Relative Auslander Correspondence via Exact Categories

Haruhisa Enomoto

Advisor: Prof. Osamu Iyama

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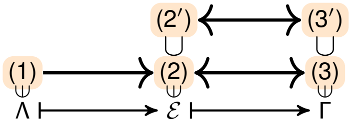

About My Thesis

My Thesis contains some new results,
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[E1] H. Enomoto, *Classifying exact categories via Wakamatsu tilting*, *J. Algebra* **485** (2017), 1–44.

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Outline

1. Introduction
   - Representation Theory of Algebras
2. Auslander Correspondence and CM Rep. Theory
   - Auslander Correspondence
   - CM Representation Theory
3. Exact Category and Main Results
   - Exact Category and Motivating Problems
   - Main Results

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What is Rep. Theory of Algebras?

k: a field.
Λ: a finite-dimensional k-algebra.
$\operatorname{mod}\Lambda$: the category of f.d. Λ-modules.

General Motivation

Want to study the structure of $\operatorname{mod}\Lambda$!

Krull-Schmidt Theorem

Every object in $\operatorname{mod}\Lambda$ *is a finite direct sum of* *indecomposable* *objects (in a unique way).*

$\rightsquigarrow$ suffices to study indec. Λ-modules

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Representation-Finite Algebras

Definition

Λ is representation-finite

$:\Leftrightarrow$ $\operatorname{mod}\Lambda$ has only finitely many indec. objects (up to isom).

Example

- k is rep-fin. (indec is only k).
- $k[X]/(X^n)$. (Jordan canonical form of nilpotent matrix)
- Path alg. of Dynkin quivers (related to Lie theory)

Auslander Correspondence:
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Statement of Auslander Correspondence

Theorem (Auslander 1971)

There exists a bijection between:

(1) *Representation-finite algebras Λ.*

(2) *Abelian categories $\mathcal{E}$ with finitely many indec. objects.*

(3) *Algebras Γ satisfying a certain homological condition ($\mathrm{gl.dim}\,\Gamma \leq 2 \leq \mathrm{dom.dim}\,\Gamma$).*

This relates rep-fin alg Λ to another class of alg. Γ!

① { Rep-fin alg } $\xrightarrow{\text{mod}}$ ② { Abelian cat } $\longleftrightarrow$ ③ {Algebras}

$$\Lambda \longmapsto \mathcal{E} := \mathrm{mod}\,\Lambda \longmapsto \Gamma$$

Bijections Between (1) and (2): Morita Theory

①		②
{ Rep-fin. alg }	$\xrightarrow{\text{mod}}$	{ Abelian cat with fin. many indec. }

$$\Lambda \longmapsto \mathcal{E} := \operatorname{mod} \Lambda$$

This bijection is related to:

Morita theory

Characterize the module category of algebra by the categorical property.

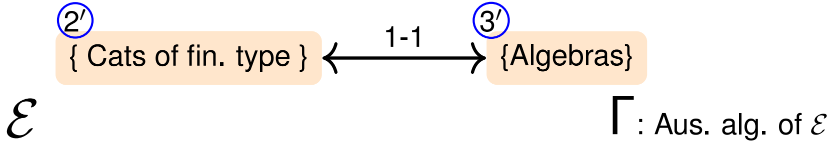
(2) and (3): Cat. of Finite Type "=" Algebras

Proposition

There exists a bijection between:

(2′) *Additive categories $\mathcal{E}$ of finite type*
(:⇔ categories with finitely many indec. objects).

(3′) *Algebra Γ (an Auslander alg. of $\mathcal{E}$).*



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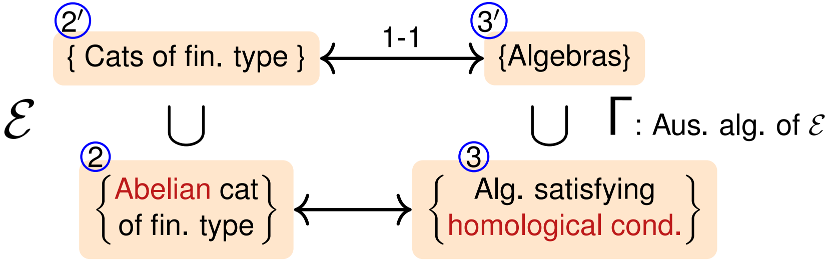
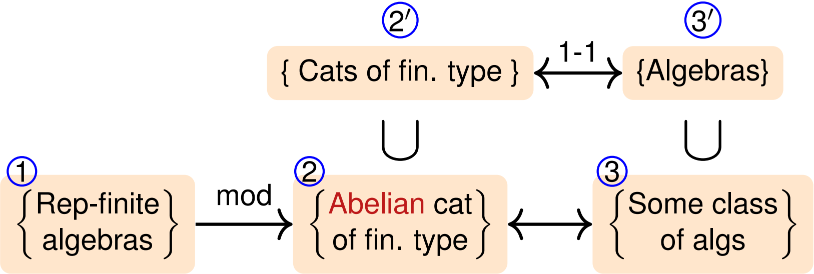


Auslander Correspondence in a Big Diagram

Bijections between (1), (2) and (3) are summerized as:



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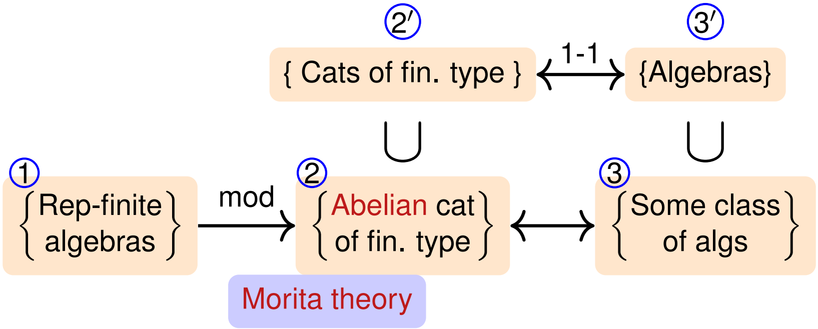

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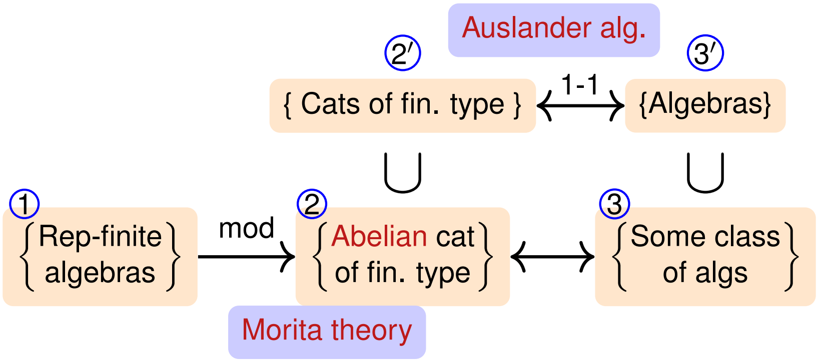

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Motivation for CM Rep. Theory

More detailed and interesting results are obtained if we restrict the class of modules we consider.

That is,

Study "good" subcategory of $\operatorname{mod}\Lambda$ (instead of $\operatorname{mod}\Lambda$ itself).

Cohen-Macaulay Rep. Theory

Today: CM rep. theory, which studies cat. $\operatorname{CM}\Lambda$ of Cohen-Macaulay modules.

Origin: Commutative rings, and their CM rep. theory is closely related to algebraic geometry.

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CM Rep. Theory for Iwanaga-Gorenstein Algebras

Definition

- Algebra Λ is Iwanaga-Gorenstein (IG) $:\Leftrightarrow \operatorname{id} \Lambda_\Lambda$ and $\operatorname{id} {}_\Lambda\Lambda$ are finite.
- $\mathsf{CM}\,\Lambda := \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^{>0}_\Lambda(X, \Lambda) = 0\}$, the category of Cohen-Macaulay Λ-modules.
- IG alg Λ is CM-finite if $\mathsf{CM}\,\Lambda$ is of finite type.

Example

- Algebra Λ with finite gl.dim. $\rightsquigarrow$ Λ is CM-fin. IG with $\mathsf{CM}\,\Lambda = \{$ f.g. projective Λ-modules $\}$.
- Self-injective algebra $\Lambda \rightsquigarrow \Lambda$ is IG, and $\mathsf{CM}\,\Lambda = \operatorname{mod}\Lambda$.

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Naive Approach to CM Auslander Corresp. Fails

① { CM-fin. IG alg } $\xrightarrow{\text{CM}}$ ②′ { Cats of fin. type } $\longleftrightarrow$ ③′ {Algebras}

$\Lambda \longmapsto \text{CM}\,\Lambda \longmapsto \Gamma$

$\Lambda' \longmapsto \text{CM}\,\Lambda$

The map “CM” is not injective!

However

Λ can be recovered from $\text{CM}\,\Lambda$

together with the structure of exact category on it.

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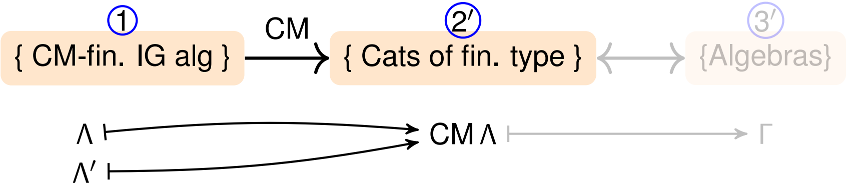

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Exact Cat. = Additive Cat. + Short Exact Seq.

Definition (Quillen 1973)

An exact category consists of a pair $(\mathcal{E}, F)$, where

- $\mathcal{E}$ is an additive category, and
- F is a class of short exact sequences in $\mathcal{E}$

satisfying some conditions.

Example

CM Λ (and other good subcat of mod Λ) naturally has the structure of exact categories.

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Modified Approach for CM Auslander Corresp.

Modified Approach

Want to seek bijections between:

(1) CM-finite IG algebras Λ.

(2) Categories $\mathcal{E}$ of finite type satisfying some conditions
+ exact structure on it.

(3) Algebra Γ satisfying some conditions
+ some information.

To this aim, we should consider:

Problem A (1) and (2): Morita theory for exact cat.

Problem B (2) and (3): Exact str. via Auslander alg.

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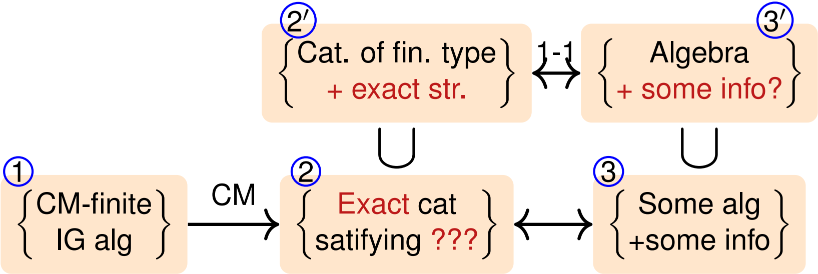
Two Problems in a Big Diagram



$$\Lambda \longmapsto \begin{array}{c}\mathcal{E} := \mathrm{CM}\,\Lambda \\ + \text{ natural exact str.}\end{array} \longmapsto \Gamma + \text{some info}$$

Two Problems in a Big Diagram

(2′) {Cat. of fin. type + exact str.} $\overset{1\text{-}1}{\longleftrightarrow}$ (3′) {Algebra + some info?}

$\cup$ $\cup$

(1) {CM-finite IG alg} $\xrightarrow{\text{CM}}$ (2) {Exact cat satifying ???} $\longleftrightarrow$ (3) {Some alg +some info}

Problem A: What's the image of CM?

$\Lambda \longmapsto$ $\mathcal{E} := \mathrm{CM}\,\Lambda$ + natural exact str. $\longmapsto \Gamma$ + some info

Two Problems in a Big Diagram

Problem B: Construct this bij.

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$$\Lambda \longmapsto \begin{matrix}\mathcal{E} := \mathrm{CM}\,\Lambda \\ + \text{ natural exact str.}\end{matrix} \longmapsto \Gamma + \text{some info}$$

Result On Problem A

Problem A

Characterize CM category of IG alg by categorical conditions!

Theorem A [E1]

For an exact category $\mathcal{E}$, the following are equivalent:

- *$\mathcal{E}$ is equivalent to* CM Λ *for some IG alg Λ.*
- *$\mathcal{E}$ is Frobenius category with progenerator and higher kernels.*

① $\left\{ \begin{array}{c} \text{CM-finite} \\ \text{IG alg} \end{array} \right\}$ $\xrightarrow{\text{CM}}$ ② $\left\{ \begin{array}{c} \text{Exact cat} \\ \text{satifying above} \end{array} \right\}$

$$\Lambda \longmapsto \mathcal{E} := \mathrm{CM}\,\Lambda$$

Result On Problem B

$\mathcal{E}$: cat. of finite type, Γ: its Auslander algebra.

Theorem B [E2]

There are bijections between the following:

- *Exact structures on $\mathcal{E}$.*
- *Sets of 2-regular simple Γ-modules.*
- *Sets of dotted arrows $\mathbb{A}$ in the graph $Q(\Gamma)$.*

(2′) $\left\{ \begin{array}{c} \text{Cat. of fin. type} \\ + \text{exact str.} \end{array} \right\}$ $\xleftrightarrow{\text{1-1}}$ $\left\{ \begin{array}{c} \text{Algebra} \\ + \text{set of dotted arrows} \end{array} \right\}$ (3′)

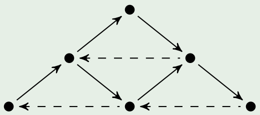
$$\mathcal{E} + \text{exact str.} \longmapsto (\Gamma, \mathbb{A})$$

Example of Theorem B: Classifying Exact Structures

Example

$\mathcal{E}$: the module category of $k[\bullet \leftarrow \bullet \leftarrow \bullet]$.

$Q(\Gamma)$:



$\exists$ 3 dotted arrow, hence

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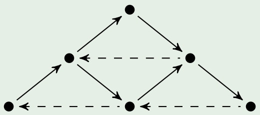

$\exists$ 3 dotted arrow, hence

$\exists$ $2^3 = 8$ exact str. on $\mathcal{E}$

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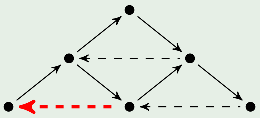

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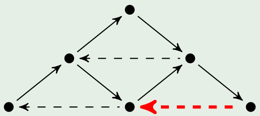

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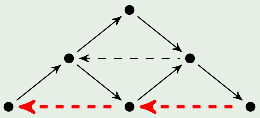

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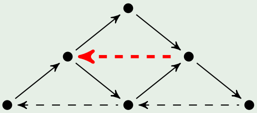

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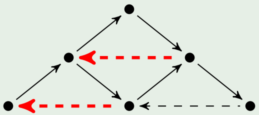

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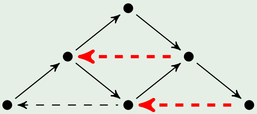

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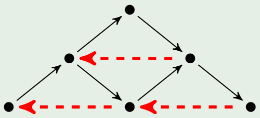

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Auslander Correspondence for CM-finite IG Algebras

Corollary [E1, E2]

There exists a bijection between the following.

(1) *CM-finite Iwanaga-Gorenstein algebras Λ.*

(2) *Frobenius exact cats $\mathcal{E}$ of finite type with higer kernels.*

(3) *Pairs $(\Gamma, \mathbb{A})$, where Γ is an algebra with finite gl.dim and $\mathbb{A}$ is a sets of cycles of dotted arrows of $Q(\Gamma)$.*

- ALL CM-finite IG algebras are obtained in this way.
- This gives systematic method to construct such algebras.

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Example of Corollary

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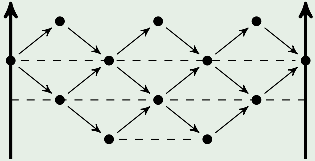

with commutativity and zero relation $\rightsquigarrow$ Γ has fin. gl.dim.
Thus $Q(\Gamma)$ has 2 cycles of dotted arrows.
$\rightsquigarrow$ We obtain $2^2 = 4$ CM-finite IG algebras Λ

$\mathbb{A}$: Orange Dotted Arrows.
Corresponding CM-finite IG Λ is the End of Red vertices.
All CM categories of Λ are the same (as additive cat)!

Example of Corollary

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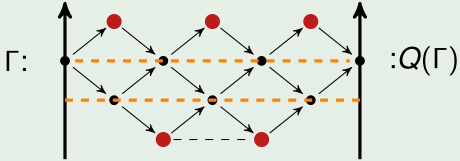

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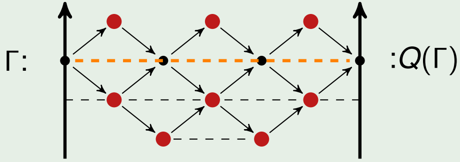

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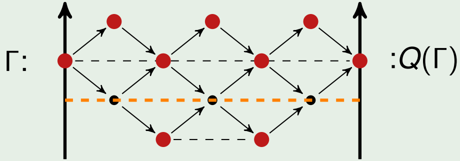

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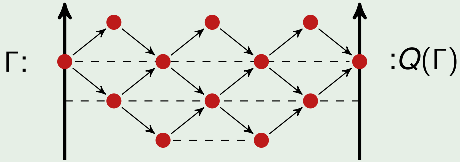

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