right and left injective dimensions of Λ are finite. It was shown in [Za, Lemma A] that $\mathsf{id}\,{}_{\Lambda}\Lambda = \mathsf{id}\,\Lambda_{\Lambda}$ holds for an Iwanaga-Gorenstein ring Λ. We call Λ an *n-Iwanaga-Gorenstein* if $\mathsf{id}\,\Lambda_{\Lambda} \leq n$. If Λ is an artin R-algebra, then Λ is Iwanaga-Gorenstein if and only if Λ_{Λ} is a cotilting module, or $\widehat{\mathsf{GP}\,\Lambda} = \mathsf{mod}\,\Lambda$ (see [AR3, Proposition 6.1] or [AR2, Proposition 4.2]).

We have the following characterization of $\mathsf{GP}\,\Lambda$ for an Iwanaga-Gorenstein ring Λ. Note that this result was mentioned in [Ka, Proposition 4].

Corollary 2.4.13. *Let $\mathcal{E}$ be a skeletally small exact category. For an integer $n \geq 0$, the following are equivalent.*

(1) *There exists an n-Iwanaga-Gorenstein ring Λ such that $\mathcal{E}$ is exact equivalent to $\mathsf{GP}\,\Lambda$.*
(2) *$\mathcal{E}$ is idempotent complete and Frobenius, has a projective generator P such that $\mathrm{End}_{\mathcal{E}}(M)$ is noetherian, and has both $(n-1)$-kernels and $(n-1)$-cokernels.*
(3) *$\mathcal{E}$ is idempotent complete and Frobenius, has a projective generator P such that $\mathrm{End}_{\mathcal{E}}(P)$ is noetherian, and has a subcategory $\mathcal{M}$ such that $\mathcal{M}$ contains P and $\mathcal{M}$ has both $(n-1)$-kernels and $(n-1)$-cokernels.*

Proof. (1) ⇒ (2): We may assume that $\mathcal{E} = \mathsf{GP}\,\Lambda$ for an Iwanaga-Gorenstein ring Λ. By Theorem 2.4.11, $\mathcal{E}$ has $(n-1)$-kernels. To show that $\mathcal{E}$ has $(n-1)$-cokernels, we only have to note that $\mathrm{Hom}_{\Lambda}(-, \Lambda) : \mathsf{GP}\,\Lambda \to \mathsf{GP}\,\Lambda^{\mathrm{op}}$ gives a duality.

(2) ⇒ (3): This is trivial.

(3) ⇒ (1): Theorem 2.4.11 gives $\mathsf{id}\,\Lambda_{\Lambda} \leq n$ and $\mathcal{E} \simeq \mathsf{GP}\,\Lambda$. Since (3) is self-dual, we have $\mathsf{id}\,{}_{\Lambda}\Lambda \leq n$, which implies Λ is n-Iwanaga-Gorenstein. □

2.5. Applications to artin R-algebras

In the previous section, we gave a necessary and sufficient condition for an exact category $\mathcal{E}$ to be exact equivalent to ${}^{\perp}U$ for some cotilting module U. As we have seen in Theorem 2.4.11, it suffices to check that $\mathcal{E}$ has enough projectives and injectives, and has higher kernels. In this section, we apply this criterion to the representation theory of artin algebras. *We always denote by Λ an artin R-algebra in this section.*

2.5.1. The category $(\mathsf{mod}\,\Lambda)/[\mathsf{Sub}\,M]$ as a torsionfree class. Recall that a subcategory $\mathcal{F}$ of $\mathsf{mod}\,\Lambda$ is said to be a *torsionfree class* if $\mathcal{F}$ is closed under extensions and submodules. It is classical that for a cotilting module $U \in \mathsf{mod}\,\Lambda$ with $\mathsf{id}\,U \leq 1$, the subcategory ${}^{\perp}U$ of $\mathsf{mod}\,\Lambda$ is a torsionfree class, see [ASS].

To state our main theorem in this subsection, let us recall the definition of ideal quotients of additive categories. Let $\mathcal{C}$ be an additive category and $\mathcal{D}$ an additive subcategory of $\mathcal{C}$. Denote by $[\mathcal{D}]$ the ideal of $\mathcal{C}$ consisting of all morphisms in $\mathcal{C}$ which factor through some objects in $\mathcal{D}$. Then the ideal quotient $\mathcal{C}/[\mathcal{D}]$ is an additive category whose objects are those in $\mathcal{C}$ and whose morphisms are given by

$$(\mathcal{C}/[\mathcal{D}])(M,N) := \mathcal{C}(M,N)/[\mathcal{D}](M,N)$$

for M and N in $\mathcal{C}$. Also see Definition A.1.1 for the notion of functorially finiteness.

THEOREM 2.5.1. *Let Λ be an artin R-algebra and $\mathcal{C}$ a functorially finite subcategory of $\mathsf{mod}\,\Lambda$ which is closed under submodules. Put $\mathcal{E} := \mathsf{mod}\,\Lambda$ and denote by $\pi : \mathcal{E} \twoheadrightarrow \mathcal{E}/[\mathcal{C}]$ the natural quotient functor. Then the following hold.*

1. *There exists an exact structure on $\mathcal{E}/[\mathcal{C}]$ induced by $\mathcal{C}$-conflations (see Appendix A.2 for details) such that $\mathcal{E}/[\mathcal{C}]$ has 0-kernels.*
2. *The exact category $\mathcal{E}/[\mathcal{C}]$ has enough projectives $\mathcal{P}$ and injectives $\mathcal{I}$.*
3. *$\mathcal{P}$ (resp. $\mathcal{I}$) is precisely the essential image of $\mathsf{add}(\Lambda, \tau^{-}\mathcal{C})$ (resp. $\mathsf{add}(D\Lambda, \tau\mathcal{C})$) under π.*
4. *$\mathcal{P}$ is of finite type if and only if $\mathcal{I}$ is of finite type.*
5. *Suppose that (4) holds. Let M (resp. N) be an object satisfying $\mathcal{P} = \mathsf{add}\,M$ (resp. $\mathcal{I} = \mathsf{add}\,N$) and put $\Gamma := \mathrm{End}_{\mathcal{E}/[\mathcal{C}]}(M)$. Then $(\mathcal{E}/[\mathcal{C}])(M,N)$ is a 1-cotilting Γ-module and $\mathcal{E}/[\mathcal{C}]$ is exact equivalent to a torsionfree class ${}^{\perp}U$.*

We omit the dual result for the case when $\mathcal{C}$ is closed under quotients. Note that this was proved in [Iy3, Proposition 5.5] in case $\mathcal{C} = \mathsf{mod}(\Lambda/I)$ for some two-sided ideal I and $(\mathsf{mod}\,\Lambda)/[\mathcal{C}]$ is of finite type.

REMARK 2.5.2. Let $\mathcal{C}$ be a subcategory of $\mathsf{mod}\,\Lambda$ which is closed under submodules. By [ASm, Proposition 4.7], the following are equivalent.

1. $\mathcal{C}$ is functorially finite.
2. $\mathcal{C}$ is contravariantly finite.
3. $\mathcal{C} = \mathsf{Sub}\,M$ for some $M \in \mathsf{mod}\,\Lambda$.

Here $\mathsf{Sub}\,M$ is the subcategory consisting of all modules cogenerated by M, in other words, the smallest additive subcategory containing M which is closed under submodules.

To begin the proof, the following elementary observation is useful. In what follows, we denote by $\pi(f) : \underline{L} \to \underline{M}$ in $\mathcal{E}/[\mathcal{C}]$ the image of $f : L \to M$ in $\mathcal{E}$ under the natural functor $\pi : \mathcal{E} \twoheadrightarrow \mathcal{E}/[\mathcal{C}]$.

PROPOSITION 2.5.3. *Let Λ be a right noetherian ring. The following are equivalent for an additive subcategory $\mathcal{C}$ of $\mathsf{mod}\,\Lambda$.*

1. *$\mathcal{C}$ is closed under quotient modules.*
2. *For every X in $\mathsf{mod}\,\Lambda$, there exists a right $\mathcal{C}$-approximation $C_X \rightarrowtail X$ which is an injection.*
3. *$\mathcal{C}$ is contravariantly finite, and epimorphisms are preserved by the natural functor $\pi : \mathsf{mod}\,\Lambda \twoheadrightarrow (\mathsf{mod}\,\Lambda)/[\mathcal{C}]$.*

PROOF. (1) $\Rightarrow$ (2): Let X be in $\mathsf{mod}\,\Lambda$. Consider the direct sum $\bigoplus_{C,f} C \to X$ where C runs over isomorphism classes of objects in $\mathcal{C}$ and f runs over all morphisms $f : C \to X$. Write C_X for its image. Since Λ is assumed to be right noetherian, C_X is a quotient module of finite direct sum of objects in $\mathcal{C}$, which implies that C_X is also in $\mathcal{C}$. It is easy to see that $C_X \rightarrowtail X$ is a right $\mathcal{C}$-approximation.

(2) $\Rightarrow$ (3): Contravariantly finiteness is obvious. Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence in $\mathsf{mod}\,\Lambda$. We will see that $\pi(g)$ is epic. So let $\varphi : N \to X$ be a morphism in $\mathsf{mod}\,\Lambda$ such that $\pi(\varphi g) = \pi(\varphi)\pi(g) = 0$ in $(\mathsf{mod}\,\Lambda)/[\mathcal{C}]$. By the definition of the quotient category, it follows that φg factors through some object in $\mathcal{C}$. It follows that φg factors through the right

$\mathcal{C}$-approximation $i : C_X \rightarrowtail X$. Thus there exists a morphism $h : M \to C_X$ which makes the following diagram commute.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \varphi} & & \\ & & & & C_X & \overset{i}{\rightarrowtail} & X & & \end{array}$$

Since $ihf = \varphi gf = 0$ and i is monic, we have $hf = 0$. Thus there exists a morphism $k : N \to C_X$ such that $h = kg$. Then $ikg = ih = \varphi g$ holds, which implies that $ik = \varphi$ since g is epic. Therefore $\pi(\varphi) = 0$ holds in $\mathcal{E}/[\mathcal{C}]$, which proves that $\pi(g)$ is epic.

(3) $\Rightarrow$ (1): Suppose that C is in $\mathcal{C}$ and $f : C \twoheadrightarrow X$ is a surjection. It follows from (3) that $\pi(f) : \underline{C} \to \underline{X}$ is epimorphism in $\mathcal{E}/[\mathcal{C}]$. However $\underline{C}$ is a zero object in $\mathcal{E}/[\mathcal{C}]$, thus $\underline{X} \cong 0$, that is, $X \in \mathcal{C}$. □

Note that if Λ is a right artinian ring, then the dual of Proposition 2.5.3 holds.

To prove Theorem 2.5.1, we have to introduce a new exact structure given by $\mathcal{C}$-conflations on $\mathsf{mod}\,\Lambda$, which induces the desired exact structure on $(\mathsf{mod}\,\Lambda)/[\mathcal{C}]$. For the notion of $(-,\mathcal{C})$-conflations, $(\mathcal{C},-)$-conflations and $\mathcal{C}$-conflations, we refer the reader to Appendix A.2.

Now let us prove Theorem 2.5.1. We divide the proof into several propositions. *For the rest of this subsection, we always assume that Λ is an artin R-algebra and that $\mathcal{C}$ is a functorially finite subcategory of $\mathcal{E} := \mathsf{mod}\,\Lambda$ which is closed under submodules.*

Proposition 2.5.4. *Endow $\mathcal{E}$ with an exact structure $\mathcal{E}_{\mathcal{C}}$. Then $\mathcal{E}/[\mathcal{C}]$ naturally inherits an exact structure from $\mathcal{E}_{\mathcal{C}}$ as in Proposition A.2.3.*

Proof. Denote by $\pi : \mathcal{E} \twoheadrightarrow \mathcal{E}/[\mathcal{C}]$ the natural functor. According to Proposition A.2.3, it suffices to show that the image of every $\mathcal{C}$-inflation under π is a monomorphism and the image of every $\mathcal{C}$-deflation is an epimorphism. The assertion for $\mathcal{C}$-inflations follows from the dual of Proposition 2.5.3.

Let $0 \to L \to M \to N \to 0$ be a $\mathcal{C}$-conflation. We show that $\underline{M} \to \underline{N}$ is an epimorphism in $\mathcal{E}/[\mathcal{C}]$. Suppose that $N \to X$ is a morphism in $\mathcal{E}$ such that $\underline{M} \to \underline{N} \to \underline{X}$ is zero. Then the composition $M \to N \to X$ factors through the left $\mathcal{C}$-approximation $M \to C^M$. This yields the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \vdots & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & C^M & \longrightarrow & X & & \end{array}$$

with exact rows. Since $\mathcal{C}$ is closed under submodules, K is in $\mathcal{C}$. Because $L \rightarrowtail M \twoheadrightarrow N$ is $(-,\mathcal{C})$-conflation, $L \to K$ factors through $L \to M$. Then it is routine to check that $N \to X$ factors through $C^M \to X$, thus $\underline{N} \to \underline{X}$ is zero in $\mathcal{E}/[\mathcal{C}]$ as desired. □

Proposition 2.5.5. *The exact category $\mathcal{E}/[\mathcal{C}]$ has 0-kernels.*

Proof. We have to show that every morphism $\pi(f) : \underline{X} \to \underline{Y}$ in $\mathcal{E}/[\mathcal{C}]$ can be factored as a $\mathcal{C}$-deflation followed by a monomorphism.

Let $f : X \to Y$ be a morphism in $\mathcal{E} = \mathsf{mod}\,\Lambda$. Then we have the factorization $X \twoheadrightarrow \operatorname{Im} f \rightarrowtail Y$. Note that $\underline{\operatorname{Im} f} \to \underline{Y}$ is monic by the dual of Proposition 2.5.3. Thus it suffices to show that $\underline{X} \to \underline{\operatorname{Im} f}$ can be factored as a $\mathcal{C}$-deflation followed by a monomorphism. Therefore we may assume that f is a surjection.

Let $C_Y \to Y$ be a right $\mathcal{C}$-approximation. Consider the pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & C_Y & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & 0 \end{array}$$

in $\mathcal{E}$. Then the right square gives a short exact sequence

$$0 \to E \to X \oplus C_Y \to Y \to 0$$

in $\mathcal{E}$. Since $C_Y \to Y$ is a right $\mathcal{C}$-approximation, clearly this sequence is a $(\mathcal{C}, -)$-conflation. To transform it into a $\mathcal{C}$-conflation, take a left $\mathcal{C}$-approximation $E \twoheadrightarrow C^E$ and consider the pushout

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & X \oplus C_Y & \xrightarrow{ca} & Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle a} & & \| & & \\ 0 & \longrightarrow & C^E & \xrightarrow{b} & F & \xrightarrow{c} & Y & \longrightarrow & 0 \end{array} \tag{2.5.1}$$

in $\mathcal{E}$. The left square gives a short exact sequence

$$0 \to E \to C^E \oplus X \oplus C_Y \to F \to 0 \tag{2.5.2}$$

in $\mathcal{E}$, which is a $(-, \mathcal{C})$-conflation since $E \twoheadrightarrow C^E$ is a left $\mathcal{C}$-approximation. Since ca is a $(\mathcal{C}, -)$-deflation, a diagram chase for (2.5.1) shows that (2.5.2) is also a $(\mathcal{C}, -)$-conflation. Thus (2.5.2) is a $\mathcal{C}$-conflation. Because $\underline{C^E}$ and $\underline{C_Y}$ is zero in $\mathcal{E}/[\mathcal{C}]$, we obtain a deflation $\pi(g) : \underline{X} \twoheadrightarrow \underline{F}$ in $\mathcal{E}_{\mathcal{C}}/[\mathcal{C}]$, where we write $a = [g, h]$.

By the commutativity of the right square in (2.5.1), we have $\pi(f) = \pi(c)\pi(g)$. Since $\pi(g)$ is a deflation, it suffices to show that $\pi(c) : \underline{F} \to \underline{Y}$ is monic in $\mathcal{E}/[\mathcal{C}]$.

Let $\varphi : Z \to F$ be a morphism such that the composition $c\varphi$ factors through some objects in $\mathcal{C}$. Since $C_Y \to Y$ is a right $\mathcal{C}$-approximation, it follows that $c\varphi$ factors through $C_Y \to Y$. Thus we have a map $d : Z \to C_Y$ such that $c\varphi = ca \circ {}^t[0, d]$. This implies that there exists a map $e : Z \to C^E$ such that $\varphi - a \circ {}^t[0, d] = be$, hence $\varphi = a \circ {}^t[0, d] + be$. Since the images of ${}^t[0, d]$ and b under π are zero in $\mathcal{E}/[\mathcal{C}]$, it follows that $\pi(\varphi) = 0$ in $\mathcal{E}/[\mathcal{C}]$, which shows that $\pi(c)$ is monic. □

Now we are in position to finish the proof of Theorem 2.5.1.

Proof of Theorem 2.5.1. (1): The assertion follows from Proposition 2.5.4 and 2.5.5.

(2): We basically follow the idea of [ASo1]. It is well-known that a short exact sequence $0 \to L \to M \to N \to 0$ is $(-, X)$-exact if and only if $(\tau^- X, -)$-exact (see, e.g. [ARS, Corollary 4.4]). Hence this sequence is $\mathcal{C}$-exact if and only if it is $(\mathsf{add}(\mathcal{C}, \tau^- \mathcal{C}), -)$-exact. It was shown in [ASo1, Theorem 1.14] that this exact structure has enough projectives and injectives if and only if $\mathsf{add}(\mathcal{C}, \tau^- \mathcal{C})$ is functorially finite. Since $\mathcal{C}$ is functorially finite in $\mathsf{mod}\,\Lambda$, it follows that so is $\tau^- \mathcal{C}$. Thus it is clear that $\mathsf{add}(\mathcal{C}, \tau^- \mathcal{C})$ is functorially finite. Therefore $\mathcal{E}_{\mathcal{C}}$ has enough projectives and injectives. By Theorem A.2.3, $\mathcal{E}_{\mathcal{C}}/[\mathcal{C}]$ also has enough projectives and injectives, thus (2) holds.

(3): Note that projective (resp. injective) objects in $\mathcal{E}_{\mathcal{C}}$ are precisely objects in $\mathsf{add}(\Lambda, \mathcal{C}, \tau^- \mathcal{C})$ (resp. $\mathsf{add}(D\Lambda, \mathcal{C}, \tau\mathcal{C})$). Thus Theorem A.2.3 immediately gives (3).

(4): This is clear from (3).

(5): By using the Morita embedding $\mathrm{Hom}_{\mathcal{E}/[\mathcal{C}]}(M, -) : \mathcal{E}/[\mathcal{C}] \to \mathsf{mod}\,\Gamma$, the assertion is immediate from Corollary 2.4.12. □

As an immediate consequence, we have the following.

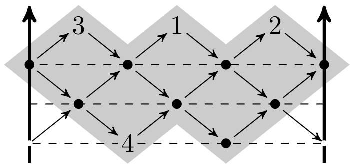
Corollary 2.5.6. *Let Λ be an artin R-algebra and S a semisimple module in $\mathsf{mod}\,\Lambda$. Put $\mathcal{E} := (\mathsf{mod}\,\Lambda)/[\mathsf{add}\, S]$ and $\Gamma := \mathrm{End}_{\mathcal{E}}(\Lambda \oplus \tau^- S)$. Then $U := \mathcal{E}(\Lambda \oplus \tau^- S, D\Lambda \oplus \tau S) \in \mathsf{mod}\,\Gamma$ is a 1-cotilting module and we have an equivalence*

$$\mathcal{E}(\Lambda \oplus \tau^- S, -) : \mathcal{E} \simeq {}^{\perp}U. \tag{2.5.3}$$

If S is simple projective and not injective, then $\mathcal{E}$ is shown to be equivalent to the subcategory of $\mathsf{mod}\,\Lambda$ consisting of modules M such that $\mathrm{Hom}_\Lambda(M, S) = 0$. In this case, Γ is an APR tilt of Λ at S and U is the corresponding APR cotilting Γ-module (see [ASS, Example 2.8(c)]) and the above equivalence (2.5.3) coincides with the classical equivalence [ASS, Theorem 3.8].

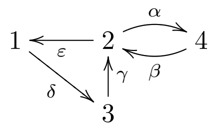
Historically, the notion of APR tilts was generalized to tilting modules, which induce an embedding of a *subcategory* of $\mathsf{mod}\,\Lambda$ into another module category by the Brenner-Butler theorem. On the other hand, this corollary (and our main theorem) gives an embedding of a certain *quotient category* of $\mathsf{mod}\,\Lambda$ into another module category. Thus our results can be regarded as another generalization of APR tilts.

Example 2.5.7. Let Λ be a self-injective Nakayama algebra of Loewy length four given by the cyclic quiver with three vertices over a field k. Let S be a simple module corresponding the white dot in the following Auslander-Reiten quiver of $\mathsf{mod}\,\Lambda$

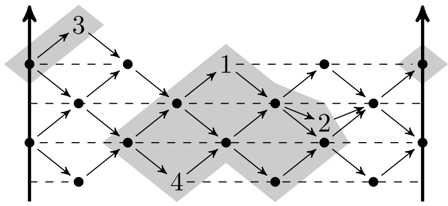


where we identify two vertical arrows. Then obviously $\mathcal{C} := \mathsf{add}\,S$ is closed under submodules. The full translation subquiver consisting of the shaded areas gives the Auslander-Reiten quiver of $(\mathsf{mod}\,\Lambda)/[\mathcal{C}]$.

Then M in Theorem 2.5.1 corresponds to the numbered vertices and Γ is given by the quiver



and relation $\beta\alpha = \gamma\alpha = \delta\gamma\varepsilon = \beta\varepsilon = \alpha\beta - \varepsilon\delta\gamma = 0$. The following diagram gives the Auslander-Reiten quiver of $\mathsf{mod}\,\Gamma$



where we identify two vertical arrows. The shaded area corresponds to the subcategory ${}^{\perp}U$ of $\mathsf{mod}\,\Gamma$ in Theorem 2.5.1. The above two shaded regions illustrate the equivalence $(\mathsf{mod}\,\Lambda)/[\mathcal{C}] \simeq {}^{\perp}U$.

Remark 2.5.8. We point out that the assumption $\mathcal{C}$ *is closed under submodules* in Theorem 2.5.1 cannot be replaced by $\mathcal{C}$ *is closed under images*. The simple example is as follows. Let Λ be a path algebra of the quiver $1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \leftarrow 5$ over a field k. Put $M := P(4)/S(1)$ where $P(4)$ is the indecomposable projective module corresponding to 4 and $S(1)$ is the simple module corresponding to 1. Then $\mathcal{C} := \mathsf{add}\,M$ is easily checked to be closed under images. On the other hand, one can easily check that $(\mathsf{mod}\,\Lambda)/[\mathcal{C}]$ does not have 0-kernels, e.g. by using Proposition 2.4.6.

2.5.2. The category $\mathsf{mod}\,\Lambda$ as an exact category. We assume that Λ is an artin R-algebra. The category $\mathsf{mod}\,\Lambda$ has various exact structures given in Definition A.2.1. An explicit classification was given in [Bua, Proposition 3.3.2].

It is natural to ask which exact structure has enough projectives or which has a projective generator. The answer of this was essentially given in [ASo1] in terms of relative homological algebra. We call a subcategory $\mathcal{M}$ of $\mathsf{mod}\,\Lambda$ a *generating subcategory* if $\mathsf{proj}\,\Lambda \subset \mathcal{M} = \mathsf{add}\,\mathcal{M}$ holds. Dually we define a *cogenerating subcategory*. A Λ-module M is called a *generator* (resp. *cogenerator*) if $\mathsf{add}\,M$ is generating (resp. cogenerating).

Proposition 2.5.9. *Let Λ be an artin R-algebra.*

(1) *There exists a bijection between exact structures on $\mathsf{mod}\,\Lambda$ with enough projectives and contravariantly finite generating subcategories of $\mathsf{mod}\,\Lambda$. It is given by sending an exact structure on $\mathsf{mod}\,\Lambda$ to the category of all projective objects, and the inverse map is given by sending a generating subcategory $\mathcal{M}$ to $(\mathsf{mod}\,\Lambda)_{(\mathcal{M},-)}$.*
(2) *The exact structure on $\mathsf{mod}\,\Lambda$ has a projective generator if and only if it has an injective cogenerator. If G is a projective generator in this exact structure, then $C := D\Lambda \oplus \tau M$ is an injective cogenerator.*

(3) *The bijection of* (1) *restricts to a bijection between exact structures with projective generators and isomorphism classes of basic generators of* $\mathsf{mod}\,\Lambda$.

Proof. This follows directly from results in [ASo1, Theorem 1.15] and [DRSS, Proposition 1.7] (or Theorem A.2.2). □

Hence the nontrivial generators yield nontrivial embeddings of $\mathsf{mod}\,\Lambda$ into another module category. Our main results in this direction is the following. This theorem was essentially proved in [ASo2, Proposition 3.26] by using the method of relative cotilting theory. We give an alternative proof of this theorem by using our results.

Theorem 2.5.10. *Let* Λ *be an artin* R*-algebra and* G *be a generator of* $\mathsf{mod}\,\Lambda$. *Put* $C := D\Lambda \oplus \tau G$, $\Gamma := \mathrm{End}_\Lambda(G)$ *and* $U := \mathrm{Hom}_\Lambda(G, C) \in \mathsf{mod}\,\Gamma$. *Then the following hold.*

(1) U *is a cotilting* Γ*-module with* $\mathsf{id}\,U = 2$ *or* 0. *The case* $\mathsf{id}\,U = 0$ *occurs only when* G *is projective.*
(2) $\mathrm{Hom}_\Lambda(G, -) : \mathsf{mod}\,\Lambda \to \mathsf{mod}\,\Gamma$ *induces an equivalence* $\mathsf{mod}\,\Lambda \simeq {}^{\perp}U$.
(3) $\mathsf{mod}\,\Lambda$ *admits an exact structure such that projective objects are precisely objects in* $\mathsf{add}\,G$ *and the equivalence* $\mathsf{mod}\,\Lambda \simeq {}^{\perp}U$ *is an exact equivalence.*
(4) $\mathrm{End}_\Lambda(G)$ *and* $\mathrm{End}_\Lambda(C)$ *are derived equivalent.*

We need the following preparation.

Lemma 2.5.11. *Let* $\mathcal{A}$ *be an abelian category. Suppose that* $\mathcal{A}$ *is endowed with some exact structure, which we denote by* $\mathcal{A}_{\mathcal{E}}$. *If* $\mathcal{A}_{\mathcal{E}}$ *has* 0*-kernels, then it coincides with the standard exact structure on* $\mathcal{A}$.

Proof. It suffices to show that any epimorphism in $\mathcal{A}_{\mathcal{E}}$ is a deflation. Let $f : Y \to Z$ be an epimorphism. Since $\mathcal{A}_{\mathcal{E}}$ has 0-kernels, there is a factorization $f = ig$ such that g is a deflation and i is a monomorphism. Since f is an epimorphism, so is i. Thus i is an isomorphism since $\mathcal{A}$ is abelian. Therefore f is a deflation. □

Proof of Theorem 2.5.10. Let G be a generator and put $\mathcal{E} := \mathsf{mod}\,\Lambda$. Then by Theorem 2.5.9, $\mathcal{E}_{(G,-)}$ has a projective generator G and an injective cogenerator C.

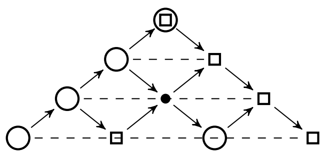
To use Corollary 2.4.12, we observe that $\mathcal{E}_{(G,-)}$ has 1-kernels, which is immediate since $\mathsf{mod}\,\Lambda$ has kernels. Hence (2)-(4) hold by Corollary 2.4.12. The remaining statement of (1) easily follows from Lemma 2.5.11. □

As an application of Theorem 2.5.10, we obtain the following result, which says that every module category of an artin R-algebra sits inside the module category of an artin R-algebra with finite global dimension, with a little modification of its exact structure.

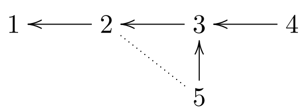
Corollary 2.5.12. *Let* Λ *be an artin* R*-algebra. Then there exist an artin* R*-algebra* Γ *with finite global dimension and a cotilting* Γ*-module* U *with* $\mathsf{id}\,U = 2$ *or* 0 *such that* $\mathsf{mod}\,\Lambda \simeq {}^{\perp}U$.

Proof. There exists a generator G of $\mathsf{mod}\,\Lambda$ whose endomorphism ring has a finite global dimension, see [Au2, Section 3]. Thus the assertion follows from Theorem 2.5.10. □

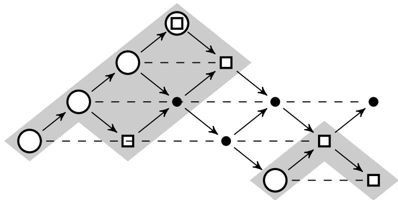
Example 2.5.13. Let Λ be a path algebra of the quiver $1 \leftarrow 2 \leftarrow 3 \leftarrow 4$ over a field k. The following diagram is the Auslander-Reiten quiver of $\mathsf{mod}\,\Lambda$.



Let G be a generator of $\mathsf{mod}\,\Lambda$ corresponding to the circles. Then the associated cogenerator C is the module indicated by the rectangles. Then $\Gamma := \mathrm{End}_\Lambda(G)$ is given by the quiver



where the dotted line indicates the zero relation. The Auslander-Reiten quiver of $\operatorname{\mathsf{mod}}\Gamma$ is given by the following diagram.



The shaded area corresponds to the essential image of the embedding $\operatorname{\mathsf{mod}}\Lambda \to \operatorname{\mathsf{mod}}\Gamma$ in Theorem 2.5.10. We keep the shapes of vertices in the quiver. In particular, the direct sum of all rectangles gives the 2-cotilting module U and $\operatorname{\mathsf{mod}}\Lambda$ is equivalent to ${}^{\perp}U$.

CHAPTER 3

Classifications of exact structures and CM-finite algebras

In this chapter, we deal with Problem B in the first chapter. More precisely, we give a classification of all exact structures on a given idempotent complete additive category. Using this, we investigate the structure of an exact category with finitely many indecomposables. We show that the relation of the Grothendieck group of such a category is generated by AR conflations. Moreover, we obtain an explicit classification of (1) Gorenstein-projective-finite Iwanaga-Gorenstein algebras, (2) Cohen-Macaulay-finite orders, and more generally, (3) cotilting modules U with ${}^{\perp}U$ of finite type.

3.1. Introduction

In the representation theory of finite-dimensional algebras, one of the most important subjects is to *classify certain categories of finite type*. Here we say that an additive k-category over a field k is *of finite type* if it has only finitely many indecomposable objects up to isomorphism. The aim of this chapter is to give a classification of exact categories of finite type, and thereby provide an explicit classification of all GP-finite Iwanaga-Gorenstein algebras. Let us explain the motivation for this.

First we recall how categories of finite type have been studied in the representation theory. It is well-known that an abelian Hom-finite k-category of finite type is nothing but the category $\mathsf{mod}\,\Lambda$ of finitely generated Λ-modules over some representation-finite k-algebra Λ (see [Ga2, 8.2] or Proposition 3.3.14 below). A classification of such algebras is one of the main problems in the representation theory of algebras, and has been studied widely by a number of papers, e.g. [Ga1, Ri, BGRS, GR]. For the case of representation-finite R-orders over a noetherian local ring R, we refer the reader to [Ar, DK, HN, LW, RVdB2, Yo]. Besides abelian categories, triangulated categories of finite type also has been investigated, e.g. in [Am]. Such triangulated categories naturally arise in the representation of algebras and in the categorification of cluster algebras.

Among other things, the observation by Auslander [Au2] is of particular importance to us when we deal with categories of finite type. Let $\mathcal{E}$ be a Hom-finite k-category of finite type and consider the algebra $\Gamma := \mathrm{End}_{\mathcal{E}}(M)$, where M is a direct sum of all non-isomorphic indecomposables in $\mathcal{E}$. This Γ is called an *Auslander algebra* of $\mathcal{E}$, and categorical properties of $\mathcal{E}$ should be related to homological properties of Γ. For example, the condition $\mathcal{E}$ being abelian is equivalent to a certain homological condition of Γ, that is, $\mathrm{gl.dim}\,\Gamma \leq 2 \leq \mathrm{dom.dim}\,\Gamma$. This is called the *Auslander correspondence*, and is now the basic and important viewpoint in the representation theory.

However, most of the algebras are representation-wild, and it is hopeless to understand the whole structure of the module category. Thus nice subcategories of module categories has attracted much attention. For example, in the representation theory of a commutative noetherian local ring R, the category $\mathsf{CM}\,R$ of Cohen-Macaulay modules has played a central role. For an Iwanaga-Gorenstein ring (two-sided noetherian ring satisfying $\mathsf{id}(\Lambda_\Lambda) = \mathsf{id}({}_\Lambda\Lambda) < \infty$), we have the natural notion of Cohen-Macaulay Λ-modules (which we call *Gorenstein-projective* Λ-modules to avoid any confusion). In the spirit of Auslander's observation, it is natural to ask whether there exists an Auslander-type correspondence for *GP-finite* Iwanaga-Gorenstein algebras (algebras Λ such that $\mathsf{GP}\,\Lambda$ is of finite type). In this thesis, we give an answer to this problem.

It is well-known that an algebra Γ is the GP-Auslander-algebra of some Iwanaga-Gorenstein algebra if and only if the global dimension of Γ is finite. In Theorem H, we classify GP-finite Iwanaga-Gorenstein algebras by algebras with finite global dimension together with their modules satisfying a certain condition, given as follows.

Definition 3.1.1. Let Γ be a two-sided noetherian ring and S a simple right Γ-module. We say that S satisfies the 2-*regular condition* if the following conditions are satisfied.

(1) $\mathsf{pd}\, S_\Gamma = 2$.
(2) $\mathrm{Ext}^i_\Gamma(S, \Gamma) = 0$ for $i = 0, 1$.
(3) $\mathrm{Ext}^2_\Gamma(S, \Gamma)$ is a simple left Γ-module.

This condition is satisfied by the simple modules over commutative regular local rings of dimension 2, and their non-commutative analogues, e.g. Artin-Schelter regular algebras, Calabi-Yau algebras, non-singular orders of dimension 2.

We interpret the 2-regular condition in terms of the translation quiver $Q(\Gamma)$. This is the usual quiver of Γ together with dotted arrows corresponding to the simple Γ-modules with the 2-regular condition (Definition 3.3.9). Using these concepts, we obtain the following classification, where a Γ-module is called *basic* if its indecomposable direct summands are pairwise non-isomorphic.

Theorem H (= Corollary 3.4.9). *There exists a bijection between the following for a field k.*

(1) *Morita equivalence classes of GP-finite Iwanaga-Gorenstein finite-dimensional k-algebras Λ.*
(2) *Equivalence classes of pairs (Γ, M), where Γ is a finite-dimensional k-algebra with finite global dimension and M is a basic semisimple Γ-module such that every simple summand of M satisfies the 2-regular condition and $\mathrm{Hom}_k(M, k) \cong \mathrm{Ext}^2_\Gamma(M, \Gamma)$ holds as left Γ-modules.*
(3) *Equivalence classes of pairs (Γ, X), where Γ is a finite-dimensional k-algebra with finite global dimension and X is a union of stable τ-orbits in $Q(\Gamma)$.*

For a pair (Γ, X) in (3), the corresponding algebra Λ in (1) is given by the endomorphism ring of projective Γ-modules which does not belong to τ-orbits in X.

As Example 3.1.2 illustrates, it provides a systematic method to construct a family of GP-finite Iwanaga-Gorenstein algebras.

To prove Theorem H, we need the notion of *exact categories*, which was introduced by Quillen [Qu] as a generalization of abelian categories. It provides an appropriate framework for a relative homological algebra and has a number of applications in many branches of mathematics, such as representation theory, algebraic topology and functional analysis. Let us explain why the use of exact structures is essential in our classification. In the classical situation, we can recover Λ (up to Morita equivalence) only from the additive structure of $\mathsf{mod}\, \Lambda$, while this fails to be true for $\mathsf{GP}\, \Lambda$; it often happens that $\mathsf{GP}\, \Lambda \simeq \mathsf{GP}\, \Gamma$ while Λ and Γ are not Morita equivalent. Even in this case, Λ can be recovered from the *exact structure* of $\mathsf{GP}\, \Lambda$, as the endomorphism ring of the progenerator of it. In Theorem H, the algebra Γ gives the additive structure of $\mathsf{GP}\, \Lambda$, while the module M or the set X gives the exact structure on it. In this setting, the quiver $Q(\Gamma)$ with dotted arrow X is nothing but the Auslander-Reiten quiver of $\mathsf{GP}\, \Lambda$.

Example 3.1.2. Let Γ be the algebra given by the quiver in Figure 1, where we identify two vertical arrows, with commutativity and zero relations indicated by dotted lines. Then Γ has finite global dimension and the translation quiver $Q(\Gamma)$ is the same as Figure 1. It has two stable τ-orbits A and B, hence we obtain four GP-finite Iwanaga-Gorenstein algebras, corresponding to the endomorphism rings of vertices which does not belong to X. Table 1 is an explicit calculation of all GP-finite Iwanaga-Gorenstein algebras Λ such that $\mathsf{GP}\, \Lambda$ is equivalent to $\mathsf{proj}\, \Gamma$.

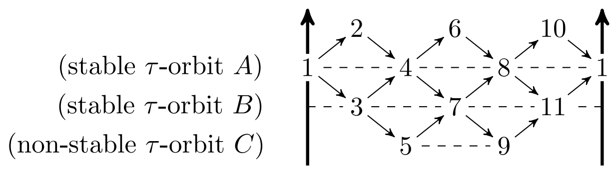


Figure 1. The quiver of Γ (or equivalently, $\mathsf{proj}\, \Gamma$).

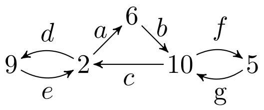
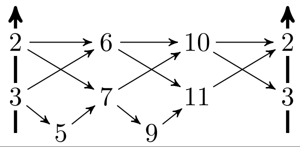
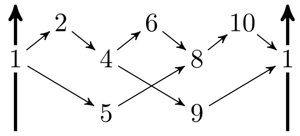
X	Quivers for Λ	Relations for Λ
$\varnothing$	Γ in Figure 1	The same relation as Γ.
$A \cup B$		$cba = ed$, $bac = gf$, $fb = ae = acb = dc = fg = cg = de = 0$.
A		Obvious commutative relations, and the zero relations of $5 \to 7 \to 9$, two paths of length two from 11, two paths of length two to 3.
B		Obvious commutative relations, and the zero relations of $5 \to 8 \to 10 \to 1$, $1 \to 2 \to 4 \to 9$, two paths of length three and four from 8 to 5 and 4, two paths of length two and three from 9 to 5 and 4.

TABLE 1. All CM-finite Iwanaga-Gorenstein algebras with its CM category $\mathsf{proj}\,\Gamma$.

To deal with exact structures, we mainly work with idempotent complete additive categories instead of categories of finite type. For such a category $\mathcal{E}$, we classify all exact structures on $\mathcal{E}$ in terms of its functor category $\mathsf{mod}\,\mathcal{E}$. The precise statement is the following, where $\mathcal{C}_2(\mathcal{E})$ is the category of $\mathcal{E}$-modules whose projective dimension and grade are equal to 2.

THEOREM I (= Theorem 3.2.7). *Let $\mathcal{E}$ be an idempotent complete additive category. Then there exists a bijection between the following two classes.*

(1) *Exact structures on $\mathcal{E}$.*
(2) *Subcategories $\mathcal{D}$ of $\mathcal{C}_2(\mathcal{E})$ satisfying the following conditions.*
  (a) *$\mathcal{D}$ is a Serre subcategory of $\mathsf{mod}\,\mathcal{E}$.*
  (b) $\mathrm{Ext}^2_{\mathcal{E}}(\mathcal{D}, \mathcal{E})$ *is a Serre subcategory of* $\mathsf{mod}\,\mathcal{E}^{\mathrm{op}}$.

It is surprising to us that such a general description of all exact structures is available. For example, the existence theorem of the largest exact structure due to [Ru2] easily follows from Theorem I in the case of idempotent complete categories.

We apply Theorem I to exact categories of finite type, which is our motivating object. When $\mathcal{E}$ is a Hom-finite idempotent complete k-category of finite type, we show that exact structures on $\mathcal{E}$ bijectively correspond to sets of simple Γ-modules satisfying the 2-regular condition, and sets of dotted arrows of $Q(\Gamma)$, where Γ is the Auslander algebra of $\mathcal{E}$ (Theorem 3.3.7 and Corollary 3.3.10). We apply this result to obtain Theorem H, and more generally, an Auslander correspondence for cotilting modules U such that ${}^{\perp}U$ is of finite type (Theorem 3.4.8). As another application, we obtain the following Auslander correspondence for representation-finite R-orders in case $\dim R \geq 2$, which improves [Iy4, Theorem 4.2.3].

THEOREM J (= Corollary 3.4.11). *Let R be a complete Cohen-Macaulay local ring with* $\dim R = d \geq 2$. *Then there exists a bijection between the following.*

(1) *Morita equivalence classes of R-orders Λ such that $\mathsf{CM}\,\Lambda$ is of finite type.*
(2) *Equivalence classes of pairs (Γ, e), where Γ is a noetherian R-algebra and e is an idempotent of Γ such that the following conditions are satisfied.*
  (a) $\mathrm{gl.dim}\,\Gamma = d$.
  (b) *Γe is maximal Cohen-Macaulay as an R-module.*
  (c) *$\underline{\Gamma} := \Gamma/\Gamma e\Gamma$ is of finite length over R.*
  (d) *$\underline{\Gamma}/(\mathrm{rad}\,\underline{\Gamma})$ is a direct sum of simple Γ-modules satisfying the 2-regular condition.*

Also, we investigate the Grothendieck group $\mathsf{K}_0(\mathcal{E})$ of an exact category $\mathcal{E}$ of finite type, and show that the relation of $\mathsf{K}_0(\mathcal{E})$ is generated by AR conflations under some mild conditions (Corollary 3.3.18). This unifies several known results by [AR2, But, Yo].

This chapter is organized as follows. In Section 2, we state and prove our main classification. In Section 3, we investigate exact categories of finite type. In Section 4, we apply our previous results to the study of the representation theory of algebras.

3.2. Classifying exact structures via Serre subcategories

In this section, we give a bijection between exact structures on an idempotent complete additive category $\mathcal{E}$ and subcategories of $\mathsf{mod}\,\mathcal{E}$ satisfying certain "2-regular condition." *Throughout this section, we always assume that $\mathcal{E}$ is an additive category.*

Recall that an additive category $\mathcal{E}$ is *idempotent complete* if every morphism $e : X \to X$ in $\mathcal{E}$ satisfying $e^2 = e$ has a kernel, or equivalently, a cokernel. For example, every subcategory of an abelian category which is closed under direct sums and direct summands are idempotent complete.

3.2.1. Preliminaries on functor categories.

Our strategy to investigate exact structures on $\mathcal{E}$ is to study the *module category* $\mathsf{Mod}\,\mathcal{E}$ over $\mathcal{E}$. Let us recall related definitions.

For an additive category $\mathcal{E}$, a *right $\mathcal{E}$-module* M is a contravariant additive functor $M : \mathcal{E}^{\mathrm{op}} \to \mathcal{A}b$ from $\mathcal{E}$ to the category of abelian groups $\mathcal{A}b$. We denote by $\mathsf{Mod}\,\mathcal{E}$ the category of right $\mathcal{E}$-modules, and morphisms in $\mathsf{Mod}\,\mathcal{E}$ are natural transformations between them. This category $\mathsf{Mod}\,\mathcal{E}$ is an abelian category with enough projectives, and projective objects are precisely direct summands of (possibly infinite) direct sums of representable functors. For simplicity, we put $P_X := \mathcal{E}(-, X) \in \mathsf{Mod}\,\mathcal{E}$ and $P^X := \mathcal{E}(X, -) \in \mathsf{Mod}\,\mathcal{E}^{\mathrm{op}}$ for any X in $\mathcal{E}$. Note that $P_{(-)} : \mathcal{E} \to \mathsf{Mod}\,\mathcal{E}$ gives the Yoneda embedding.

We say that a $\mathcal{E}$-module M is *finitely generated* if there exists an epimorphism $P_X \twoheadrightarrow M$ for some X in $\mathcal{E}$. Throughout this chapter, we often use the fact that *$\mathcal{E}$ is idempotent complete if and only if the essential image of the Yoneda embedding $P_{(-)} : \mathcal{E} \to \mathsf{Mod}\,\mathcal{E}$ consists of all finitely generated projective $\mathcal{E}$-modules.* We denote by $\mathsf{mod}\,\mathcal{E}$ the category of finitely generated $\mathcal{E}$-modules. We also denote by $\mathsf{mod}_1\,\mathcal{E}$ the category of finitely presented $\mathcal{E}$-modules, that is, the modules M such that there exists an exact sequence $P_X \to P_Y \to M \to 0$ for some X, Y in $\mathcal{E}$.

We define a contravariant functor $\mathrm{Hom}_{\mathcal{E}}(-, \mathcal{E}) : \mathsf{Mod}\,\mathcal{E} \to \mathsf{Mod}\,\mathcal{E}^{\mathrm{op}}$ by the following way: For M in $\mathsf{Mod}\,\mathcal{E}$, the left $\mathcal{E}$-module $\mathrm{Hom}_{\mathcal{E}}(M, \mathcal{E}) : \mathcal{E} \to \mathcal{A}b$ is the composition of the Yoneda embedding $\mathcal{E} \to \mathsf{Mod}\,\mathcal{E}$ and $(\mathsf{Mod}\,\mathcal{E})(M, -) : \mathsf{Mod}\,\mathcal{E} \to \mathcal{A}b$. Note that $\mathrm{Hom}_{\mathcal{E}}(-, \mathcal{E})$ is a left exact functor satisfying $\mathrm{Hom}_{\mathcal{E}}(P_{(-)}, \mathcal{E}) \simeq P^{(-)}$. We denote by $\mathrm{Ext}^i_{\mathcal{E}}(-, \mathcal{E}) : \mathsf{Mod}\,\mathcal{E} \to \mathsf{Mod}\,\mathcal{E}^{\mathrm{op}}$ the i-th right derived functor of $\mathrm{Hom}_{\mathcal{E}}(-, \mathcal{E})$.

Using these concepts, we can interpret kernel-cokernel pairs in terms of modules over $\mathcal{E}$. Here we say that a complex $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{E}$ is a *kernel-cokernel pair* if f is a kernel of g and g is a cokernel of f.

PROPOSITION 3.2.1. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a complex in $\mathcal{E}$. Put $M := \mathrm{Coker}(P_g) \in \mathsf{Mod}\,\mathcal{E}$. Then the following hold.*

(1) *g is an epimorphism in $\mathcal{E}$ if and only if $\mathrm{Hom}_{\mathcal{E}}(M, \mathcal{E}) = 0$.*
(2) *$X \xrightarrow{f} Y \xrightarrow{g} Z$ is a kernel-cokernel pair if and only if the following conditions are satisfied.*
    (a) *$0 \to P_X \xrightarrow{P_f} P_Y \xrightarrow{P_g} P_Z \to M \to 0$ is exact.*
    (b) *$\mathrm{Ext}^i_{\mathcal{E}}(M, \mathcal{E}) = 0$ for $i = 0, 1$.*

PROOF. (1) We have an exact sequence $0 \to \mathrm{Hom}_{\mathcal{E}}(M, \mathcal{E}) \to P^Z \xrightarrow{P^g} P^Y$. Thus g is an epimorphism if and only if P^g is a monomorphism if and only if $\mathrm{Hom}_{\mathcal{E}}(M, \mathcal{E}) = 0$.

(2) The condition (a) is equivalent to that f is a kernel of g. Under (a), the condition (b) is equivalent to that $0 \to P^Z \xrightarrow{P^g} P^Y \xrightarrow{P^f} P^X$ is exact, which holds precisely when g is a cokernel of f. □

We need the following technical lemma later.

LEMMA 3.2.2. *Suppose that $\mathcal{E}$ is idempotent complete and there exists an exact sequence*

$$0 \to P_X \to P_Y \to P_Z \to M \to 0$$

in $\mathsf{Mod}\,\mathcal{E}$. For any morphism $h : B \to C$ with $\mathrm{Coker}(P_h) \cong M$, there exists an object A in $\mathcal{E}$ such that $\mathrm{Ker}\, P_h \cong P_A$.

PROOF. Schanuel's lemma shows that $\mathrm{Ker}\, P_h \oplus P_Y \oplus P_C \cong P_X \oplus P_B \oplus P_Z$, which clearly implies that $\mathrm{Ker}\, P_h$ is finitely generated projective. Since $\mathcal{E}$ is idempotent complete, the assertion holds. □

3.2.2. Construction of maps. First we fix some notations which we need to describe our main theorem. The following observation follows immediately from Proposition 3.2.1.

LEMMA 3.2.3. *For an object M in $\mathsf{Mod}\,\mathcal{E}$, the following are equivalent.*

(1) *There exists a kernel-cokernel pair $X \to Y \xrightarrow{g} Z$ in $\mathcal{E}$ such that M is isomorphic to $\mathrm{Coker}(P_g)$.*
(2) *There exists an exact sequence $0 \to P_X \to P_Y \to P_Z \to M \to 0$ in $\mathsf{Mod}\,\mathcal{E}$ and $\mathrm{Ext}^i_{\mathcal{E}}(M, \mathcal{E}) = 0$ for $i = 0, 1$.*

We denote by $\mathcal{C}_2(\mathcal{E})$ the subcategory of $\mathsf{Mod}\,\mathcal{E}$ consisting of $\mathcal{E}$-modules satisfying the above equivalent conditions. This class of modules play an indispensable role throughout this chapter.

LEMMA 3.2.4. *The category $\mathcal{C}_2(\mathcal{E})$ is closed under direct summands in $\mathsf{Mod}\,\mathcal{E}$.*

PROOF. We denote by $\mathsf{mod}_2\,\mathcal{E}$ the subcategory of $\mathsf{Mod}\,\mathcal{E}$ consisting of all objects M such that there exists an exact sequence

$$P_2 \to P_1 \to P_0 \to M \to 0$$

in $\mathsf{Mod}\,\mathcal{E}$, where each P_i is finitely generated projective.

In [Ki, Lemma 2.5(a)], it was shown that $\mathsf{mod}_2\,\mathcal{E}$ is closed under summands in $\mathsf{Mod}\,\mathcal{E}$. Note that $\mathcal{C}_2(\mathcal{E})$ is equal to the intersection of two subcategories of $\mathsf{mod}_2\,\mathcal{E}$:

(1) the subcategory consisting of all objects whose projective dimensions are at most 2 and
(2) the subcategory consisting of all objects M such that $\mathrm{Ext}^i_{\mathcal{E}}(M, \mathcal{E}) = 0$ for $i = 0, 1$.

Since these subcategories are closed under direct summands in $\mathsf{Mod}\,\mathcal{E}$, so is $\mathcal{C}_2(\mathcal{E})$. □

LEMMA 3.2.5. *The functor $\mathrm{Ext}^2_{\mathcal{E}}(-, \mathcal{E})$ induces a duality of exact categories $\mathcal{C}_2(\mathcal{E}) \to \mathcal{C}_2(\mathcal{E}^{\mathrm{op}})$.*

PROOF. The same proof as in [Iy2, 6.2(1)] applies here. □

By definition, the category $\mathcal{C}_2(\mathcal{E})$ is closely related to kernel-cokernel pairs of $\mathcal{E}$. Indeed, we have two maps between them, as we shall construct below.

Suppose that F is a class of kernel-cokernel pairs in $\mathcal{E}$. We say that a complex $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an *F-exact* if it belongs to F. In this case, we say that f is an *F-monomorphism* and that g is an *F-epimorphism.*

DEFINITION 3.2.6. (1) Let $\mathcal{D}$ be a subcategory of $\mathcal{C}_2(\mathcal{E})$. We denote by $F(\mathcal{D})$ the class of all complexes $X \xrightarrow{f} Y \xrightarrow{g} Z$ which satisfy the following condition:
There exists an exact sequence $0 \to P_X \xrightarrow{P_f} P_Y \xrightarrow{P_g} P_Z \to M \to 0$ in $\mathsf{Mod}\,\mathcal{E}$ such that M belongs to $\mathcal{D}$.

(2) Let F be a class of kernel-cokernel pairs in $\mathcal{E}$. We denote by $\mathcal{D}(F)$ the subcategory of $\mathsf{Mod}\,\mathcal{E}$ consisting of all objects M which satisfy the following condition:
There exists an F-exact complex $X \xrightarrow{f} Y \xrightarrow{g} Z$ satisfying $M \cong \mathrm{Coker}(P_g)$.

The category $\mathcal{D}(F)$ can be seen as the category of *contravariant defects* of F-exact sequences, in the sense of Auslander (see [ARS, IV.4]). Note that by Lemma 3.2.3, the following hold.

- Every complex $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $F(\mathcal{D})$ is a kernel-cokernel pair.
- Every object M in $\mathcal{D}(F)$ is in $\mathcal{C}_2(\mathcal{E})$.

3.2.3. Main theorem. In this subsection, we will state our main theorem and give a proof. Recall that a *Serre subcategory* of an exact category $\mathcal{E}$ is an additive subcategory $\mathcal{D}$ of $\mathcal{E}$ such that for all conflations $L \rightarrowtail M \twoheadrightarrow N$ in $\mathcal{E}$, the object M belongs to $\mathcal{D}$ if and only if both L and N belong to $\mathcal{D}$.

THEOREM 3.2.7. *Let $\mathcal{E}$ be an idempotent complete additive category. Then there exist mutually inverse bijections between the following two classes.*

(1) *Exact structures F on $\mathcal{E}$.*
(2) *Subcategories $\mathcal{D}$ of $\mathcal{C}_2(\mathcal{E})$ satisfying the following conditions.*
  (a) *$\mathcal{D}$ is a Serre subcategory of $\operatorname{mod}\mathcal{E}$.*
  (b) *$\operatorname{Ext}^2_{\mathcal{E}}(\mathcal{D}, \mathcal{E})$ is a Serre subcategory of $\operatorname{mod}\mathcal{E}^{\mathrm{op}}$.*

The map from (1) to (2) is given by $F \mapsto \mathcal{D}(F)$ and from (2) to (1) by $\mathcal{D} \mapsto F(\mathcal{D})$ (see Definition 3.2.6).

To prove this, we first show that the maps in Definition 3.2.6 induces a one-to-one correspondence between wider classes than in Theorem 3.2.7.

PROPOSITION 3.2.8. *Let $\mathcal{E}$ be an additive category. Then the maps in Definition 3.2.6 induce mutually inverse bijections between the following two classes.*

(1) *Classes of kernel-cokernel pairs F in $\mathcal{E}$ satisfying the following conditions.*
  (a) *F is closed under homotopy equivalences of complexes.*
  (b) *F is closed under direct sums of complexes.*
  (c) *F is closed under direct summands of complexes.*
  (d) *F is not empty.*
(2) *Subcategories $\mathcal{D}$ of $\mathcal{C}_2(\mathcal{E})$ satisfying the following condition.*
  (a) *$\mathcal{D}$ is closed under direct sums.*
  (b) *$\mathcal{D}$ is closed under direct summands.*
  (c) *$\mathcal{D}$ is not empty.*

PROOF. First we see that the maps in Definition 3.2.6 induces well-defined maps between (1) and (2).

(1) → (2): Suppose that F is a class of kernel-cokernel pairs in $\mathcal{E}$ satisfying the conditions of (1). We will see that $\mathcal{D}(F)$ satisfies the conditions of (2). Clearly $\mathcal{D}(F)$ satisfies (a) and (c) by the conditions (1)(b) and (1)(d) respectively. Hence it suffices to show that $\mathcal{D}$ is closed under direct summands.

Suppose that $M_1 \oplus M_2$ is in $\mathcal{D}(F)$. Then $M_1 \oplus M_2$ is an object of $\mathcal{C}_2(\mathcal{E})$, which is closed under summands by Lemma 3.2.4. Therefore both M_1 and M_2 are in $\mathcal{C}_2(\mathcal{E})$. This gives exact sequences

$$0 \to P_{X_i} \xrightarrow{P_{f_i}} P_{Y_i} \xrightarrow{P_{g_i}} P_{Z_i} \to M_i \to 0$$

for some kernel-cokernel pairs $X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i$ in $\mathcal{E}$ with $i = 1, 2$. By taking the direct sum of these two complexes, we obtain a kernel-cokernel pair

$$X_1 \oplus X_2 \xrightarrow{f_1 \oplus f_2} Y_1 \oplus Y_2 \xrightarrow{g_1 \oplus g_2} Z_1 \oplus Z_2. \tag{3.2.1}$$

Thereby we obtain the following exact sequence.

$$0 \to P_{X_1 \oplus X_2} \xrightarrow{P_{f_1} \oplus P_{f_2}} P_{Y_1 \oplus Y_2} \xrightarrow{P_{g_1} \oplus P_{g_2}} P_{Z_1 \oplus Z_2} \to M_1 \oplus M_2 \to 0$$

On the other hand, since $M_1 \oplus M_2$ is in $\mathcal{D}(F)$, there exists an F-exact complex

$$X \xrightarrow{f} Y \xrightarrow{g} Z \tag{3.2.2}$$

in $\mathcal{E}$ such that

$$0 \to P_X \xrightarrow{P_f} P_Y \xrightarrow{P_g} P_Z \to M_1 \oplus M_2 \to 0$$

is exact. It is standard that two projective resolutions are homotopy equivalent to each other. Thus the Yoneda lemma implies that (3.2.1) and (3.2.2) are homotopy equivalent to each other as three-term complexes in $\mathcal{E}$. Because F is closed under homotopy equivalences, (3.2.1) is F-exact.

By the condition (1)(c), we conclude that each g_i is an F-epimorphism, which shows that M_1 and M_2 are in $\mathcal{D}(F)$.

(2) $\to$ (1): This is immediate and we leave the details to the reader.

Now we will show that the maps in Definition 3.2.6 are in fact inverse to each other. It is easy to see that $\mathcal{D} = \mathcal{D}(F(\mathcal{D}))$ in general. (Note that *subcategories* are always assumed to be closed under isomorphisms.)

Suppose F is a class of kernel-cokernel pairs in $\mathcal{E}$ satisfying the conditions of (1). Put $\mathcal{D} := \mathcal{D}(F)$. Then clearly we have $F(\mathcal{D}) \supset F$. We will prove $F(\mathcal{D}) \subset F$.

Suppose that we have an $F(\mathcal{D})$-exact sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z. \tag{3.2.3}$$

Put $M := \operatorname{Coker}(P_g)$. Then M is contained in $\mathcal{D}$. Thus there exists an F-exact sequence

$$X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \tag{3.2.4}$$

such that

$$0 \to P_{X'} \xrightarrow{P_{f'}} P_{Y'} \xrightarrow{P_{g'}} P_{Z'} \to M \to 0$$

is exact. Since the Yoneda embedding of (3.2.3) also yields a projective resolution for M, it follows that (3.2.3) and (3.2.4) are homotopy equivalent to each other. Therefore (3.2.3) is an F-exact sequence, since F is closed under homotopy equivalences. □

Now we are in the position to prove Theorem 3.2.7. First we recall the axiom of exact categories following [Bü, Definition 2.1].

DEFINITION 3.2.9. Let F be a class of kernel-cokernel pairs in $\mathcal{E}$ which is closed under isomorphisms. Consider the following conditions.

(Ex0) For all objects $X \in \mathcal{E}$, the complex $0 \to X = X$ is F-exact.
(Ex1) The class of F-epimorphisms are closed under compositions.
(Ex2) The pullback of an F-epimorphisms along an arbitrary morphism exists and yields an F-epimorphism.

We say that $(\mathcal{E}, F)$ *is an exact category* if F satisfies (Ex0)-(Ex2) and $(\text{Ex0})^{\text{op}}$-$(\text{Ex2})^{\text{op}}$. In this case, F is called an *exact structure* on $\mathcal{E}$, and we just call $\mathcal{E}$ an exact category if F is clear from context.

In an exact category $(\mathcal{E}, F)$, we use the terminologies *conflations*, *deflations* and *inflations* instead of F-exact sequences, F-epimorphisms and F-monomorphisms respectively.

LEMMA 3.2.10. *Let $(\mathcal{E}, F)$ be an exact category. Then F satisfies all the conditions of Proposition 3.2.8(1).*

PROOF. The condition (1)(d) is satisfied by definition. The condition (1)(a) can be proved by the Gabriel-Quillen embedding. We refer the reader to [Bü, Proposition 2.9] and [Bü, Corollary 2.18] for the conditions (1)(b) and (1)(c) respectively. □

The following proposition is a technical part of the proof of Theorem 3.2.7.

PROPOSITION 3.2.11. *Suppose that $\mathcal{E}$ is idempotent complete and a class F satisfies the conditions of Proposition 3.2.8* (1). *Put $\mathcal{D} := \mathcal{D}(F)$. Then the following are equivalent.*

(1) *F is an exact structure on $\mathcal{E}$.*
(2) *$\mathcal{D}$ is a Serre subcategory of* $\mathsf{mod}\,\mathcal{E}$ *and* $\operatorname{Ext}^2_{\mathcal{E}}(\mathcal{D}, \mathcal{E})$ *is a Serre subcategory of* $\mathsf{mod}\,\mathcal{E}^{\text{op}}$.

PROOF. (1) $\Rightarrow$ (2): First we show that $\mathcal{D}$ is a Serre subcategory of $\mathsf{mod}\,\mathcal{E}$. Let

$$0 \to M_1 \to M \xrightarrow{k} M_2 \to 0$$

be a short exact sequence in $\mathsf{mod}\,\mathcal{E}$.

Assume that M_1 and M_2 are in $\mathcal{D}$. We will see that $M \in \mathcal{D}$. By the definition of $\mathcal{D}$, there exists an F-exact sequence $X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i$ such that

$$0 \to P_{X_i} \xrightarrow{P_{f_i}} P_{Y_i} \xrightarrow{P_{g_i}} P_{Z_i} \to M_i \to 0$$

is exact for each $i = 1, 2$. By the horseshoe lemma, we obtain an exact commutative diagram in $\mathsf{Mod}\,\mathcal{E}$ of the following form

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_{X_1} & \xrightarrow{P_{f_1}} & P_{Y_1} & \xrightarrow{P_{g_1}} & P_{Z_1} & \longrightarrow & M_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_{X_1 \oplus X_2} & \xrightarrow{P_f} & P_{Y_1 \oplus Y_2} & \xrightarrow{P_g} & P_{Z_1 \oplus Z_2} & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle k} & & \\
0 & \longrightarrow & P_{X_2} & \xrightarrow{P_{f_2}} & P_{Y_2} & \xrightarrow{P_{g_2}} & P_{Z_2} & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

where the columns except the right-most one are split exact. Since $P_{(-)} : \mathcal{E} \to \mathsf{Mod}\,\mathcal{E}$ is fully faithful by the Yoneda Lemma, we obtain the following commutative diagram

$$\begin{array}{ccccc}
X_1 & \overset{f_1}{\rightarrowtail} & Y_1 & \overset{g_1}{\twoheadrightarrow} & Z_1 \\
\downarrow & & \downarrow & & \downarrow \\
X_1 \oplus X_2 & \xrightarrow{f} & Y_1 \oplus Y_2 & \xrightarrow{g} & Z_1 \oplus Z_2 \\
\downarrow & & \downarrow & & \downarrow \\
X_2 & \overset{f_2}{\rightarrowtail} & Y_2 & \overset{g_2}{\twoheadrightarrow} & Z_2
\end{array}$$

in $\mathcal{E}$. The top and bottom rows are F-exact, each of the three columns is split exact and $gf = 0$. Thus we can apply 3×3 lemma in exact categories, see [Bü, Corollary 3.6]. Thus the middle row is also F-exact, which implies that M is in $\mathcal{D}$.

Next suppose that M is in $\mathcal{D}$. We first see that M_1 is in $\mathcal{D}$. We have an F-exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that $0 \to P_X \xrightarrow{P_f} P_Y \xrightarrow{P_g} P_Z \xrightarrow{h} M \to 0$ is exact. Since M_1 is in $\mathsf{mod}\,\mathcal{E}$, we have a surjection $P_W \to M_1 \to 0$ for some W in $\mathcal{E}$. This yields the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & P_W & \longrightarrow & M_1 & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle P_\varphi} & & \downarrow & & \\
0 & \longrightarrow & P_X & \xrightarrow{P_f} \; P_Y \; \xrightarrow{P_g} & P_Z & \xrightarrow{h} & M & \longrightarrow & 0
\end{array}$$

Since F is an exact structure on $\mathcal{E}$, there exists a pullback diagram

$$\begin{array}{ccccc}
X & \overset{a}{\rightarrowtail} & E & \overset{b}{\twoheadrightarrow} & W \\
\| & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \varphi} \\
X & \overset{f}{\rightarrowtail} & Y & \overset{g}{\twoheadrightarrow} & Z
\end{array} \tag{3.2.5}$$

in $\mathcal{E}$ where the top row is F-exact. By using the universal property of pullbacks, one can easily check that

$$0 \to P_X \xrightarrow{P_a} P_E \xrightarrow{P_b} P_W \to M_1 \to 0$$

is exact. Thus M_1 is in $\mathcal{D}$.

On the other hand, it is well-known that the complex $E \xrightarrow{{}^t[b,-c]} W \oplus Y \xrightarrow{[\varphi, g]} Z$ corresponding to the pullback square in (3.2.5) is F-exact (see e.g. [Bü, Proposition 2.12]). Moreover, one can easily check that

$$P_W \oplus P_Y \xrightarrow{[P_\varphi, P_g]} P_Z \xrightarrow{kh} M_2 \to 0$$

is exact, where kh is the composition $h : P_Z \twoheadrightarrow M$ and $k : M \twoheadrightarrow M_2$. Therefore N is in $\mathcal{D}$, which completes the proof that $\mathcal{D}$ is a Serre subcategory of $\mathsf{mod}\,\mathcal{E}$.

Note that an object M in $\operatorname{mod}\mathcal{E}^{\mathrm{op}}$ is contained in $\operatorname{Ext}^2_{\mathcal{E}}(\mathcal{D},\mathcal{E})$ if and only if there exists an F-exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{E}$ such that $0 \to P^Z \xrightarrow{P^g} P^Y \xrightarrow{P^f} P^X \to M \to 0$ is exact. Hence $\operatorname{Ext}^2_{\mathcal{E}}(\mathcal{D},\mathcal{E})$ is a Serre subcategory of $\operatorname{mod}\mathcal{E}^{\mathrm{op}}$ by the dual argument.

(2) $\Rightarrow$ (1): By duality, it suffices to show that F satisfies (Ex0)-(Ex2). Note that (Ex0) automatically holds by the condition (1)(d) in Proposition 3.2.8.

(Ex1) Let $A \xrightarrow{f} B \xrightarrow{g} C$ and $X \xrightarrow{h} C \xrightarrow{k} D$ be F-exact sequences. We show that kg is also an F-epimorphism. We have the following commutative diagram with exact rows

$$\begin{array}{ccccccccccc}
 & & & & & & 0 & & & & \\
 & & & & & & \downarrow & & & & \\
 & & & & & & P_X & = & P_X & & \\
 & & & & & & \downarrow{\scriptstyle P_h} & & \downarrow & & \\
0 & \longrightarrow & P_A & \xrightarrow{P_f} & P_B & \xrightarrow{P_g} & P_C & \longrightarrow & L & \longrightarrow & 0 \\
 & & & & \| & & \downarrow{\scriptstyle P_k} & & \downarrow & & \\
 & & & & P_B & \xrightarrow{P_{kg}} & P_D & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & N & = & N & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & &
\end{array}$$

where L and N are in $\mathcal{D}$ by the definition of $\mathcal{D}$. The right-most column is exact by diagram chasing. Since $\mathcal{D}$ is a Serre subcategory of $\operatorname{mod}\mathcal{E}$, we immediately have that M is in $\mathcal{D}$. In particular, M is contained in $\mathcal{C}_2(\mathcal{E})$. By Lemma 3.2.2, there exists a kernel-cokernel pair $Y \xrightarrow{l} B \xrightarrow{kg} D$ such that $0 \to P_Y \xrightarrow{P_l} P_B \xrightarrow{P_{kg}} P_D \to M \to 0$ is exact. Thus kg is an $F(\mathcal{D})$-epimorphism. Since $F(\mathcal{D}) = F$, the morphism kg is an F-epimorphism.

(Ex2) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be an F-exact sequence and $h : W \to Z$ an arbitrary morphism in $\mathcal{E}$. Then we have the commutative diagram

$$\begin{array}{ccccccccccc}
 & & & & & & & & 0 & & \\
 & & & & & & & & \downarrow & & \\
 & & & & & & P_W & \longrightarrow & L & \longrightarrow & 0 \\
 & & & & & & \downarrow{\scriptstyle P_h} & & \downarrow & & \\
0 & \longrightarrow & P_X & \xrightarrow{P_f} & P_Y & \xrightarrow{P_g} & P_Z & \xrightarrow{a} & M & \longrightarrow & 0 \\
 & & & & & & & & \downarrow{\scriptstyle b} & & \\
 & & & & & & & & N & & \\
 & & & & & & & & \downarrow & & \\
 & & & & & & & & 0 & &
\end{array}$$

where all the rows and columns are exact. Since M is in $\mathcal{D}$ by the definition, L and N are also in $\mathcal{D}$. In particular, N is contained in $\mathcal{C}_2(\mathcal{E})$. On the other hand, we have that $P_W \oplus P_Y \xrightarrow{[P_h, P_g]} P_Z \xrightarrow{ba} N \to 0$ is exact. Thus by Lemma 3.2.2, there exists an exact sequence

$$0 \longrightarrow P_E \xrightarrow{\left[\begin{smallmatrix} P_k \\ -P_l \end{smallmatrix}\right]} P_W \oplus P_Y \xrightarrow{[P_h, P_g]} P_Z \longrightarrow N \xrightarrow{ba} 0$$

in $\operatorname{Mod}\mathcal{E}$. It is standard that this exact sequence yields a pullback diagram

$$\begin{array}{ccc}
E & \xrightarrow{k} & W \\
\downarrow{\scriptstyle l} & & \downarrow{\scriptstyle h} \\
Y & \xrightarrow{g} & Z
\end{array} \tag{3.2.6}$$

in $\mathcal{E}$. Thus the existence of the pullback has been proved. By the universal property of the pullback (3.2.6), there exists a complex $X \xrightarrow{i} E \xrightarrow{k} W$ in $\mathcal{E}$ such that

$$0 \to P_X \xrightarrow{P_i} P_E \xrightarrow{P_k} P_W \to L \to 0$$

is exact. Thus the complex $X \xrightarrow{i} E \xrightarrow{k} W$ belongs to $F(\mathcal{D}) = F$, which implies that k is an F-epimorphism as desired. □

PROOF OF THEOREM 3.2.7. It is immediate from Proposition 3.2.11 and Lemma 3.2.10. □

The following description of projective objects in $\mathcal{E}$ in terms of $\mathcal{D}$ is needed later.

PROPOSITION 3.2.12. *Let $\mathcal{E}$ be an idempotent complete exact category and $\mathcal{D}$ the corresponding Serre subcategory of $\operatorname{mod}\mathcal{E}$ given in Theorem 3.2.7. Then for an object Z in $\mathcal{E}$, the following are equivalent.*

(1) *W is projective in $\mathcal{E}$.*
(2) *$M(W) = 0$ for all object $M \in \mathcal{D}$.*

PROOF. The category $\mathcal{D}$ consists of all functors M such that $P_Y \xrightarrow{P_g} P_Z \to M \to 0$ is exact for some deflation $g : Y \twoheadrightarrow Z$, so (2) is equivalent to that $P_Y(W) \xrightarrow{P_g(W)} P_Z(W)$ is surjective for every deflation g. This occurs if and only if W is a projective object in $\mathcal{E}$. □

Next let us consider the particular case when the simpler classification is available. This includes the case when $\mathcal{E}$ is abelian, or more generally, quasi-abelian (see [Ru1] for the detail).

LEMMA 3.2.13. *Let $\mathcal{E}$ be an idempotent complete category. Then the following are equivalent.*

(1) *$\mathcal{C}_2(\mathcal{E})$ and $\mathcal{C}_2(\mathcal{E}^{\mathrm{op}})$ are Serre subcategories of $\operatorname{mod}\mathcal{E}$ and $\operatorname{mod}\mathcal{E}^{\mathrm{op}}$ respectively.*
(2) *The class of all kernel-cokernel pairs in $\mathcal{E}$ defines an exact structure of $\mathcal{E}$.*

In this case, there exists a bijection between the following two classes.

(1) *Exact structures on $\mathcal{E}$.*
(2) *Serre subcategories of $\mathcal{C}_2(\mathcal{E})$.*

PROOF. Let F be the class of all kernel-cokernel pairs in $\mathcal{E}$. Then $\mathcal{D}(F) = \mathcal{C}_2(\mathcal{E})$ holds, thus Proposition 3.2.11 applies. The latter part is clear from Theorem 3.2.7. □

Now we will show that a more familiar description for $\mathcal{D}$ is available if $\mathcal{E}$ has enough projectives. First we recall the notation of the projectively stable category of an exact category. Let $\mathcal{E}$ be an exact category with enough projective objects. Denote by $[\mathcal{P}](X, Y)$ the set of all morphisms from X to Y which factor through projective objects in $\mathcal{E}$. Then $[\mathcal{P}]$ is a two-sided ideal of $\mathcal{E}$ and we denote by $\underline{\mathcal{E}} := \mathcal{E}/[\mathcal{P}]$, which we call the *projectively stable category*.

LEMMA 3.2.14. *Let $\mathcal{E}$ be an exact category with enough projectives and $\mathcal{D}$ the subcategory of $\operatorname{mod}\mathcal{E}$ corresponding to all conflations (see Definition 3.2.6). Then $\mathcal{D} \simeq \operatorname{mod}_1 \underline{\mathcal{E}}$ holds.*

PROOF. We have an embedding $\operatorname{mod}_1 \underline{\mathcal{E}} \to \operatorname{mod}\mathcal{E}$ and denote its essential image by $\mathcal{D}'$. We show that the following conditions are equivalent for an object M in $\operatorname{mod}\mathcal{E}$.

(1) $M \in \mathcal{D}$.
(2) There exist a deflation $g : Y \to Z$ in $\mathcal{E}$ and an exact sequence

$$\underline{\mathcal{E}}(-, Y) \xrightarrow{(-)\circ g} \underline{\mathcal{E}}(-, Z) \to M \to 0. \tag{3.2.7}$$

(3) $M \in \mathcal{D}'$.

(1) ⇔ (2): This is easily shown by diagram chasing.
(2) ⇒ (3): Clear.
(3) ⇒ (2): Suppose that M is in $\mathcal{D}'$. Since M is finitely presented $\underline{\mathcal{E}}$-module, we have an exact sequence of the form (3.2.7) for some $g : Y \to Z$. By assumption, there exists a deflation $\varphi : P \twoheadrightarrow Z$ for some projective object P. Using this, we may replace g by $[g, \varphi] : Y \oplus P \to Z$, which is a deflation. This proves the claim. □

Combining Lemma 3.2.13 and 3.2.14, we immediately obtain the following conclusion, which gives a generalization of Buan's result [Bua, Proposition 3.3.2], where $\mathcal{E}$ was assumed to be $\mathsf{mod}\,\Lambda$ for some artin algebra Λ.

COROLLARY 3.2.15. *Let $\mathcal{E}$ be an idempotent complete additive category such that the class of all kernel-cokernel pairs defines an exact structure with enough projectives (e.g. abelian category with enough projectives). Denote by $\underline{\mathcal{E}}$ the projectively stable category in this exact structure. Then there exists a bijection between the following two classes.*

(1) *Exact structures on $\mathcal{E}$.*
(2) *Serre subcategories of $\mathsf{mod}_1\,\underline{\mathcal{E}}$.*

3.3. Exact categories of finite type

We use our previous results to classify idempotent complete exact categories of finite type. For an additive category $\mathcal{E}$, an object M is called an *additive generator* of $\mathcal{E}$ if $\mathsf{add}\,M = \mathcal{E}$ holds, where $\mathsf{add}\,M$ is the subcategory of $\mathcal{E}$ consisting of all direct summands of finite direct sums of M. We call that an additive category $\mathcal{E}$ is *of finite type* if it has an additive generator.

We say that an additive category $\mathcal{E}$ is a *Krull-Schmidt category* if every object in $\mathcal{E}$ is a finite direct sum of indecomposable objects whose endomorphism rings are local. For the basics of Krull-Schmidt categories, we refer the reader to [Kr]. An additive category $\mathcal{E}$ is Krull-Schmidt if and only if $\mathcal{E}$ is idempotent complete and $\mathrm{End}_{\mathcal{E}}(X)$ is semiperfect for every $X \in \mathcal{E}$. If $\mathcal{E}$ is Krull-Schmidt, then $\mathcal{E}$ is of finite type precisely when $\mathcal{E}$ has finitely many indecomposables up to isomorphism.

Let M be an additive generator of $\mathcal{E}$ and $\Gamma := \mathrm{End}_{\mathcal{E}}(M)$. Then we have a fully faithful functor $\mathcal{E}(M,-) : \mathcal{E} \to \mathsf{Mod}\,\Gamma$ to the category of right Γ-modules. Its essential image coincides with the category $\mathsf{proj}\,\Gamma$ of finitely generated projective Γ-modules precisely when $\mathcal{E}$ is idempotent complete. Thus, when we deal with an idempotent complete additive category of finite type, we may assume that $\mathcal{E} = \mathsf{proj}\,\Gamma$ for some ring Γ. To avoid technical complications, *we restrict our attention to the case when Γ is a noetherian ring.* Note that $\mathsf{proj}\,\Gamma$ is Krull-Schmidt if and only if Γ is semiperfect.

3.3.1. Basic properties. We start with reformulating our result in Section 3 in terms of the ring Γ, where we use the same notation $\mathcal{C}_2(\Gamma)$:

$$\mathcal{C}_2(\Gamma) := \{M \in \mathsf{mod}\,\Gamma \mid \mathsf{pd}\,M_\Gamma \leq 2 \text{ and } \mathrm{Ext}^i_\Gamma(M,\Gamma) = 0 \text{ for } i = 0,1\}.$$

Note that every non-zero module M in $\mathcal{C}_2(\Gamma)$ satisfies $\mathsf{pd}\,M_\Gamma = 2$.

THEOREM 3.3.1. *Let Γ be a noetherian ring and put $\mathcal{E} := \mathsf{proj}\,\Gamma$. Then there exists a bijection between the following two classes.*

(1) *Exact structures F on $\mathcal{E}$.*
(2) *Subcategories $\mathcal{D}$ of $\mathcal{C}_2(\Gamma)$ satisfying the following condition.*
   (a) *$\mathcal{D}$ is a Serre subcategory of $\mathsf{mod}\,\Gamma$.*
   (b) *$\mathrm{Ext}^2_\Gamma(\mathcal{D},\Gamma)$ is a Serre subcategory of $\mathsf{mod}\,\Gamma^{\mathrm{op}}$.*

The correspondence is given as follows.

- *For a given $\mathcal{D}$, a complex $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{E}$ is in F if and only if $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to M \to 0$ is an exact sequence in $\mathsf{mod}\,\Gamma$ with some M in $\mathcal{D}$.*
- *For a given F, a Γ-module $M \in \mathsf{mod}\,\Gamma$ is in $\mathcal{D}$ if and only if there exists a deflation g in $\mathcal{E}$ with $M \cong \mathrm{Coker}\,g$.*

When we deal with exact structures on $\mathsf{proj}\,\Gamma$, the 2-regular condition for simple modules (see Definition 3.1.1) are quite essential. For the set $\mathcal{S}$ of simple Γ-modules, we denote by $\mathsf{Filt}\,\mathcal{S}$ the subcategory of $\mathsf{mod}\,\Gamma$ consisting of all modules M such that M has finite length and all composition factors of M are contained in $\mathcal{S}$. The following observation is useful.

LEMMA 3.3.2. *Let Γ be a noetherian ring and $\mathcal{S}$ a set of simple Γ-modules. Then the following are equivalent.*

(1) *Every module in $\mathcal{S}$ satisfies the 2-regular condition.*

(2) $\mathcal{D} := \mathsf{Filt}\,\mathcal{S}$ *is contained in* $\mathcal{C}_2(\Gamma)$ *and satisfies the conditions of Theorem 3.3.1(2).*
(3) *There exists a subcategory* $\mathcal{D}$ *of* $\mathcal{C}_2(\Gamma)$ *containing* $\mathcal{S}$ *such that* $\mathcal{D}$ *satisfies the conditions of Theorem 3.3.1(2).*

PROOF. (1) ⇒ (2): Clearly $S \in \mathcal{C}_2(\Gamma)$ holds for every $S \in \mathcal{S}$. Since $\mathcal{C}_2(\Gamma)$ is closed under extensions in $\mathsf{mod}\,\Gamma$, we have that $\mathsf{Filt}\,S$ is a Serre subcategory of $\mathsf{mod}\,\Gamma$ contained in $\mathcal{C}_2(\Gamma)$. Since $\mathrm{Ext}^2_\Gamma(-,\Gamma)$ gives an exact duality $\mathcal{C}_2(\Gamma) \simeq \mathcal{C}_2(\Gamma^{\mathrm{op}})$ and $\mathrm{Ext}^2_\Gamma(S,\Gamma)$ is simple for every $S \in \mathcal{S}$, we have $\mathrm{Ext}^2_\Gamma(\mathcal{D},\Gamma) = \mathsf{Filt}\,\mathrm{Ext}^2_\Gamma(\mathcal{S},\Gamma)$. Thus this category is a Serre subcategory of $\mathsf{mod}\,\Gamma^{\mathrm{op}}$, which shows that $\mathsf{Filt}\,\mathcal{S}$ satisfies the conditions of Theorem 3.3.1(2).

(2) ⇒ (3): Obvious.

(3) ⇒ (1): Let S be a simple module contained in $\mathcal{S}$. It follows from $S \in \mathcal{C}_2(\Gamma)$ that $\mathsf{pd}\,S_\Gamma = 2$ and $\mathrm{Ext}^i_\Gamma(S,\Gamma) = 0$ for $i = 0, 1$. Notice that $\mathrm{Ext}^2_\Gamma(-,\Gamma)$ gives the duality between $\mathcal{D}$ and $\mathrm{Ext}^2_\Gamma(\mathcal{D},\Gamma)$, both of which are abelian categories. Thus it is immediate that $\mathrm{Ext}^2_\Gamma(S,\Gamma)$ is a simple object in the abelian category $\mathrm{Ext}^2_\Gamma(\mathcal{D},\Gamma)$, which implies that $\mathrm{Ext}^2_\Gamma(S,\Gamma)$ is a simple left Γ-module. Therefore S satisfies the 2-regular condition. □

Every additive category admits a trivial exact structure whose conflations are split exact sequences. We have the following criterion on the existence of non-trivial exact structures on $\mathsf{proj}\,\Gamma$.

PROPOSITION 3.3.3. *Let* Γ *be a noetherian ring. Then* $\mathsf{proj}\,\Gamma$ *admits a non-trivial exact structure if and only if there exists a simple* Γ*-module satisfying the* 2*-regular condition.*

PROOF. Suppose that there exists a simple Γ-module S satisfying the 2-regular condition. Then Lemma 3.3.2 and Theorem 3.3.1 imply the existence of a non-trivial exact structure on $\mathsf{proj}\,\Gamma$.

Conversely, suppose that $\mathsf{proj}\,\Gamma$ has a non-trivial exact structure. By Theorem 3.3.1, we have a non-zero Serre subcategory $\mathcal{D}$ of $\mathsf{mod}\,\Gamma$. Since any non-zero Γ-module in $\mathsf{mod}\,\Gamma$ has a surjection onto simple Γ-module, $\mathcal{D}$ contains at least one simple Γ-module S. Then Lemma 3.3.2 implies that S satisfies the 2-regular condition. □

EXAMPLE 3.3.4. Let R be a commutative noetherian local ring. Then Proposition 3.3.3 implies that there exists a non-trivial exact structure on $\mathsf{proj}\,R$ if and only if R is a regular local ring of dimension 2. In this case, $\mathsf{proj}\,R$ has exactly one non-trivial exact structures. In this exact structure, $P_2 \xrightarrow{f} P_1 \xrightarrow{g} P_0$ is a conflation if and only if f is a kernel of g and $\mathrm{Coker}(g)$ is of finite length over R.

3.3.2. Admissible exact structures. We introduce a nice class of exact categories which is completely controlled by the simple modules satisfying the 2-regular condition. For a ring Γ, we denote by $\mathsf{f.l.}\,\Gamma$ the subcategory of $\mathsf{mod}\,\Gamma$ consisting of Γ-modules of finite length. Similarly, we denote by $\mathsf{f.l.}\,\mathcal{E}$ the category consisting of $\mathcal{E}$-modules of finite length.

DEFINITION 3.3.5. Let $(\mathcal{E}, F)$ be an exact category and $\mathcal{D}$ the subcategory of $\mathsf{mod}\,\mathcal{E}$ corresponding to F under Theorem 3.2.7. We say that F is admissible if $\mathcal{D} \subset \mathsf{f.l.}\,\mathcal{E}$ holds.

Let $(\mathsf{proj}\,\Gamma, F)$ be an exact category for a noetherian ring Γ and $\mathcal{D}$ the subcategory of $\mathsf{mod}\,\Gamma$ corresponding to F under Theorem 3.3.1. Then it is clear that F is admissible if and only if $\mathcal{D} \subset \mathsf{f.l.}\,\Gamma$ holds. Therefore, for an artinian ring Γ, every exact structure on $\mathsf{proj}\,\Gamma$ is admissible.

We show that the admissibility is left-right symmetric under some assumptions.

PROPOSITION 3.3.6. *Let* $(\mathsf{proj}\,\Gamma, F)$ *be an exact category for a noetherian ring* Γ*. Then it is admissible if and only if* $(\mathsf{proj}\,\Gamma^{\mathrm{op}}, F^{\mathrm{op}})$ *is admissible.*

PROOF. Let $\mathcal{D}$ be the subcategory of $\mathsf{mod}\,\Gamma$ corresponding to F under Theorem 3.3.1. Then $\mathrm{Ext}^2_\Gamma(\mathcal{D},\Gamma)$ is the subcategory corresponding to F^{op}. Since F is admissible, every object in $\mathcal{D}$ has finite length as a Γ-module. Thus $\mathcal{D}$ is an abelian category in which every object has finite length. Because $\mathrm{Ext}^2_\Gamma(\mathcal{D},\Gamma)$ is dual to $\mathcal{D}$, every object in $\mathrm{Ext}^2_{\mathcal{E}}(\mathcal{D},\mathcal{E})$ has finite length in the abelian category $\mathrm{Ext}^2_\Gamma(\mathcal{D},\Gamma)$. Since $\mathrm{Ext}^2_\Gamma(\mathcal{D},\Gamma)$ is a Serre subcategory of $\mathsf{mod}\,\Gamma^{\mathrm{op}}$, we have $\mathrm{Ext}^2_\Gamma(\mathcal{D},\Gamma) \subset \mathsf{f.l.}\,\Gamma^{\mathrm{op}}$. □

Using this notion, we can classify all admissible exact structures on $\mathsf{proj}\,\Gamma$.

THEOREM 3.3.7. *Let Γ be a noetherian ring. For $\mathcal{E} := \mathsf{proj}\,\Gamma$, there exists a bijection between the following two classes.*

(1) *Admissible exact structures F on $\mathcal{E}$.*
(2) *Sets $\mathcal{S}$ of isomorphism classes of simple Γ-modules satisfying the 2-regular condition.*

The correspondence is given as follows.

- *For a given $\mathcal{S}$, a complex $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{E}$ is a conflation if and only if there exists an exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to M \to 0$ in $\mathsf{mod}\,\Gamma$ with M in $\mathsf{Filt}\,\mathcal{S}$.*
- *For a given F, a simple Γ-module S is in $\mathcal{S}$ if and only if there exists a deflation $g : Y \to Z$ in $\mathcal{E}$ such that $\mathrm{Coker}(g) \cong S$ in $\mathsf{mod}\,\Gamma$.*

PROOF. By Theorem 3.3.1, there exists a bijection between (1) and

(3) Subcategories $\mathcal{D}$ of $\mathcal{C}_2(\Gamma)$ satisfying the following condition.
  (a) $\mathcal{D}$ is a Serre subcategory of $\mathsf{mod}\,\Gamma$ satisfying $\mathcal{D} \subset \mathsf{f.l.}\,\Gamma$.
  (b) $\mathrm{Ext}^2_\Gamma(\mathcal{D}, \Gamma)$ is a Serre subcategory of $\mathsf{mod}\,\Gamma^{\mathrm{op}}$.

We have mutually inverse bijections between (2) and (3) as follows; For a given $\mathcal{S}$ in (2), we put $\mathcal{D} := \mathsf{Filt}\,\mathcal{S}$, and for a given $\mathcal{D}$ in (3), we denote by $\mathcal{S}$ the set of simple modules contained in $\mathcal{D}$. By Lemma 3.3.2, these maps are well-defined. These are mutually inverse to each other by definition. □

Next we will focus on the case over the ground ring R. *In the rest of this subsection, we fix a commutative noetherian complete local ring R.* For an additive R-category $\mathcal{E}$ for a commutative noetherian ring R, we say that $\mathcal{E}$ is *Hom-noetherian* (resp. *Hom-finite*) if the R-module $\mathcal{E}(X, Y)$ is finitely generated (resp. of finite length) for every $X, Y \in \mathcal{E}$. Every Hom-noetherian idempotent complete R-category is Krull-Schmidt. For an exact R-category $\mathcal{E}$, we say that $\mathcal{E}$ is *Ext-noetherian* (resp. *Ext-finite*) if for every object X and Y, the R-module $\mathrm{Ext}^1_{\mathcal{E}}(X, Y)$ is finitely generated (resp. of finite length).

The following figure illustrates the relationship between these concepts.

$$R \text{ is artinian} \Longrightarrow \begin{array}{c}\text{enough proj.}\\ \text{(or inj.)}\end{array} \Longrightarrow \text{Ext-noeth.} \overset{\text{finite type}}{\rightleftarrows} \text{Ext-finite} \underset{\text{enough inj.}}{\overset{\text{finite type}}{\rightleftarrows}} \text{admissible}$$

We will prove these implication in Proposition 3.3.8 and Corollary 3.3.15. We refer the reader to Appendix B.2 for the results and notions we need in the proof.

PROPOSITION 3.3.8. *Let $\mathcal{E}$ be a Hom-noetherian idempotent complete exact R-category. Then the following hold.*

(1) *If $\mathcal{E}$ has either enough projectives or enough injectives, then $\mathcal{E}$ is Ext-noetherian.*
(2) *If $\mathcal{E}$ is Ext-noetherian and $\mathcal{E}$ is of finite type, then $\mathcal{E}$ is Ext-finite.*
(3) *If $\mathcal{E}$ is Ext-finite and $\mathcal{E}$ is of finite type, then $\mathcal{E}$ is admissible. Conversely, if $\mathcal{E}$ is admissible and has enough injectives, then $\mathcal{E}$ is Ext-finite.*

PROOF. (1) This is clear since extension groups can be computed by projective resolutions or injective coresolutions.

(2) If $\mathcal{E}$ is of finite type, then $\mathcal{E}$ has AR conflations by Corollary B.1.4. Since $\mathcal{E}$ is Ext-noetherian, Proposition B.2.1 implies that $\mathcal{E}$ is Ext-finite.

(3) Suppose that $\mathcal{E}$ is Ext-finite and of finite type. Take W to be an additive generator of $\mathcal{E}$ and $\Gamma := \mathrm{End}_{\mathcal{E}} W$. For any conflation $X \xrightarrow{f} Y \xrightarrow{g} Z$, the cokernel of $g \circ (-) : \mathcal{E}(W, Y) \to \mathcal{E}(W, Z)$ is a submodule of $\mathrm{Ext}^1_{\mathcal{E}}(W, X)$, thus is of finite length over R, or equivalently, over Γ. Therefore $\mathcal{E}$ is admissible.

Conversely, suppose that the exact structure on $\mathcal{E}$ is admissible and $\mathcal{E}$ has enough injectives. For an object $X \in \mathcal{E}$, take a conflation $X \xrightarrow{f} I \xrightarrow{g} Z$ such that I is injective. Then

$\mathcal{E}(-, I) \xrightarrow{g\circ(-)} \mathcal{E}(-, Z) \to \mathrm{Ext}^1_{\mathcal{E}}(-, X) \to 0$ is exact, hence $\mathrm{Ext}^1_{\mathcal{E}}(-, X)$ is of finite length. It follows that $\mathrm{Ext}^1_{\mathcal{E}}(W, X)$ is of finite length over R for any W in $\mathcal{E}$. □

Next we interpret Theorem 3.3.7 in terms of the quiver of Γ. Recall that for a noetherian R-algebra Γ, the valued quiver $Q(\Gamma)$ is defined as follows, where $\mathcal{J}$ denotes the radical of $\mathsf{proj}\,\Gamma$.

(1) The set of vertices is $\mathsf{ind}(\mathsf{proj}\,\Gamma)$, that is, the isomorphism classes of all indecomposable projective right Γ-modules.
(2) We draw an arrow from P to Q if $\mathcal{J}(P, Q)/\mathcal{J}^2(P, Q) \neq 0$ with a valuation $(d_{P,Q}, d'_{P,Q})$, where $d_{P,Q}$ (resp. $d'_{P,Q}$) is the dimension of $\mathcal{J}(P, Q)/\mathcal{J}^2(P, Q)$ as a k_P-vector space (resp. k_Q-vector space). Here $k_P := \mathrm{End}_{\mathcal{E}}(P)/\operatorname{rad}\mathrm{End}_{\mathcal{E}}(P)$ and $k_Q := \mathrm{End}_{\mathcal{E}}(Q)/\operatorname{rad}\mathrm{End}_{\mathcal{E}}(Q)$.

We introduce a *translation* on this quiver $Q(\Gamma)$.

Definition 3.3.9. Let Γ be a noetherian R-algebra. The translation τ of $P \in Q(\Gamma)$ is defined when $P/\operatorname{rad}P$ satisfies the 2-regular condition. In this case, τP is the projective module Q such that $\mathrm{Hom}_\Gamma(Q, \Gamma)$ is a projective cover of the simple left Γ-module $\mathrm{Ext}^2_\Gamma(P/\operatorname{rad}P, \Gamma)$. We draw a *dotted arrow* from P to τP whenever τP is defined.

This construction yields a valued translation quiver (see [ARS]). Since we will not use valued arrows, we omit the proof.

Now admissible exact structures on $\mathsf{proj}\,\Gamma$ can be visually classified by the dotted arrows.

Corollary 3.3.10. *Let Γ be a noetherian R-algebra over a noetherian complete local ring R. Then there exists a bijection between the following two classes.*

(1) *Admissible exact structures on* $\mathsf{proj}\,\Gamma$.
(2) *Sets of dotted arrows in* $Q(\Gamma)$.

Moreover, the Auslander-Reiten quiver of the exact category $\mathcal{E}$ is given by the quiver $Q(\Gamma)$ with the dotted arrows chosen in (2).

Proof. Note that each dotted arrow $\tau P \leftarrow\!\!-\,- P$ in $Q(\Gamma)$ bijectively corresponds to the simple Γ-module $P/\operatorname{rad}P$ satisfying the 2-regular condition by the definition. Thus Theorem 3.3.7 implies the assertion. □

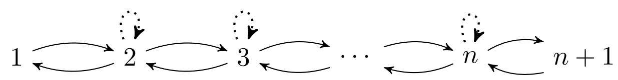
We end this subsection by giving examples of Corollary 3.3.10.

Example 3.3.11. Let Γ be the algebra defined in Example 3.1.2 the introduction. Then $Q(\Gamma)$ coincides with Figure 1 (all valuations are trivial, i.e. $(1, 1)$ and all dotted lines are interpreted as arrows from right to left). It has seven dotted arrows, thus $\mathsf{proj}\,\Gamma$ has $2^7 = 128$ exact structures.

Example 3.3.12. Let k be a field, $R := k[\![t]\!]$ the ring of formal power series over a field k and

$$\Gamma := \begin{bmatrix} R & R & \cdots & R & R \\ (t) & R & \cdots & R & R \\ (t^2) & (t) & \ddots & R & R \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ (t^n) & \cdots & (t^2) & (t) & R \end{bmatrix}.$$

Then $Q(\Gamma)$ is given by the following quiver.

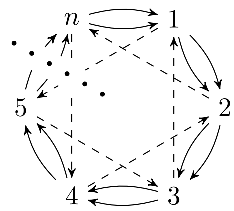


Note that Γ is the Auslander order for an R-order

$$\Lambda := \begin{bmatrix} R & R \\ (t^n) & R \end{bmatrix}$$

hence there is an equivalence $\mathsf{proj}\,\Gamma \simeq \mathsf{CM}\,\Lambda$, and the above translation quiver gives the Auslander-Reiten quiver of $\mathsf{CM}\,\Lambda$. Corollary 3.3.10 shows that $\mathsf{proj}\,\Gamma$ has four admissible exact structures, and the usual exact structure on $\mathsf{CM}\,\Lambda$ corresponds to the set of two dotted arrows.

EXAMPLE 3.3.13. Let $T = k[\![x, y]\!]$ be the ring of formal power series in two variables over a field k of characteristic 0, and let $R = T^{(n)}$ be the n-th Veronese subring of T, that is, $R = k[\![x^n, x^{n-1}y, \cdots, y^n]\!]$. Put $\Gamma := \mathrm{End}_R(T)$. Then all simple Γ-modules satisfy the 2-regular condition, and the translation quiver $Q(\Gamma)$ is the following.



Here the double arrows are interpreted as the single arrows with valuation $(2, 2)$. Note that $\mathsf{add}(T_R) = \mathsf{CM}\, R$ holds, thus $\mathsf{proj}\,\Gamma \simeq \mathsf{CM}\, R$ (see [Yo, Chapter 10] for the detail). Therefore Corollary 3.3.10 implies that $\mathsf{CM}\, R$ has 2^n admissible exact structures.

3.3.3. Enough projectivity and admissibility. We collect some properties about the relation between enough projectivity and admissibility of exact R-categories. *Throughout this subsection, we fix a commutative noetherian complete local ring R.*

By Proposition 3.3.8(3), If $\mathcal{E}$ is a Hom-noetherian idempotent complete exact R-category of finite type which has enough projectives, then $\mathcal{E}$ is admissible. The following gives a criterion when the converse holds.

PROPOSITION 3.3.14. *Let $\mathcal{E} := (\mathsf{proj}\,\Gamma, F)$ be an admissible exact category for a semiperfect noetherian ring Γ. Take an idempotent $e \in \Gamma$ such that $e\Gamma$ is an additive generator of projective objects in $\mathcal{E}$. Then the following hold.*

(1) *$\mathcal{D} = \mathsf{f.l.}(\Gamma/\Gamma e\Gamma)$ holds, where $\mathcal{D}$ is the subcategory of $\mathsf{mod}\,\Gamma$ corresponding to F under Theorem 3.3.1.*
(2) *$\mathcal{E}$ has enough projectives if and only if $\Gamma/\Gamma e\Gamma$ is in $\mathsf{f.l.}\,\Gamma$.*

PROOF. (1) Since Γ is semiperfect, $\mathcal{E}$ is a Krull-Schmidt exact category. By Proposition 3.2.12, an object P in $\mathcal{E}$ is projective if and only if $\mathrm{Hom}_\Gamma(P, \mathcal{D}) = 0$. Thus an indecomposable object in $\mathcal{E}$ is projective in $\mathcal{E}$ if and only if it is the projective cover of a simple Γ-module which is not contained in $\mathcal{D}$. Therefore, a simple Γ-module S is contained in $\mathcal{D}$ if and only if $\mathrm{Hom}_{\mathcal{E}}(e\Gamma, S) = 0$, that is, $Se = 0$. Thus, for a Γ-module $M \in \mathsf{mod}\,\Gamma$, it follows that $M \in \mathcal{D}$ holds if and only if $Me = 0$ and $M \in \mathsf{f.l.}\,\Gamma$, which implies the assertion.

(2) Recall that a morphism $g : P \to X$ in $\mathcal{E}$ is a deflation if and only if $\mathrm{Coker}(g)$ in $\mathsf{mod}\,\Gamma$ is in $\mathcal{D}$, and every projective resolution of $M \in \mathcal{D}$ yields a conflation in $\mathcal{E}$. Thus $\mathcal{E}$ has enough projectives if and only if there exists an exact sequence

$$P \xrightarrow{g} \Gamma \to M \to 0 \tag{3.3.1}$$

in $\mathsf{mod}\,\Gamma$ for some objects $P \in \mathsf{add}(e\Gamma)$ and $M \in \mathcal{D}$. Suppose that this holds. Then $\mathrm{Im}(g) \subset \Gamma e\Gamma$ holds, so we have a surjection $M \twoheadrightarrow \Gamma/\Gamma e\Gamma$. Since M is in $\mathcal{D}$, it follows that M is of finite length, thus so is $\Gamma/\Gamma e\Gamma$. Conversely, suppose that $\Gamma/\Gamma e\Gamma$ has finite length. Then $\Gamma/\Gamma e\Gamma$ is contained in $\mathcal{D} = \mathsf{f.l.}(\Gamma/\Gamma e\Gamma)$. Since Γ is noetherian, $\Gamma e\Gamma$ is a finitely generated as a right Γ-module. Therefore there exists an exact sequence of the form (3.3.1) with $M = \Gamma/\Gamma e\Gamma$. □

Consequently, we have the following interesting consequences. It is remarkable that we use purely module-theoretical argument to show non-trivial properties of exact categories.

COROLLARY 3.3.15. *Let $\mathcal{E}$ be a Hom-noetherian idempotent complete admissible exact R-category of finite type. Then the following holds.*

(1) *If R is artinian, then $\mathcal{E}$ has enough projectives and injectives.*
(2) *$\mathcal{E}$ has enough projectives if and only if $\mathcal{E}$ has enough injectives.*

PROOF. (1) Immediate from Proposition 3.3.14(2).

(2) It suffices to show the "only if" part. We may assume $\mathcal{E} = (\mathsf{proj}\,\Gamma, F)$ for a noetherian R-algebra Γ. Denote by $\mathcal{D} = \mathsf{Filt}\,\mathcal{S}$ the Serre subcategory of $\mathsf{mod}\,\Gamma$ which corresponds to F under Theorem 3.3.7, and take idempotents e and f in Γ such that $e\Gamma$ (resp. $f\Gamma$) is an additive generator of projective objects (resp. injective objects) in $\mathcal{E}$. Put $\underline{\Gamma} := \Gamma/\Gamma e\Gamma$ and $\overline{\Gamma} := \Gamma/\Gamma f\Gamma$. We know from Proposition 3.3.14(2) that $\underline{\Gamma}$ is of finite length, and it suffices to show that $\overline{\Gamma}$ is of finite length.

Recall that we have a duality $\mathrm{Ext}^2_\Gamma(-,\Gamma) : \mathcal{D} \simeq \mathrm{Ext}^2_\Gamma(\mathcal{D},\Gamma)$. On the other hand, by Proposition 3.3.14(1), we have $\mathcal{D} = \mathsf{f.l.}\,\underline{\Gamma}$ and $\mathrm{Ext}^2_\Gamma(\mathcal{D},\Gamma) = \mathsf{f.l.}\,\overline{\Gamma}^{\mathrm{op}}$. Therefore we have a duality $F : \mathsf{f.l.}\,\overline{\Gamma}^{\mathrm{op}} \simeq \mathsf{f.l.}\,\underline{\Gamma}$ between two abelian categories.

Observe that $\mathsf{f.l.}\,\underline{\Gamma}$ has an injective cogenerator $\mathrm{Hom}_R(\Gamma, I)$ by the duality $\mathrm{Hom}_R(-,I) : \mathsf{f.l.}\,\underline{\Gamma} \to \mathsf{f.l.}\,\underline{\Gamma}^{\mathrm{op}}$, where I is an injective hull of $R/\operatorname{rad} R$. Thus the abelian category $\mathsf{f.l.}\,\overline{\Gamma}^{\mathrm{op}}$ has a projective generator, which we denote by P.

Suppose that $\overline{\Gamma}$ is not of finite length over R. In particular, we have the following infinite chain of proper surjections

$$\cdots \twoheadrightarrow \overline{\Gamma}/\operatorname{rad}^3\overline{\Gamma} \twoheadrightarrow \overline{\Gamma}/\operatorname{rad}^2\overline{\Gamma} \twoheadrightarrow \overline{\Gamma}/\operatorname{rad}\overline{\Gamma}.$$

in $\mathsf{f.l.}\,\overline{\Gamma}^{\mathrm{op}}$. Take a surjection $f_1 : P^k \twoheadrightarrow \overline{\Gamma}/\operatorname{rad}\overline{\Gamma}$. Then this lifts to morphisms $f_i : P^k \to \overline{\Gamma}/\operatorname{rad}^i\overline{\Gamma}$. Since the kernel of $\overline{\Gamma}/\operatorname{rad}^i\overline{\Gamma} \twoheadrightarrow \overline{\Gamma}/\operatorname{rad}^{i-1}\overline{\Gamma}$ is contained in $\operatorname{rad}\overline{\Gamma}/\operatorname{rad}^i\overline{\Gamma}$, it follows that f_i is surjective for each i, which contradicts the fact that P^k has finite length. □

REMARK 3.3.16. If R is not artinian, $\mathcal{E}$ does not necessarily has enough projectives. For example, consider Example 3.3.13. Then $\mathsf{proj}\,\Gamma$ has the exact structure corresponding to all the dotted arrows. In this exact structure, there exists no non-zero projective object.

3.3.4. AR conflations and the Grothendieck group. In this subsection, we investigate the Grothendieck group of exact categories. Several papers showed that the relation of the Grothendieck group $\mathsf{K}_0(\mathcal{E})$ is generated by AR sequences when $\mathcal{E}$ is a particular exact category of finite type, e.g. [AR2, But, Yo]. Our aim in this subsection is to unify these results.

Let $\mathcal{E}$ be a Krull-Schmidt category. First we recall the following basic concepts in the AR theory. A morphism $g : Y \to Z$ in $\mathcal{E}$ is called *right almost split* if g is not a retraction and any non-retraction $h : W \to Z$ factors through g. Dually we define *left almost split*. We say that a conflation $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{E}$ is an *AR conflation* if f is left almost split and g is right almost split. We say that $\mathcal{E}$ *has AR conflations* if for every indecomposable non-projective object Z there exists an AR conflation ending at Z, and for every indecomposable non-injective object X there exists an AR conflation starting at X. For further properties of AR conflations, we refer the reader to Appendix B.1. In Corollary B.1.4, we will prove that Krull-Schmidt exact category $\mathcal{E}$ has AR conflations if $\mathcal{E}$ is of finite type and the endomorphism ring of an additive generator of $\mathcal{E}$ is noetherian.

Next we introduce some notation concerning the Grothendieck group. For a Krull-Schmidt exact category $\mathcal{E}$, let $\mathsf{G}(\mathcal{E})$ be the free abelian group $\bigoplus_{[X]\in \mathsf{ind}\,\mathcal{E}} \mathbb{Z}\cdot[X]$ generated by the set $\mathsf{ind}\,\mathcal{E}$ of isomorphism classes of indecomposable objects in $\mathcal{E}$. We denote by $\mathsf{Ex}(\mathcal{E})$ the subgroup of $\mathsf{G}(\mathcal{E})$ generated by

$$\{[X] - [Y] + [Z] \mid \text{there exists a conflation } X \rightarrowtail Y \twoheadrightarrow Z \text{ in } \mathcal{E}\}.$$

We call the quotient group $\mathsf{K}_0(\mathcal{E}) := \mathsf{G}(\mathcal{E})/\mathsf{Ex}(\mathcal{E})$ the *Grothendieck group* of $\mathcal{E}$. We denote by $\mathsf{AR}(\mathcal{E})$ the subgroup of $\mathsf{Ex}(\mathcal{E})$ generated by

$$\{[X] - [Y] + [Z] \mid \text{there exists an AR conflation } X \rightarrowtail Y \twoheadrightarrow Z \text{ in } \mathcal{E}\}.$$

Now we prove the main result about the relation of the Grothendieck groups.

THEOREM 3.3.17. *Let $\mathcal{E}$ be a Krull-Schmidt exact category of finite type such that the endomorphism ring of an additive generator of $\mathcal{E}$ is noetherian. If the exact structure on $\mathcal{E}$ is admissible, then* $\mathsf{Ex}(\mathcal{E}) = \mathsf{AR}(\mathcal{E})$ *holds.*

PROOF. Since $\mathsf{AR}(\mathcal{E}) \subset \mathsf{Ex}(\mathcal{E})$ always holds, we will check $\mathsf{Ex}(\mathcal{E}) \subset \mathsf{AR}(\mathcal{E})$. Let Γ be the endomorphism ring of an additive generator of $\mathcal{E}$. Then $\mathcal{E} \simeq \mathsf{proj}\,\Gamma$ holds, so we may assume that $\mathcal{E} = \mathsf{proj}\,\Gamma$ for a semiperfect noetherian ring Γ. Denote by $\mathcal{S}$ the set of simple Γ-modules corresponding to the exact structure on $\mathcal{E} = \mathsf{proj}\,\Gamma$ under Theorem 3.3.7.

Let $X_1 \xrightarrow{f_1} Y_1 \xrightarrow{g_1} Z_1$ be a conflation in $\mathcal{E}$ and $M := \mathrm{Coker}(g_1)$ in $\mathsf{mod}\,\Gamma$. Then M is in $\mathsf{Filt}\,\mathcal{S}$ by Theorem 3.3.7. We show $[X_1] - [Y_1] + [Z_1] \in \mathsf{AR}(\mathcal{E})$. Suppose that there exists another conflation $X_2 \xrightarrow{f_2} Y_2 \xrightarrow{g_2} Z_2$ in $\mathcal{E}$ such that $M \cong \mathrm{Coker}(g_2)$. Then we have the exact sequences

$$0 \to X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i \to M \to 0$$

in $\mathsf{mod}\,\Gamma$ for each $i = 1, 2$. Thus Schanuel's lemma shows that $X_1 \oplus Y_2 \oplus Z_1 \cong X_2 \oplus Y_1 \oplus Z_2$, which implies that $[X_1] - [Y_1] + [Z_1] = [X_2] - [Y_2] + [Z_2]$ in $\mathsf{G}(\mathcal{E})$. Thus it suffices to show the following claim to prove our theorem.

Claim: For any $M \in \mathsf{Filt}\,\mathcal{S}$, there exists at least one exact sequence in $\mathsf{mod}\,\Gamma$

$$0 \to X \to Y \to Z \to M \to 0$$

with $X, Y, Z \in \mathsf{proj}\,\Gamma$ and $[X] - [Y] + [Z] \in \mathsf{AR}(\mathcal{E})$.

We will show this claim by induction on $l(M)$, the length of M as a Γ-module. Suppose that $l(M) = 1$, that is, $M \in \mathcal{S}$. Take the minimal projective resolution $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to M \to 0$ of M with $X, Y, Z \in \mathsf{proj}\,\Gamma = \mathcal{E}$. By the proof of Proposition B.1.3, we have that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is indeed an AR-conflation in $\mathcal{E}$. Thus $[X] - [Y] + [Z] \in \mathsf{AR}(\mathcal{E})$.

Now suppose that $l(M) > 1$. Take a simple Γ-module $M_1 \in \mathcal{S}$ which is a submodule of M. Then we have an exact sequence $0 \to M_1 \to M \to M_2 \to 0$, where M_1 and M_2 are in $\mathsf{Filt}\,\mathcal{S}$. Since $l(M_1), l(M_2) < l(M)$, we have the corresponding projective resolutions $0 \to X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i \to M_i \to 0$ such that $[X_i] - [Y_i] + [Z_i] \in \mathsf{AR}(\mathcal{E})$ for $i = 1, 2$ by induction hypothesis. By the horseshoe lemma, we obtain a projective resolution $0 \to X_1 \oplus X_2 \xrightarrow{f_1 \oplus f_2} Y_1 \oplus Y_2 \xrightarrow{g_1 \oplus g_2} Z_1 \oplus Z_2 \to M \to 0$. Then we have

$$[X_1 \oplus X_2] - [Y_1 \oplus Y_2] + [Z_1 \oplus Z_2] = \sum_{i=1,2} ([X_i] - [Y_i] + [Z_i]) \in \mathsf{AR}(\mathcal{E}),$$

which completes the proof of the claim. □

COROLLARY 3.3.18. *Let R be a noetherian complete local ring and $\mathcal{E}$ a Hom-noetherian idempotent complete exact R-category of finite type. Suppose either R is artinian or $\mathcal{E}$ has enough projectives. Then $\mathsf{Ex}(\mathcal{E}) = \mathsf{AR}(\mathcal{E})$ holds.*

PROOF. In both cases, $\mathcal{E}$ is an admissible exact category by Proposition 3.3.8. Thus Theorem 3.3.17 applies. □

3.4. Classifications of CM-finite algebras

In this section, we apply our previous results to the representation theory of Iwanaga-Gorenstein algebras and orders. More generally, we study the left perpendicular category ${}^{\perp}U$ for a cotilting module U. *Throughout this section, we fix a commutative noetherian complete local ring R.*

3.4.1. Cotilting modules. First, we introduce the notion of *cotilting module* following [AR3]. Let Λ be a noetherian ring and $U \in \mathsf{mod}\,\Lambda$ a Λ-module. We denote by ${}^{\perp}U$ the subcategory of $\mathsf{mod}\,\Lambda$ consisting of all modules X satisfying $\mathrm{Ext}_{\Lambda}^{>0}(X, U) = 0$. Since ${}^{\perp}U$ is an extension-closed subcategory of $\mathsf{mod}\,\Lambda$, we always regard ${}^{\perp}U$ as an exact category.

DEFINITION 3.4.1. We say that U is a *cotilting module* if it satisfies the following conditions.

(C1) $\mathsf{id}\,U_{\Lambda}$ is finite.
(C2) $\mathrm{Ext}_{\Lambda}^{>0}(U, U) = 0$.
(C3) ${}^{\perp}U$ has enough injectives $\mathsf{add}\,U$, that is, for every X in ${}^{\perp}U$, there exists an exact sequence

$$0 \to X \to U^0 \to Y \to 0$$

in $\mathsf{mod}\,\Lambda$ with $Y \in {}^{\perp}U$ and $U^0 \in \mathsf{add}\,U$.

We shall see in Proposition 3.4.4 that if we restrict to R-orders, then our definition of cotilting modules coincides with the usual one, e.g. in [Iy4]. From our definition, the following property is immediate. Here we say that an object X in an exact category $\mathcal{E}$ is a *projective generator* (resp. *injective cogenerator*) if $\mathcal{E}$ has enough projectives $\mathsf{add}\, X$ (resp. enough injectives $\mathsf{add}\, X$).

PROPOSITION 3.4.2. *Let Λ be a noetherian ring and $U \in \mathsf{mod}\,\Lambda$ a cotilting Λ-module. Then ${}^{\perp}U$ is an exact category with a projective generator Λ and an injective cogenerator U.*

Let R be a Cohen-Macaulay local ring admitting a canonical module ω, for example, complete Cohen-Macaulay local ring. For a noetherian R-algebra Λ, we denote by $\mathsf{CM}\,\Lambda$ the subcategory of $\mathsf{mod}\,\Lambda$ consisting of modules which are maximal Cohen-Macaulay as R-modules. A noetherian R-algebra Λ is called an *R-order* if $\Lambda \in \mathsf{CM}\,\Lambda$ holds. For an R-order Λ, there exists a duality $D_d := \mathrm{Hom}_R(-,\omega) : \mathsf{CM}\,\Lambda \simeq \mathsf{CM}\,\Lambda^{\mathrm{op}}$. It is immediate that $\mathsf{CM}\,\Lambda$ is an extension-closed subcategory of $\mathsf{mod}\,\Lambda$ with a projective generator Λ and an injective cogenerator $D_d\Lambda$.

We prepare the following well-known properties of R-orders.

LEMMA 3.4.3. *Let R be a d-dimensional complete Cohen-Macaulay local ring, Λ an R-order and $M \in \mathsf{CM}\,\Lambda$. Then the following holds.*

(1) $\mathsf{CM}\,\Lambda = {}^{\perp}(D_d\Lambda)$ *holds, and* $\mathsf{id}(D_d\Lambda)_\Lambda = d$.

(2) $\mathsf{id}\, M_\Lambda \geq d$ *holds, and* $\mathsf{id}\, M_\Lambda \leq m$ *if and only if* $\mathrm{Ext}_\Lambda^{>m-d}(X, M) = 0$ *for all* $X \in \mathsf{CM}\,\Lambda$.

PROOF. Both follow from [GN1, Proposition 1.1]. □

PROPOSITION 3.4.4. *Let R be a d-dimensional complete Cohen-Macaulay local ring, Λ a noetherian R-algebra and U a finitely generated Λ-module.*

(1) *Suppose that Λ is an R-order and $U \in \mathsf{CM}\,\Lambda$. Then U is a cotilting module if and only if U satisfies the condition* (C1), (C2) in Definition 3.4.1 and the following.

(C3′) There exists an exact sequence

$$0 \to U_n \to \cdots \to U_1 \to U_0 \to D_d\Lambda \to 0$$

in $\mathsf{mod}\,\Lambda$ for some n such that $U_i \in \mathsf{add}\, U$ for each i.

In particular, U is a cotilting module with $\mathsf{id}\, U_\Lambda \leq d$ if and only if $\mathsf{add}\, U = \mathsf{add}\, D_d\Lambda$.

(2) Suppose that U is a cotilting Λ-module. Then ${}^{\perp}U \subset \mathsf{CM}\,\Lambda$ holds if and only if Λ is an R-order and $U \in \mathsf{CM}\,\Lambda$ holds.

PROOF. (1) Suppose that U satisfies (C1), (C2) and (C3′). Then the same proof as in [AR3, Theorem 5.4] implies that (C3) holds (see [Iy4, Proposition 3.2.2] for the order case).

Conversely, suppose that $U \in \mathsf{CM}\,\Lambda$ satisfies (C1)-(C3). First we show ${}^{\perp}U \subset \mathsf{CM}\,\Lambda$. Let X be in ${}^{\perp}U$. Then by (C3), we have an exact sequence $0 \to X \to U^0 \to U^1 \to \cdots \to U^{d-1}$ with U^i in $\mathsf{add}\, U$ for each i. Since each U^i is maximal Cohen-Macaulay as an R-module, it follows from the depth lemma [BH, Proposition 1.2.9] that X is in $\mathsf{CM}\,\Lambda$.

Next we will see that (C3′) holds. Put $\mathcal{X} := {}^{\perp}U$ and denote by $\widehat{U}$ the subcategory of $\mathsf{mod}\,\Lambda$ consisting of modules Y such that there exists an exact sequence

$$0 \to U_n \to \cdots \to U_1 \to U_0 \to Y \to 0$$

in $\mathsf{mod}\,\Lambda$ for some n with $U_i \in \mathsf{add}\, U$ for each i. Then the Auslander-Buchweitz theory implies that $Y \in \widehat{U}$ if $\mathrm{Ext}_\Lambda^{>0}(\mathcal{X}, Y) = 0$ (see [AB, Proposition 3.6] or Corollary A.1.3 for the detail). Since $\mathcal{X} \subset \mathsf{CM}\,\Lambda$ and $\mathrm{Ext}_\Lambda^{>0}(\mathsf{CM}\,\Lambda, D_d\Lambda) = 0$, we obtain $D_d\Lambda \in \widehat{U}$, which implies (C3′).

It follows from Lemma 3.4.3(2) that $\mathsf{id}\, U_\Lambda \leq d$ if and only if U is an injective object in $\mathsf{CM}\,\Lambda$. Thus the remaining assertion easily follows from the definition and (C3′).

(2) If we have ${}^{\perp}U \subset \mathsf{CM}\,\Lambda$, then in particular Λ and U are in $\mathsf{CM}\,\Lambda$, which in particular implies that Λ is an R-order. Thus the "only if" part follows. The the "if" part has already shown in the proof of (1). □

A noetherian ring Λ is called *Iwanaga-Gorenstein* if both $\mathsf{id}(\Lambda_\Lambda)$ and $\mathsf{id}({}_\Lambda\Lambda)$ are finite. By [Za, Lemma A], we have $\mathsf{id}(\Lambda_\Lambda) = \mathsf{id}({}_\Lambda\Lambda)$ in this case. For an Iwanaga-Gorenstein ring Λ, a Λ-module $X \in \mathsf{mod}\,\Lambda$ is called *Gorenstein-projective* if X is in $\mathsf{GP}\,\Lambda := {}^{\perp}\Lambda$. Then $\mathsf{GP}\,\Lambda$ is a Frobenius exact category with a projective generator Λ. An Iwanaga-Gorenstein ring Λ is *GP-finite* if $\mathsf{GP}\,\Lambda$ is of finite type. These concepts are special cases of cotilting modules, as the following proposition shows. If Λ is an R-order, then this can be easily proved by using Proposition 3.4.4.

PROPOSITION 3.4.5. *Let Λ be a noetherian ring.*

(1) *Λ is Iwanaga-Gorenstein if and only if Λ_Λ is a cotilting Λ-module.*
(2) *Let $U \in \mathsf{mod}\,\Lambda$ be a cotilting Λ-module. Then ${}^\perp U$ is Frobenius if and only if Λ is Iwanaga-Gorenstein and $\mathsf{add}\,U = \mathsf{proj}\,\Lambda$ holds.*

PROOF. (1) First we show the "only if" part. Suppose that Λ is Iwanaga-Gorenstein. Then Λ_Λ clearly satisfies (C1) and (C2). Moreover (C3) follows from the fact that ${}^\perp\Lambda = \mathsf{GP}\,\Lambda$ is the Frobenius category with an injective cogenerator Λ.

To show the "if" part, we use the main result of [HH]. We refer the reader to [HH] and references therein for the unexplained concepts. In [HH], it was shown that Λ is Iwanaga-Gorenstein if every module in $\mathsf{mod}\,\Lambda$ has a finite Gorenstein-projective dimension. Thus it suffices to see that this is the case if Λ_Λ is cotilting, which follows from Proposition 2.4.2.

(2) Clear from (1) and Proposition 3.4.2. □

In Chapter 2, it was characterized when a given exact category is exact equivalent to the exact category of the form ${}^\perp U$ for a cotilting module U over a noetherian ring. For an integer $n \geq 1$, recall that $\mathcal{E}$ *has n-kernels* if for every morphism $f : X \to Y$ in $\mathcal{E}$, there exists a complex

$$0 \to X_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X \xrightarrow{f} Y$$

in $\mathcal{E}$ such that the following diagram is exact.

$$0 \to \mathcal{E}(-, X_n) \xrightarrow{\mathcal{E}(-,f_n)} \cdots \xrightarrow{\mathcal{E}(-,f_2)} \mathcal{E}(-, X_1) \xrightarrow{\mathcal{E}(-,f_1)} \mathcal{E}(-, X) \xrightarrow{\mathcal{E}(-,f)} \mathcal{E}(-, Y)$$

PROPOSITION 3.4.6 (= Corollary 2.4.12). *Suppose that $\mathcal{E} := (\mathsf{proj}\,\Gamma, F)$ is an exact category for a noetherian ring Γ and $n \geq 2$ is an integer.*

(1) *The following are equivalent.*
   (a) *There exist a noetherian ring Λ and a cotilting Λ-module U with $\mathsf{id}\,U_\Lambda \leq n$ such that ${}^\perp U$ is exact equivalent to $\mathcal{E}$.*
   (b) *$\mathcal{E}$ has projective generators and injective cogenerators, and* $\operatorname{gl.dim}\Gamma \leq n$ *holds.*
   (c) *$\mathcal{E}$ has projective generators, injective cogenerators, and $(n-1)$-kernels.*
(2) *Suppose that the condition in* (1) *holds.* Take a projective generator P and an injective cogenerator I, and put $\Lambda := \operatorname{End}_{\mathcal{E}}(P)$ and $U := \mathcal{E}(P, I) \in \mathsf{mod}\,\Lambda$. Then Λ and U satisfies (1) and $\mathcal{E}(P, -)$ gives an exact equivalence $\mathcal{E} \simeq {}^\perp U$.

PROOF. Note that the endomorphism ring of every object in $\mathcal{E}$ is noetherian, see e.g. [Sa, Proposition 2.3]. Thus Theorem 2.4.11 applies. □

3.4.2. Classifications for noetherian R-algebras. To state our classifications, it is convenient to introduce the following terminology.

DEFINITION 3.4.7. Let R be a noetherian complete local ring.

(1) We say that a pair (Λ, U) is an *n-cotilting pair* if Λ is a noetherian R-algebra and $U \in \mathsf{mod}\,\Lambda$ is a cotilting Λ-module with $\mathsf{id}\,U_\Lambda \leq n$. Cotilting pairs (Λ_1, U_1) and (Λ_2, U_2) are said to be *equivalent* if there exists an equivalence $\mathsf{mod}\,\Lambda_1 \to \mathsf{mod}\,\Lambda_2$ which induces an equivalence $\mathsf{add}\,U_1 \to \mathsf{add}\,U_2$.
(2) We say that a pair (Γ, X) is an *algebra with dotted arrows* if Γ is a noetherian R-algebra and X is a set of dotted arrows of $Q(\Gamma)$ (see Definition 3.3.9). Two such pairs (Γ_1, X_1) and (Γ_2, X_2) are said to be *equivalent* if there exists an equivalence $\mathsf{proj}\,\Gamma_1 \to \mathsf{proj}\,\Gamma_2$ such that X_1 corresponds to X_2 under the isomorphism $Q(\Gamma_1) \to Q(\Gamma_2)$. For an algebra with dotted arrows (Γ, X), let P_{X} be the direct sum of indecomposable projective Γ-modules which are not sources of dotted arrows in X. We fix an idempotent $e_{\mathsf{X}} \in \Gamma$ satisfying $e_{\mathsf{X}}\Gamma = P_{\mathsf{X}}$.

The following main theorem classifies an n-cotilting pair of finite type by algebras with finite global dimension and set of dotted arrows.

THEOREM 3.4.8. *Let R be a noetherian complete local ring. There exists a bijection between the following for $n \geq 2$.*

(1) *Equivalence classes of n-cotilting pairs (Λ, U) such that ${}^{\perp}U$ is of finite type.*
(2) *Equivalence classes of algebras with dotted arrows (Γ, X) such that $\mathrm{gl.dim}\,\Gamma \leq n$ and $\Gamma/\Gamma e_{\mathsf{X}}\Gamma$ is of finite length over R.*
(3) *Exact equivalence classes of Hom-noetherian idempotent complete exact R-category $\mathcal{E}$ of finite type such that $\mathcal{E}$ has enough projectives, enough injectives and $(n-1)$-kernels.*

PROOF. By Corollary 3.3.10, we have the bijection between (2) and the following.

(4) Exact equivalence classes of $(\mathsf{proj}\,\Gamma, F)$, where Γ is a noetherian R-algebra with $\mathrm{gl.dim}\,\Gamma \leq n$ and F is an exact structure on $\mathsf{proj}\,\Gamma$ with enough projectives and injectives.

In fact, Proposition 3.3.8 shows that F in (4) is admissible, and Proposition 3.3.14 and Corollary 3.3.15 implies that F has enough projectives and injectives if and only if $\Gamma/\Gamma e_{\mathsf{X}}\Gamma$ is of finite length.

By Proposition 3.4.6, we have the bijections between (2), (3) and (4). Finally, we have maps between (1) and (3) as follows. For an n-cotilting pair (Λ, U) in (1), we put $\mathcal{E} := {}^{\perp}U$. For an exact category $\mathcal{E}$ in (3), there exists an n-cotilting pair (Λ, U) such that ${}^{\perp}U$ is exact equivalent to $\mathcal{E}$ by Proposition 3.4.6, which satisfies the condition of (1). The straightforward argument shows that these maps are mutually inverse to each other. □

Next we shall apply Theorem 3.4.8 to GP-finite Iwanaga-Gorenstein algebras. For a translation quiver Q, we consider a *τ-orbit* (that is, a connected component of the graph Q' consisting of the same vertices as Q and all the dotted translation arrows of Q). We say that a τ-orbit is *stable* if every vertex on it is both a source and a target of dotted arrows.

COROLLARY 3.4.9. *Let R be a noetherian complete local ring. There exists a bijection between the following for $n \geq 2$.*

(1) *Morita equivalence classes of GP-finite Iwanaga-Gorenstein noetherian R-algebras Λ with $\mathsf{id}\,\Lambda \leq n$.*
(2) *Equivalence classes of algebras with dotted arrows (Γ, X) satisfying the following.*
   (a) $\mathrm{gl.dim}\,\Gamma \leq n$.
   (b) $\Gamma/\Gamma e_{\mathsf{X}}\Gamma$ *is of finite length over R.*
   (c) X *is a union of stable τ-orbits in $Q(\Gamma)$.*

PROOF. By Proposition 3.4.5, it suffices to show that the pair (Γ, F) in Theorem 3.4.8(2) gives the Frobenius exact structure on $\mathsf{proj}\,\Gamma$ if and only if X in Theorem 3.4.8(2) is a union of stable τ-orbits. Since the exact category $\mathsf{proj}\,\Gamma$ has enough projectives and injectives by Proposition 3.3.14(2) and Corollary 3.3.15, the exact structure F is Frobenius if and only if the class of projectives and that of injectives coincide. For a set of dotted arrows X, an indecomposable object M in $\mathsf{proj}\,\Gamma$ is not projective (resp. not injective) in F if and only if there exists a dotted arrow in X starting at (resp. ending at) M. Therefore the class of indecomposable non-projective objects and that of indecomposable non-injective objects coincide if and only if X is a union of some stable τ-orbits, which clearly implies the assertion. □

We refer the reader to Example 3.1.2 in Section 1 for the detailed example of this classification.

3.4.3. Classifications for R-orders. Restricting Theorem 3.4.8 to the case of R-orders, we obtain the corresponding result as follows.

COROLLARY 3.4.10. *Let R be a complete Cohen-Macaulay local ring. Then there exists a bijection between the following for $n \geq 2$.*

(1) *Equivalence classes of n-cotilting pairs (Λ, U) such that Λ is an R-order, U is in $\mathsf{CM}\,\Lambda$ and ${}^{\perp}U$ is of finite type.*
(2) *Equivalence classes of algebras with dotted arrows (Γ, X) satisfying the following.*
   (a) $\mathrm{gl.dim}\,\Gamma \leq n$.

(b) $\Gamma/\Gamma e_{\mathsf{X}}\Gamma$ *is of finite length over* R.
(c) Γe_{X} *is maximal Cohen-Macaulay as an* R*-module.*

PROOF. We check that the bijections in Theorem 3.4.8 restricts to this case. By Proposition 3.4.4(2), we may replace the equivalence class (1) with

(1′) Equivalence classes of n-cotilting pairs (Λ, U) such that ${}^{\perp}U$ is of finite type and ${}^{\perp}U \subset \mathsf{CM}\,\Lambda$ holds.

Suppose that (Λ, U) and (Γ, X) correspond to each other under Theorem 3.4.8. It suffices to show that ${}^{\perp}U \subset \mathsf{CM}\,\Lambda$ if and only if Γe_{X} is maximal Cohen-Macaulay as an R-module. Notice that $\mathsf{add}\, e_{\mathsf{X}}\Gamma$ is the category of all projective objects in the exact category $(\mathsf{proj}\,\Gamma, F)$, thus $e_{\mathsf{X}}\Gamma$ is a projective generator of it. Hence we may assume that $\Lambda = \mathrm{End}_{\Gamma}(e_{\mathsf{X}}\Gamma) = e_{\mathsf{X}}\Gamma e_{\mathsf{X}}$. In this situation, we have an embedding $\mathrm{Hom}_{\Gamma}(e_{\mathsf{X}}\Gamma, -) : \mathsf{proj}\,\Gamma \to \mathsf{mod}\,\Lambda$ whose essential image is ${}^{\perp}U$. It follows that ${}^{\perp}U = \mathsf{add}\,\mathrm{Hom}_{\Gamma}(e_{\mathsf{X}}\Gamma, \Gamma) = \mathsf{add}\,\Gamma e_{\mathsf{X}}$. Therefore Γe_{X} is maximal Cohen-Macaulay as an R-module if and only if ${}^{\perp}U \subset \mathsf{CM}\,\Lambda$ holds. □

This gives the following Auslander-type correspondence for Cohen-Macaulay-finite R-orders with $\dim R \geq 2$. It largely improves [Iy4, Theorem 4.2.3].

COROLLARY 3.4.11. *Let* R *be a complete Cohen-Macaulay local ring with* $\dim R = d \geq 2$. *Then there exists a bijection between the following.*

(1) *Morita equivalence classes of* R*-orders* Λ *such that* $\mathsf{CM}\,\Lambda$ *is of finite type.*
(2) *Equivalence classes of algebras with dotted arrows* (Γ, X) *satisfying the following.*
   (a) $\mathrm{gl.dim}\,\Gamma = d$.
   (b) $\Gamma/\Gamma e_{\mathsf{X}}\Gamma$ *is of finite length over* R.
   (c) Γe_{X} *is maximal Cohen-Macaulay as an* R*-module.*

PROOF. Let us apply Corollary 3.4.10 to the case $n = d$. By Proposition 3.4.4, it suffices to show that for a noetherian R-algebra Γ satisfying the conditions of (2)(c), we have $\mathrm{gl.dim}\,\Gamma \geq d$. Every noetherian R-algebra Γ satisfies $\dim_R \Gamma \leq \mathrm{gl.dim}\,\Gamma$, see e.g. [GN2, Corollary 3.5(4)]. Since Γe_{X} is a direct summand of Γ and Γe_{X} is maximal Cohen-Macaulay, we have $d = \dim_R \Gamma e_{\mathsf{X}} \leq \dim_R \Gamma \leq \mathrm{gl.dim}\,\Gamma$. □

APPENDIX A

Some results needed in Chapter 2

In this chapter, we collect some results which we used in Chapter 2.

A.1. The Auslander-Buchweitz theory for exact categories

In this section, we shall study the Auslander-Buchweitz approximation theory, developed in [AB], in the context of exact categories. This is a useful tool to investigate cotilting subcategories. Our results in this appendix are used in Section 4.

First let us recall the following important notions.

Definition A.1.1. Let $\mathcal{C}$ be an additive category and $\mathcal{D}$ an additive subcategory of $\mathcal{C}$.

(1) A morphism $f : D_X \to X$ in $\mathcal{C}$ is said to be a *right $\mathcal{D}$-approximation* if D_X is in $\mathcal{D}$ and every morphism $D \to X$ with $D \in \mathcal{D}$ factors through f.
(2) $\mathcal{D}$ is said to be *contravariantly finite* if every object in $\mathcal{C}$ has a right $\mathcal{D}$-approximation.

Dually we define a *left $\mathcal{D}$-approximation* and a *covariantly finite* subcategories.

(3) $\mathcal{D}$ is said to be *functorially finite* if $\mathcal{C}$ is both contravariantly finite and covariantly finite.

The Auslander-Buchweitz theory gives a systematic method to provide right and left approximations by a certain subcategory. The following is an exact category version of [AB, Theorem 1.1] and we present a proof for the convenience of the reader.

Proposition A.1.2. *Let $\mathcal{E}$ be an exact category and $\mathcal{X}$ an extension-closed subcategory of $\mathcal{E}$. Suppose that $\mathcal{X}$ has enough injectives $\mathcal{W}$. Then the following hold.*

(1) *For any $C \in \widehat{\mathcal{X}}^n$, there exist conflations*

$$Y_C \rightarrowtail X_C \xrightarrow{f} C, \tag{A.1.1}$$

$$C \overset{g}{\rightarrowtail} Y^C \twoheadrightarrow X^C, \tag{A.1.2}$$

with $X_C, X^C \in \mathcal{X}$, $Y_C \in \widehat{\mathcal{W}}^{n-1}$ and $Y^C \in \widehat{\mathcal{W}}^n$.

(2) *If $\mathcal{E}$ has enough projectives and $\mathcal{X}$ is a preresolving subcategory of $\mathcal{E}$, then f is a right $\mathcal{X}$-approximation for C and g is a left $\widehat{\mathcal{W}}$-approximation for C.*

Proof. (1): The proof is by induction on n. Suppose that $C \in \widehat{\mathcal{X}}^0 = \mathcal{X}$. Then $0 \rightarrowtail C \twoheadrightarrow C$ gives (A.1.1). Since $\mathcal{X}$ has enough injective objects, we have a conflation $C \rightarrowtail W \twoheadrightarrow C'$ with W in $\mathcal{W}$ and C' in $\mathcal{X}$, which is the desired conflation (A.1.2).

Now let $n \geq 0$ be an integer and C in $\widehat{\mathcal{X}}^{n+1}$. By the definition of $\widehat{\mathcal{X}}^{n+1}$, there exists a conflation $D \rightarrowtail X \twoheadrightarrow C$ such that D is in $\widehat{\mathcal{X}}^n$ and X is in $\mathcal{X}$. By the induction hypothesis, we have conflations $Y_D \rightarrowtail X_D \twoheadrightarrow D$ and $D \rightarrowtail Y^D \twoheadrightarrow X^D$ with $X_D, X^D \in \mathcal{X}$, $Y_D \in \widehat{\mathcal{W}}^{n-1}$ and $Y^D \in \widehat{\mathcal{W}}^n$. Then we have the following pushout diagram.

$$\begin{array}{ccccc} D & \rightarrowtail & X & \twoheadrightarrow & C \\ \downarrow & & \downarrow & & \| \\ Y^D & \rightarrowtail & E & \twoheadrightarrow & C \\ \downarrow & & \downarrow & & \\ X^D & = & X^D & & \end{array}$$

Since $\mathcal{X}$ is closed under extensions, E is in $\mathcal{X}$ by the middle column, hence the middle row gives (A.1.1). Because $\mathcal{X}$ has enough injective objects, we obtain a conflation $E \rightarrowtail W \twoheadrightarrow F$ in $\mathcal{X}$ with $W \in \mathcal{W}$ and $F \in \mathcal{X}$, which induces the following diagram.

$$\begin{array}{ccccc} Y^D & \rightarrowtail & E & \twoheadrightarrow & C \\ \| & & \downarrow & & \downarrow \\ Y^D & \rightarrowtail & W & \twoheadrightarrow & G \\ & & \downarrow & & \downarrow \\ & & F & = & F \end{array}$$

Thus G is in $\widehat{\mathcal{W}}^n$ by the middle row, and the right column gives (A.1.2).

(2): Since $\mathcal{E}$ has enough projectives and $\mathcal{X}$ is a preresolving subcategory of $\mathcal{E}$, it follows that $\mathrm{Ext}^{>0}_{\mathcal{E}}(\mathcal{X}, \mathcal{W})$ vanishes. Then it is easy to check $\mathrm{Ext}^{>0}_{\mathcal{E}}(\mathcal{X}, \widehat{\mathcal{W}}) = 0$, and by using the long exact sequences of Ext one can easily show that f and g are approximations. ☐

COROLLARY A.1.3. *Let $\mathcal{E}$ be an exact category with enough projectives and $\mathcal{X}$ a preresolving subcategory of $\mathcal{E}$ with enough injectives $\mathcal{W}$. Suppose that $\widehat{\mathcal{X}} = \mathcal{E}$ holds. Then $\mathcal{X}$ is contravariantly finite and $\mathcal{X}^{\perp} = \widehat{\mathcal{W}}$ is covariantly finite. Moreover we have $\mathcal{X} \cap \mathcal{X}^{\perp} = \mathcal{W}$ and $\mathsf{add}\, \mathcal{X} = {}^{\perp}\mathcal{W} = {}^{\perp}\widehat{\mathcal{W}}$. If $\widehat{\mathcal{X}}^n = \mathcal{E}$, then $\mathcal{X}^{\perp} = \widehat{\mathcal{W}}^n$.*

PROOF. We first show $\mathcal{X} \cap \mathcal{X}^{\perp} = \mathcal{W}$. Since $\mathcal{X}$ is a preresolving subcategory of $\mathcal{E}$, it follows that $\mathcal{W} \subset \mathcal{X}^{\perp}$ holds. Let $X \in \mathcal{X} \cap \mathcal{X}^{\perp}$. Since $\mathcal{X}$ has enough injectives $\mathcal{W}$, there exists a conflation $X \rightarrowtail W \twoheadrightarrow X'$ with W in $\mathcal{W}$ and X' in $\mathcal{X}$. Then this sequence splits because $\mathrm{Ext}^1_{\mathcal{E}}(X', X) = 0$. Thus X is contained in $\mathsf{add}\, W$, which implies that X is injective in $\mathcal{X}$. Therefore X is in $\mathcal{W}$.

Next we show $\mathcal{X}^{\perp} = \widehat{\mathcal{W}}$. Note that $\widehat{\mathcal{W}} \subset \mathcal{X}^{\perp}$ holds, so it suffices to prove $\mathcal{X}^{\perp} \subset \widehat{\mathcal{W}}$. Let C be in $\mathcal{X}^{\perp}$. By Proposition A.1.2, we have a conflation $Y_C \rightarrowtail X_C \twoheadrightarrow C$ with X_C in $\mathcal{X}$ and Y_C in $\widehat{\mathcal{W}}$. Since $\mathcal{X}^{\perp}$ is clearly closed under extensions, we have $X_C \in \mathcal{X} \cap \mathcal{X}^{\perp} = \mathcal{W}$. It follows from the definition of $\widehat{\mathcal{W}}$ that C is in $\widehat{\mathcal{W}}$. Note that in case $\widehat{\mathcal{X}}^n = \mathcal{E}$, we may assume that Y_C is in $\widehat{\mathcal{W}}^{n-1}$, thus C is actually in $\widehat{\mathcal{W}}^n$.

Finally we shall show $\mathsf{add}\, \mathcal{X} = {}^{\perp}\mathcal{W} = {}^{\perp}\widehat{\mathcal{W}}$. Clearly $\mathsf{add}\, \mathcal{X} \subset {}^{\perp}\mathcal{W} = {}^{\perp}\widehat{\mathcal{W}}$ holds. Let C be in ${}^{\perp}\widehat{\mathcal{W}}$. Then by Proposition A.1.2, we have a conflation $Y_C \rightarrowtail X_C \twoheadrightarrow C$ with Y_C in $\widehat{\mathcal{W}}$ and X_C in $\mathcal{X}$. Then $C \in {}^{\perp}\widehat{\mathcal{W}}$ implies that this sequence splits, which shows that C is a summand of X_C. Consequently, C is in $\mathsf{add}\, \mathcal{X}$, which completes the proof. ☐

Immediately we obtain the following criterion for two preresolving subcategories to be the same up to summands.

COROLLARY A.1.4. *Let $\mathcal{E}$ be an exact category with enough projectives, and let $\mathcal{X}$ and $\mathcal{X}'$ be preresolving subcategories of $\mathcal{E}$ with enough injectives $\mathcal{W}$ and $\mathcal{W}'$ respectively. If $\widehat{\mathcal{X}} = \widehat{\mathcal{X}'} = \mathcal{E}$ and $\mathsf{add}\, \mathcal{W} = \mathsf{add}\, \mathcal{W}'$ hold, then $\mathsf{add}\, \mathcal{X} = \mathsf{add}\, \mathcal{X}'$.*

PROOF. The assertion is clear since in this situation $\mathsf{add}\, \mathcal{X} = {}^{\perp}\mathcal{W}$ holds by Corollary A.1.3. ☐

For an application to cotilting subcategories we studied in Section 4, we have the following result.

PROPOSITION A.1.5. *Let $\mathcal{W}$ be an n-cotilting subcategory of $\mathcal{E}$. Then $\mathsf{X}_{\mathcal{W}}$ is contravariantly finite. Furthermore we have $\mathsf{X}_{\mathcal{W}}^{\perp} = \widehat{\mathcal{W}} = \widehat{\mathcal{W}}^n$ and $\mathsf{X}_{\mathcal{W}} = {}^{\perp}\mathcal{W} = {}^{\perp}\widehat{\mathcal{W}}$.*

PROOF. The assertions follow from Corollary A.1.3 (apply to the case $\mathcal{X} := \mathsf{X}_{\mathcal{W}}$), Proposition 2.3.2 and 2.4.2. ☐

A.2. Constructions of exact structures

In this section, we collect two methods to construct new exact structures from a given one. One is to change exact structures on exact categories, and the other is to give a natural exact structures to quotient categories of exact categories.

In what follows, we denote by $\mathcal{E}$ an exact category and by $\mathcal{C}$ an additive subcategory of $\mathcal{E}$. Our aim is to construct a new exact structure in which objects in $\mathcal{C}$ behave as projective or injective objects.

We remark that similar results were given in [DRSS] in the case of artin R-algebras, based on the theory of relative homological algebra developed by Auslander-Solberg [ASo1, ASo2, ASo3].

DEFINITION A.2.1. Let $L \rightarrowtail M \twoheadrightarrow N$ be a conflation in $\mathcal{E}$.

(1) It is called a $(\mathcal{C},-)$-*conflation* if $\mathcal{E}(C,M) \to \mathcal{E}(C,N)$ is surjective for all C in $\mathcal{C}$. In this case $L \rightarrowtail M$ is called a $(\mathcal{C},-)$-*inflation* and $M \twoheadrightarrow N$ is called a $(\mathcal{C},-)$-*deflation*.
(2) It is called a $(-,\mathcal{C})$-*conflation* if $\mathcal{E}(M,C) \to \mathcal{E}(L,C)$ is surjective for all $C \in \mathcal{C}$.
(3) It is called a $\mathcal{C}$-*conflation* if it is both a $(\mathcal{C},-)$-conflation and a $(-,\mathcal{C})$-conflation.

In the obvious way we define the terms $(-,\mathcal{C})$-*inflation*, $(-,\mathcal{C})$-*deflation*, $\mathcal{C}$-*inflation* and $\mathcal{C}$-*deflation*.

THEOREM A.2.2. *Let $\mathcal{E}$ be an exact category and $\mathcal{C}$ an additive subcategory of $\mathcal{E}$. Then the class of all $(\mathcal{C},-)$-conflations (resp. $(-,\mathcal{C})$-conflations, $\mathcal{C}$-conflations) defines a new exact structure on $\mathcal{E}$.*

PROOF. We first show that all $(\mathcal{C},-)$-conflations defines a new exact structure on $\mathcal{E}$. It suffices to check Keller's axiom (Ex0), (Ex1), (Ex2) and (Ex2)$^{\mathrm{op}}$ in [Ke, Appendix A]. Note that the class of all $(\mathcal{C},-)$-conflations are clearly closed under isomorphisms and (Ex0) "the identity map of zero object is $(\mathcal{C},-)$-deflations" is trivial.

(Ex1) *The composition of two $(\mathcal{C},-)$-deflations is a $(\mathcal{C},-)$-deflation.*

Suppose that $X \twoheadrightarrow Y$ and $Y \twoheadrightarrow Z$ be $(\mathcal{C},-)$-deflations. From the definition, $X \twoheadrightarrow Y$ and $Y \twoheadrightarrow Z$ are deflations in $\mathcal{E}$. Thus the composition $X \twoheadrightarrow Y \twoheadrightarrow Z$ is also a deflation in $\mathcal{E}$. Then the claim follows since the composition $\mathcal{E}(\mathcal{C},X) \twoheadrightarrow \mathcal{E}(\mathcal{C},Y) \twoheadrightarrow \mathcal{E}(\mathcal{C},Z)$ is surjective.

(Ex2) *The class of $(\mathcal{C},-)$-deflations is stable under pullbacks.*

Suppose that $L \rightarrowtail M \twoheadrightarrow N$ be a $(\mathcal{C},-)$-conflation and that $X \to N$ is an arbitrary morphism. Since $\mathcal{E}$ is an exact category, there exists a pullback diagram

$$\begin{array}{ccccc} L & \rightarrowtail & E & \twoheadrightarrow & X \\ \| & & \downarrow & & \downarrow \\ L & \rightarrowtail & M & \twoheadrightarrow & N \end{array}$$

where two rows are conflations. We should check the above row is also a $(\mathcal{C},-)$-conflation. It is equivalent to say that any morphism $C \to X$ factors through $E \twoheadrightarrow X$ for any $C \in \mathcal{C}$. Let $C \to X$ be a morphism with C in $\mathcal{C}$. Since $M \twoheadrightarrow N$ is a $(\mathcal{C},-)$-deflation, the composition $C \to X \to N$ factors through $M \twoheadrightarrow N$. Since the right square is a pullback diagram, there exists a morphism $C \to E$ such that $C \to E \to X$ is equal to $C \to X$.

(Ex2)$^{\mathrm{op}}$ *The class of $(\mathcal{C},-)$-inflations is stable under pushouts.*

Suppose that $L \rightarrowtail M \twoheadrightarrow N$ is a $(\mathcal{C},-)$-conflation, and $L \to X$ is an arbitrary morphism. Since $\mathcal{E}$ is an exact category, There exists a pullback diagram

$$\begin{array}{ccccc} L & \rightarrowtail & M & \twoheadrightarrow & N \\ \downarrow & & \downarrow & & \| \\ X & \rightarrowtail & E & \twoheadrightarrow & N \end{array}$$

where two rows are conflations. We should show that any morphism $C \to N$ factors through $E \twoheadrightarrow N$ for $C \in \mathcal{C}$. This is easy because $C \to N$ factors through $M \twoheadrightarrow N$ and $M \twoheadrightarrow N$ factors through $E \twoheadrightarrow N$.

Thus we have proved that the class of $(\mathcal{C},-)$-conflations defines an exact structure on $\mathcal{E}$.

Dually the class of $(-,\mathcal{C})$-conflations also defines another exact structure on $\mathcal{E}$. It is easy to check that the intersection of two exact structures gives another exact structure, which implies that the class of $\mathcal{C}$-conflations also defines an exact structure on $\mathcal{E}$. □

Next we consider the ideal quotients of an exact category. The following observation gives a natural way to introduce an exact structure to the ideal quotient of $\mathcal{E}$.

PROPOSITION A.2.3. [DI, Theorem 3.6] *Let $\mathcal{E}$ be an exact category and $\mathcal{C}$ an additive subcategory of $\mathcal{E}$. Denote by $\pi : \mathcal{E} \twoheadrightarrow \mathcal{E}/[\mathcal{C}]$ the natural functor. Suppose that every object in $\mathcal{C}$ is both projective and injective. Then the following are equivalent.*

(1) *$\mathcal{E}/[\mathcal{C}]$ is an exact category whose conflations are precisely the essential images of conflations in $\mathcal{E}$ under π.*
(2) *The images of inflations in $\mathcal{E}$ under π are monomorphisms and the images of deflations in $\mathcal{E}$ under π are epimorphisms.*

In this case, if moreover $\mathcal{E}$ has enough projectives $\mathcal{P}$, then $\mathcal{E}/[\mathcal{C}]$ has enough projectives $\mathsf{add}\,\pi(\mathcal{P})$.

PROOF. We only prove the last assertion. It is clear from the definition of the exact structure on $\mathcal{E}/[\mathcal{C}]$ that the images of objects in $\mathcal{P}$ under π are projective in $\mathcal{E}/[\mathcal{C}]$. For any object X in $\mathcal{E}$, we have a conflation $\Omega X \rightarrowtail P \twoheadrightarrow X$ in $\mathcal{E}$ with P being projective. Sending it by π, we obtain a conflation $\underline{\Omega X} \rightarrowtail \underline{P} \twoheadrightarrow \underline{X}$. From this it follows that $\mathcal{E}/[\mathcal{C}]$ has enough projectives, and that projective objects are precisely objects in $\mathsf{add}\,\pi(\mathcal{P})$. $\square$

APPENDIX B

The AR theory for exact categories over a noetherian ring

This chapter contains a brief discussion of the Auslander-Reiten theory for exact categories which are Hom-noetherian over a noetherian complete local ring R. We show that the existence of AR conflations is closely related to the AR duality and dualizing R-varieties (Theorem B.2.5). These observations shed new light on the result on isolated singularities by Auslander [Au4], see Remark B.2.6.

B.1. Existence of AR conflations

Let $\mathcal{E}$ be a Krull-Schmidt category. We denote by $\mathcal{J}$ the Jacobson radical of $\mathcal{E}$. We need later the following easy lemma about kernel-cokernel pairs in Krull-Schmidt categories.

Lemma B.1.1. *Suppose that $X \to Y \to Z$ is a kernel-cokernel pair in $\mathcal{E}$. Then this complex is isomorphic to the direct sum of kernel-cokernel pairs of the following forms.*

(1) *$A = A \to 0$ for some A in $\mathcal{E}$.*
(2) *$0 \to B = B$ for some B in $\mathcal{E}$.*
(3) *$X' \to Y' \to Z'$ with all morphisms in $\mathcal{J}$.*

Let $\mathcal{E}$ be a Krull-Schmidt exact category. Recall that a conflation $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{E}$ is an *AR conflation* if f is left almost split and g is right almost split. It immediately follows from the definition that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an AR conflation, then f is left minimal, that is, any $\varphi : Y \to Y$ satisfying $\varphi f = f$ is an automorphism. Dually g is right minimal in this case. Also one can prove the uniqueness of the AR conflation as in the classical case.

The following classical observation is useful for us. For an object X in $\mathcal{E}$, we put $S_X := P_X/\mathcal{J}(-,X) \in \mathsf{Mod}\,\mathcal{E}$ and $S^X := P^X/\mathcal{J}(X,-) \in \mathsf{Mod}\,\mathcal{E}^{\mathrm{op}}$.

Proposition B.1.2. *Let $\mathcal{E}$ be a Krull-Schmidt category. Then the following hold.*

(1) *The map $X \mapsto S_X$ gives a bijection between $\mathsf{ind}\,\mathcal{E}$ and the set of isomorphism classes of simple object in $\mathsf{Mod}\,\mathcal{E}$.*
(2) *A morphism $Y \xrightarrow{g} Z$ is right almost split if and only if Z is indecomposable and $P_Y \xrightarrow{P_g} P_Z \to S_Z \to 0$ is exact.*
(3) *There exists a right almost split morphism ending at Z if and only if S_Z is finitely presented.*

Proof. It is well-known, see [Au3] for example. □

The following proposition says that the existence of the right almost split map ensures the existence of the AR conflation.

Proposition B.1.3. *Let Z be an indecomposable non-projective object in $\mathcal{E}$. Then the following are equivalent.*

(1) *There exists a right almost split morphism to Z.*
(2) *There exists an AR conflation $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$ ending at Z.*

Proof. (1) ⇒ (2): Since Z is not projective, there exists some object M in $\mathcal{D}$ such that $M(Z) \neq 0$ by Proposition 3.2.12. This means that there exists some non-zero morphism $a : P_Z \to M$ by the Yoneda lemma. Since $\operatorname{Im} a$ is finitely generated and $\mathcal{D}$ is a Serre subcategory of $\mathsf{mod}\,\mathcal{E}$, we have that $\operatorname{Im} a$ is also in $\mathcal{D}$. Since P_Z has a unique maximal subobject $\mathcal{J}(-,Z)$, it is easy to see that the cokernel of the composition $\mathcal{J}(-,Z) \to P_Z \to \operatorname{Im} a$ is isomorphic to S_Z. Since $\mathcal{J}(-,Z)$ is finitely generated by (1), we have $S_Z \in \mathcal{D}$. Therefore, we obtain a conflation $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that g is right almost split. Now we may assume that f is in $\mathcal{J}$ by Lemma B.1.1. This clearly implies that g is right minimal and the classical argument shows that f is left almost split.

(2) $\Rightarrow$ (1): Obvious from the definition. □

We say that $\mathcal{E}$ *has AR conflations* if for every indecomposable non-projective object Z there exists an AR conflation ending at Z, and for every indecomposable non-injective object X there exists an AR conflation starting at X.

COROLLARY B.1.4. *Suppose that $\mathcal{E} := \mathsf{proj}\,\Lambda$ is an exact category for a semiperfect noetherian ring Λ. Then $\mathcal{E}$ has AR conflations.*

PROOF. We have $\mathsf{Mod}\,\mathcal{E} \simeq \mathsf{Mod}\,\Lambda$, so every simple object in this category is finitely presented because Λ is noetherian. Thus the assertion holds by Propositions B.1.2 and B.1.3. □

B.2. AR conflations, the AR duality and dualizing varieties

Next we investigate the relationship between AR conflations and the AR duality in the general setting. *In what follows, we denote by R a noetherian complete local ring.*

We denote by I the injective envelope of the simple R-module $R/\operatorname{rad} R$, and by $D : \mathsf{Mod}\,R \to \mathsf{Mod}\,R$ the Matlis dual $D = \operatorname{Hom}_R(-, I)$. It is well-known that this induces a duality $D : \mathsf{noeth}\,R \leftrightarrow \mathsf{art}\,R$ and $D : \mathsf{f.l.}\,R \leftrightarrow \mathsf{f.l.}\,R$, where $\mathsf{noeth}\,R$, $\mathsf{art}\,R$ and $\mathsf{f.l.}\,R$ denotes the subcategory of $\mathsf{Mod}\,R$ consisting of R-modules which are noetherian, artinian and of finite length respectively.

From now on, we assume that $\mathcal{E}$ is a Hom-noetherian idempotent complete exact R-category. Recall that a morphism $f : W \to Z$ in $\mathcal{E}$ is *projectively trivial* if for each object X, the induced map $\operatorname{Ext}^1_{\mathcal{E}}(Z,X) \to \operatorname{Ext}^1_{\mathcal{E}}(W,X)$ is zero. It is easy to see that f is projectively trivial if and only if f factors through every deflation $Y \twoheadrightarrow Z$. If $\mathcal{E}$ has enough projectives, then f is projectively trivial if and only if f factors through some projective object. Denote by $\mathcal{P}(X,Y)$ the set of all projectively trivial morphisms from X to Y. Then $\mathcal{P}$ is a two-sided ideal of $\mathcal{E}$ and we put $\underline{\mathcal{E}} := \mathcal{E}/\mathcal{P}$, which we call the *projectively stable category*. Dually we define the notion of *injectively trivial* morphisms and the *injectively stable category* $\overline{\mathcal{E}} := \mathcal{E}/\mathcal{I}$.

The following theorem is an exact category version of [RVdB1, Proposition I.2.3], and it generalizes [LNP, Theorem 3.6] and [Ji, Proposition 2.4]. Also see [LNP, IN] for related work. Note that we do not assume that R is artinian or $\mathcal{E}$ is Hom-finite. This enables us to give another proof of the one implication of Auslander's theorem about isolated singularities, see Remark B.2.6

PROPOSITION B.2.1. *Suppose that $\mathcal{E}$ is Hom-noetherian Krull-Schmidt exact R-category such that either (i) $\mathcal{E}$ is Ext-noetherian or (ii) $\overline{\mathcal{E}}$ is Hom-finite. Let Z be an indecomposable non-projective object in $\mathcal{E}$. Then the following are equivalent.*

(1) *There exists a right almost split morphism to Z.*
(2) *There exists an AR conflation $X \rightarrowtail Y \twoheadrightarrow Z$ ending at Z.*
(3) *The functor $D\operatorname{Ext}^1_{\mathcal{E}}(Z,-) : \overline{\mathcal{E}} \to \mathsf{Mod}\,R$ is representable.*

Moreover, if $X \rightarrowtail Y \twoheadrightarrow Z$ is an AR conflation, then $D\operatorname{Ext}^1_{\mathcal{E}}(Z,-) \simeq \overline{\mathcal{E}}(-,X)$ holds, and thus $\operatorname{Ext}^1_{\mathcal{E}}(Z,W)$ and $\overline{\mathcal{E}}(W,X)$ are of finite length over R for every $W \in \mathcal{E}$.

PROOF. (1) $\Leftrightarrow$ (2): This is Proposition B.1.3.

(2) $\Rightarrow$ (3): Suppose that $\eta : X \rightarrowtail Y \twoheadrightarrow Z$ is an AR conflation. We regard $0 \neq \eta \in \operatorname{Ext}^1_{\mathcal{E}}(Z,X)$. Since I is a cogenerator of $\mathsf{Mod}\,R$, there exists a non-zero morphism $\gamma : \operatorname{Ext}^1_{\mathcal{E}}(Z,X) \to I$ in $\mathsf{Mod}\,R$ such that $\gamma(\eta) \neq 0$. Then we have a morphism

$$\langle -,- \rangle_W : \overline{\mathcal{E}}(W,X) \otimes_R \operatorname{Ext}^1_{\mathcal{E}}(Z,W) \to I, \qquad (\overline{h}, \mu) \mapsto \gamma(h\mu)$$

in $\mathsf{Mod}\,R$, where $h\mu$ is the image of μ by the induced morphism $\operatorname{Ext}^1_{\mathcal{E}}(Z,f) : \operatorname{Ext}^1_{\mathcal{E}}(Z,W) \to \operatorname{Ext}^1_{\mathcal{E}}(Z,X)$. Then the same proof as in [Ji, Lemma 2.1] applies here to show that $\langle -,- \rangle_W$ is non-degenerate in both variables. Therefore, we have two injections $\overline{\mathcal{E}}(W,X) \hookrightarrow D\operatorname{Ext}^1_{\mathcal{E}}(Z,W)$ and $\operatorname{Ext}^1_{\mathcal{E}}(Z,W) \hookrightarrow D\overline{\mathcal{E}}(W,X)$ in $\mathsf{Mod}\,R$. They are obviously isomorphisms if (ii) $\overline{\mathcal{E}}$ is Hom-finite.

Suppose that (i) $\mathcal{E}$ is Ext-noetherian. Then $D\operatorname{Ext}^1_{\mathcal{E}}(Z,W)$ is artinian, thus $\overline{\mathcal{E}}(W,X)$ is of finite length. Therefore two injections are isomorphisms and both of $\overline{\mathcal{E}}(W,X)$ and $\operatorname{Ext}^1_{\mathcal{E}}(Z,W)$ are of finite length over R. The naturality in W is clear from a direct calculation, so we obtain an isomorphism of functors $D\operatorname{Ext}^1_{\mathcal{E}}(Z,-)\simeq\overline{\mathcal{E}}(-,X)$.

(3) $\Rightarrow$ (2): The proof of [Ji, Lemma 2.2] applies here. □

In the rest of the appendix, we show that the existence of AR conflations are closely related to the notion of dualizing R-varieties, introduced in [AR1] and [AR4].

Definition B.2.2. Let $\mathcal{C}$ be a Hom-finite idempotent complete R-category. We say that $\mathcal{C}$ is a *dualizing R-variety* if it satisfies both of the following.

1. For any finitely presented R-functor $F:\mathcal{C}^{\mathrm{op}}\to\mathsf{f.l.}\,R$, the composition functor $DF:\mathcal{C}^{\mathrm{op}}\to\mathsf{f.l.}\,R\to\mathsf{f.l.}\,R$ is finitely presented.
2. For any finitely presented R-functor $G:\mathcal{C}\to\mathsf{f.l.}\,R$, the composition functor $DF:\mathcal{C}\to\mathsf{f.l.}\,R\to\mathsf{f.l.}\,R$ is finitely presented.

The following properties are immediate, which we state without proofs.

Lemma B.2.3. *Let $\mathcal{C}$ be a dualizing R-variety. Then the following hold.*

1. *The category of finitely presented functors $\mathsf{mod}_1\,\mathcal{C}$ and $\mathsf{mod}_1\,\mathcal{C}^{\mathrm{op}}$ are abelian categories.*
2. *D induces a duality $D:\mathsf{mod}_1\,\mathcal{C}\leftrightarrow\mathsf{mod}_1\,\mathcal{C}^{\mathrm{op}}$.*

The following technical lemma is elementary.

Lemma B.2.4. *Let $\mathcal{E}$ be an exact category with enough projectives and injectives. Then $\mathsf{mod}_1\,\underline{\mathcal{E}}$ is an abelian category with enough projectives and injectives. The functor $\underline{\mathcal{E}}\to\mathsf{mod}_1\,\underline{\mathcal{E}}$ defined by $X\mapsto\underline{\mathcal{E}}(-,X)$ gives the equivalence between $\underline{\mathcal{E}}$ and the category of projective objects in $\mathsf{mod}_1\,\underline{\mathcal{E}}$, and the functor $\overline{\mathcal{E}}\to\mathsf{mod}_1\,\underline{\mathcal{E}}$ defined by $X\mapsto\operatorname{Ext}^1_{\mathcal{E}}(-,X)$ gives the equivalence between $\overline{\mathcal{E}}$ and the category of injective objects in $\mathsf{mod}_1\,\underline{\mathcal{E}}$.*

Proof. To show that $\mathsf{mod}_1\,\underline{\mathcal{E}}$ is abelian, it suffices to show that $\underline{\mathcal{E}}$ has weak kernels, see [Au1]. Let $g:Y\to Z$ be any morphism in $\mathcal{E}$. Since $\mathcal{E}$ has enough projectives, we have the following pullback diagram with P being projective.

$$\begin{array}{ccccc} Z' & \rightarrowtail & E & \overset{a}{\twoheadrightarrow} & Y \\ \| & & \downarrow b & & \downarrow g \\ Z' & \rightarrowtail & P & \overset{c}{\twoheadrightarrow} & Z \end{array} \tag{B.2.1}$$

It is straightforward to check that a gives the weak kernel of $\underline{\varphi}\in\underline{\mathcal{E}}(X,Y)$.

The remaining assertion follows from the same argument as in [Iy4, Theorem 2.2.2(1)]. □

Now we are in position to prove our main results in this appendix, which is an exact category version of [AR4, Proposition 2.2].

Theorem B.2.5. *Let $\mathcal{E}$ be a Hom-noetherian idempotent complete exact R-category with enough projectives and injectives. The following are equivalent.*

1. *The category $\mathcal{E}$ has AR conflations.*
2. *$\underline{\mathcal{E}}$ is a dualizing R-variety.*
3. *$\overline{\mathcal{E}}$ is a dualizing R-variety.*
4. *There exist mutually inverse equivalences $\tau:\underline{\mathcal{E}}\rightleftarrows\overline{\mathcal{E}}:\tau^-$ such that we have natural isomorphisms $D\underline{\mathcal{E}}(\tau^-X,Z)\cong\operatorname{Ext}^1_{\mathcal{E}}(Z,X)\cong D\overline{\mathcal{E}}(X,\tau Z)$.*

In this case, $\mathcal{E}$ is Ext-finite and $\underline{\mathcal{E}}$ and $\overline{\mathcal{E}}$ are Hom-finite.

Proof. (1) $\Rightarrow$ (2): First note that $\underline{\mathcal{E}}$ is Hom-finite by Proposition B.2.1. Let $F\in\mathsf{mod}_1\,\underline{\mathcal{E}}$ be a finitely presented functor. We have an exact sequence $\underline{\mathcal{E}}(-,Y)\to\underline{\mathcal{E}}(-,Z)\to F\to 0$ in $\mathsf{Mod}\,\underline{\mathcal{E}}$, which is induced from $g:Y\to Z$ in $\mathcal{E}$ by the Yoneda lemma. Consider the diagram (B.2.1). It is standard that the right square gives a conflation $E\xrightarrow{{}^t[b,-a]}P\oplus Y\xrightarrow{[c,g]}Z$ (see e.g.

[Bü, Proposition 2.12]). Hence by replacing g by $[c, g]$, we may assume that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a conflation with $\underline{\mathcal{E}}(-, Y) \xrightarrow{(-)\circ g} \underline{\mathcal{E}}(-, Z) \to F \to 0$ being exact.

By the long exact sequence of Ext,

$$\underline{\mathcal{E}}(-, Y) \xrightarrow{(-)\circ g} \underline{\mathcal{E}}(-, Z) \to \operatorname{Ext}^1_{\mathcal{E}}(-, X) \xrightarrow{\operatorname{Ext}^1_{\mathcal{E}}(-,f)} \operatorname{Ext}^1_{\mathcal{E}}(-, Y)$$

is exact in $\mathsf{Mod}\,\underline{\mathcal{E}}$. Thus we have an exact sequence

$$0 \to F \to \operatorname{Ext}^1_{\mathcal{E}}(-, X) \xrightarrow{\operatorname{Ext}^1_{\mathcal{E}}(-,f)} \operatorname{Ext}^1_{\mathcal{E}}(-, Y).$$

Since $D : \mathsf{Mod}\,\underline{\mathcal{E}} \to \mathsf{Mod}\,\underline{\mathcal{E}}^{\mathrm{op}}$ is a contravariant exact functor, we obtain an exact sequence

$$D\operatorname{Ext}^1_{\mathcal{E}}(-, Y) \to D\operatorname{Ext}^1_{\mathcal{E}}(-, X) \to DF \to 0.$$

By (1) and Proposition B.2.1, the functors $D\operatorname{Ext}^1_{\mathcal{E}}(-, Y)$ and $D\operatorname{Ext}^1_{\mathcal{E}}(-, X)$ are representable, which shows that DF is finitely presented.

Next suppose that $G \in \mathsf{mod}_1\,\underline{\mathcal{E}}^{\mathrm{op}}$ is a finitely presented functor. We have an exact sequence $\underline{\mathcal{E}}(B, -) \to \underline{\mathcal{E}}(A, -) \to G \to 0$ in $\mathsf{Mod}\,\underline{\mathcal{E}}^{\mathrm{op}}$. Applying D, we obtain an exact sequence $0 \to DG \to D\underline{\mathcal{E}}(A, -) \to D\underline{\mathcal{E}}(B, -)$ in $\mathsf{Mod}\,\underline{\mathcal{E}}$. We will show that $D\underline{\mathcal{E}}(A, -) \simeq \operatorname{Ext}^1_{\mathcal{E}}(-, \tau A)$ for some τA in $\mathcal{E}$. Without loss of generality, we may assume that A has no non-zero projective summands. Then we have a direct sum of AR conflations $\tau A \rightarrowtail E \twoheadrightarrow A$ in $\mathcal{E}$. By Proposition B.2.1, $D\underline{\mathcal{E}}(A, -) \simeq \operatorname{Ext}^1_{\mathcal{E}}(-, \tau A)$ holds. Thus we obtain the following exact sequence in $\mathsf{Mod}\,\underline{\mathcal{E}}$.

$$0 \to DG \to \operatorname{Ext}^1_{\mathcal{E}}(-, \tau A) \to \operatorname{Ext}^1_{\mathcal{E}}(-, \tau B)$$

Since $\underline{\mathcal{E}}$ has weak kernels by Lemma B.2.4, the subcategory $\mathsf{mod}_1\,\underline{\mathcal{E}}$ is closed under kernels in $\mathsf{Mod}\,\underline{\mathcal{E}}$ (see [Au1]). Also Lemma B.2.4 shows that $\operatorname{Ext}^1_{\mathcal{E}}(-, \tau A)$ is finitely presented. Thus DG is also finitely presented.

(2) $\Rightarrow$ (4): Since $\underline{\mathcal{E}}$ is a dualizing R-variety, we have a duality $D : \mathsf{mod}_1\,\underline{\mathcal{E}}^{\mathrm{op}} \simeq \mathsf{mod}_1\,\underline{\mathcal{E}}$ between abelian categories. It induces a duality $D : \mathcal{P} \simeq \mathcal{I}$, where $\mathcal{P}$ (resp. $\mathcal{I}$) denotes the category of projective objects in $\mathsf{mod}_1\,\underline{\mathcal{E}}$ (resp. injective objects in $\mathsf{mod}_1\,\underline{\mathcal{E}}^{\mathrm{op}}$). By Lemma B.2.4, we have a equivalence $\mathcal{I} \simeq \overline{\mathcal{E}}$ and the contravariant Yoneda embedding $\underline{\mathcal{E}} \simeq \mathcal{P}$. Define the equivalence τ by the composition $\underline{\mathcal{E}} \simeq \mathcal{P} \simeq \mathcal{I} \simeq \overline{\mathcal{E}}$ and denote by τ^- its quasi-inverse. Then it follows from Lemma B.2.4 that (4) holds.

(4) $\Rightarrow$ (1): Obvious from Proposition B.2.1.

By duality, (3) is also equivalent to all the other conditions. □

This theorem gives an application about the relation between AR conflations and isolated singularities shown in [Au4]. For a complete Cohen-Macaulay local ring R, we say that an R-order Λ *has at most an isolated singularity* if $\operatorname{gl.dim} \Lambda \otimes_R R_{\mathfrak{p}} = \operatorname{ht} \mathfrak{p}$ holds for every non-maximal prime ideal $\mathfrak{p}$ of R. It is well-known that an R-order Λ has at most an isolated singularity if and only if $\underline{\mathsf{CM}}\,\Lambda$ is Hom-finite.

Remark B.2.6. Suppose that $\mathsf{CM}\,\Lambda$ has AR conflations. Then $\underline{\mathsf{CM}}\,\Lambda$ is a dualizing R-variety by Theorem B.2.5, hence in particular $\underline{\mathsf{CM}}\,\Lambda$ is Hom-finite, and therefore Λ has at most an isolated singularity. This gives a simple conceptual proof of the "only if" part of the main theorem of [Au4]; $\mathsf{CM}\,\Lambda$ has AR conflations if and only if Λ has at most an isolated singularity.

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