# Computing various objects of an algebra from the poset of torsion classes

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## References

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(60 minutes : iPad
30 minutes : Demo in Computer

§ Setting  k : field  $\Lambda$ : f.d. k-alg

$\mathrm{mod}\,\Lambda$ : the cat of f.g. right $\Lambda$-mods.

Def  $\mathcal{C} \subseteq \mathrm{mod}\,\Lambda$ : torsion class (tors)

$:\Leftrightarrow$ $\mathcal{C}$ : closed under quotients & ext.

Q. If tors $\Lambda$ is given as a poset, then what can we do?

A Many things!

Application

Research based on computer experiment

## §1. Preliminaries.

**Def**

- $\mathcal{C} \subseteq \operatorname{mod}\Lambda$, $\mathsf{T}(\mathcal{C})$ : the smallest tors containing $\mathcal{C}$ $\quad$ ($\mathsf{F}(\mathcal{C})$) (torf)

  $\rightsquigarrow$ $(\mathsf{T}(\mathcal{C}), \mathcal{C}^{\perp})$ : torsion pair

  $({}^{\perp}\mathcal{C}, \mathsf{F}(\mathcal{C}))$ : ——.

- $\operatorname{tors}\Lambda$ : the poset of torsion classes ordered by inclusion

  $\rightsquigarrow$ $\operatorname{tors}\Lambda$ : a complete lattice,

$$\bigwedge \mathcal{T}_i := \bigcap \mathcal{T}_i$$

$$\bigvee \mathcal{T}_i := \operatorname{Filt}\left(\bigcup \mathcal{T}_i\right)$$

**Def**

- $\mathcal{U} \subseteq \mathcal{T}$ in $\operatorname{tors}\Lambda$

  $\rightsquigarrow$ its heart $\mathcal{H}_{[\mathcal{U},\mathcal{T}]}$ is

$$\mathcal{H}_{[\mathcal{U},\mathcal{T}]} = \mathcal{T} \cap \mathcal{U}^{\perp}$$

$$(= \text{"}\mathcal{T} - \mathcal{U}\text{"})$$

- $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is a torsion heart if $\exists\ \mathcal{U} \subseteq \mathcal{T}$ in $\operatorname{tors}\Lambda$ s.t. $\mathcal{C} = \mathcal{H}_{[\mathcal{U},\mathcal{T}]}$

**Ex**

- $\mathcal{T}$ : tors $\Rightarrow$ $\mathcal{H}_{[0,\mathcal{T}]} = \mathcal{T}$

- $\mathcal{F}$ : torf $\Rightarrow$ $\mathcal{H}_{[{}^{\perp}\mathcal{F}, \operatorname{mod}\Lambda]} = \mathcal{F}$

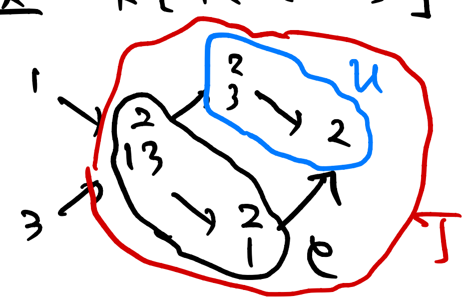
**Def** $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is

(1) wide $\iff$ closed under ker, cok, ext.

(2) ICE-closed $\iff$ —— Image, Cok, Ext. ┘

**Prop [ES]** Every ICE-closed subcat is a torsion heart. ┘

**Ex** $k[1 \leftarrow 2 \rightarrow 3]$




$\Rightarrow$ $\mathcal{C}$ : ICE

$\mathcal{C} = \mathcal{H}_{[\mathcal{U},\mathcal{T}]}$ ┘

tors. heart

**Def**

t-heart Λ, ICE Λ, wide Λ

: the posets of these subcats.

$$
\begin{tikzcd}[column sep=small]
 & \text{t-heart}\,\Lambda & \\
\text{ICE}\,\Lambda \arrow[ur, phantom, "\subset" sloped] & & \text{IKE}\,\Lambda \arrow[ul, phantom, "\supset" sloped] \\
 & \text{wide}\,\Lambda \arrow[ul, phantom, "\subset" sloped] \arrow[ur, phantom, "\supset" sloped] & \\
\text{tors}\,\Lambda \arrow[uu, phantom, "\cup" description] & & \text{torf}\,\Lambda \arrow[uu, phantom, "\cup" description] \\
 & \text{Serre}\,\Lambda \arrow[ul, phantom, "\subset" sloped] \arrow[ur, phantom, "\supset" sloped] \arrow[uu, phantom, "\cup" description] &
\end{tikzcd}
$$

**Problem**

Characterize torsion hearts categorically

## § Key Facts

**Fact 1.**

∃ lattice-theoretic characterization of itv $[\mathcal{U}, \mathcal{T}]$ in tors Λ

s.t. its heart is

(i) wide [AP]

(ii) ICE-closed [ES]

**Fact 2.** [DIRRT]

$\mathcal{C}_1, \mathcal{C}_2 \in$ t-heart Λ, then

$\mathcal{C}_1 \subseteq \mathcal{C}_2 \iff \operatorname{brick} \mathcal{C}_1 \subseteq \operatorname{brick} \mathcal{C}_2$,

where

$\operatorname{brick} \mathcal{C} := \{ B \in \mathcal{C} \mid \underbrace{B \in \operatorname{brick} \Lambda}_{\operatorname{End}_\Lambda(B) \text{ is a division ring}} \}$

**Def** $\mathbb{P}$ : poset

$\vec{H}(P)$ : v+x $p \in P$

Hasse quiver

arrow $p \to q$

$\iff p > q, \ \nexists r, \ p > r > q$

Def

(1) For $\mathcal{T} \to \mathcal{U}$ in $\vec{\mathcal{H}}(\mathrm{tors}\,\Lambda)$

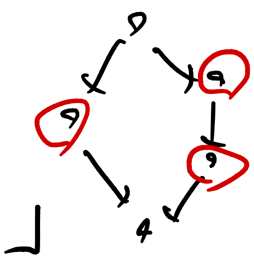
$\exists !$ $B \in \mathrm{brick}\,\mathcal{H}[\mathcal{U}, \mathcal{T}]$
[DIRRT]

$B$ : the brick label of this arrow.

(2) $L$ : a complete lattice,

$j \in L$ : join-irreducible

$\Leftrightarrow$ 「 $j = \bigvee X$ for $X \subseteq L$ $\Rightarrow j \in X$ 」



$\overset{\Rightarrow}{\underset{L:\text{fin.}}{\Leftarrow}}$ $\#\{j \to \text{ in } \vec{\mathcal{H}}(L)\} = 1$

∘ $\text{j-irr}\, L := \{\text{join-irr in } L\}$

( Dually meet-irr, $\text{m-irr}\, L$ )

Fact 3 [DIRRT]

$\exists$ bij

$\text{j-irr}(\mathrm{tors}\,\Lambda) \overset{1-1}{\longleftrightarrow} \mathrm{brick}\,\Lambda \overset{1-1}{\longleftrightarrow} \text{m-irr}(\mathrm{tors}\,\Lambda)$

$\mathcal{T}(B) \longleftarrow B \longmapsto {}^{\perp}B$

label $\left(\mathcal{T}\xrightarrow{B}\right) \longmapsto B \longleftarrow \left(\mathcal{U} \overset{B}{\swarrow}\right)$

Denote by $\kappa : \text{j-irr}(\mathrm{tors}\,\Lambda) \xrightarrow{\sim} \text{m-irr}(\mathrm{tors}\,\Lambda)$

: its composition.

Fact 4. [BTZ]

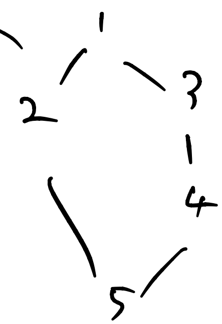
For $\mathcal{T} \in \text{j-irr}(\mathrm{tors}\,\Lambda)$ with

$\mathcal{T} \to \mathcal{T}_*$ : Hasse arrow.

Then

$\kappa(\mathcal{T}) = \max\{\mathcal{U} \in \mathrm{tors}\,\Lambda \mid \mathcal{T} \cap \mathcal{U} = \mathcal{T}_*\}$ 」

$$\boxed{\begin{array}{l} \circ\ \kappa\, \mathcal{T}(B) = {}^{\perp}B \\ \quad\quad \text{for } B \in \mathrm{brick}\,\Lambda \end{array}}$$




$\kappa(2) = 3$.

§ Results

Put $L := \mathrm{tors}\,\Lambda$ and

suppose a poset $L$ is given.

Idea Use $\text{j-irr}\, L$

instead of $\mathrm{brick}\,\Lambda$ !

Def

(1) $\kappa : \text{j-irr}\, L \xrightarrow{\sim} \text{m-irr}\, L$
defined by Fact 4.

(2) $\text{itv}\, L := \{[a,b] \mid a \le b \text{ in } L\}$

(3) $\text{j-brick} : \text{itv}\, L \longrightarrow 2^{\text{j-irr}\, L}$ :
power set
by $\text{j-brick}\,[a,b] := \{ j \in \text{j-irr}\, L \mid j \le b,\ \kappa(j) \ge a \}$

Thm 1.
We have a poset isom

$$\text{j-brick}(\text{itv}\, L) \simeq \text{f-heart}\, \Lambda$$

full subposet $- \bigcap\!|$ $2^{\text{j-irr}\, L}$

∵ ETS

$$\begin{array}{ccc} \text{itv}\, L & \xrightarrow{\mathcal{H}(-)} & \text{f-heart}\, \Lambda \\ {\scriptstyle \text{j-brick}} \downarrow & & \downarrow {\scriptstyle \text{brick}} \\ 2^{\text{j-irr}\, L} & \simeq & 2^{\text{brick}\, \Lambda} \end{array}$$

poset emb

Fact 2

$( T(B) \longleftrightarrow B )$

i.e., $B \in \mathcal{H}[\mathcal{U}, \mathcal{T}]$

$\Longleftrightarrow \underline{T(B) \le \mathcal{T}}$ and $\kappa T(B) \ge \mathcal{U}$

∵ $B \in \mathcal{T} \cap \mathcal{U}^{\perp}$

$\Longleftrightarrow B \in \mathcal{T}$ and $B \in \mathcal{U}^{\perp}$

$\Longleftrightarrow \underline{T(B) \le \mathcal{T}}$ and $F(B) \le \mathcal{U}^{\perp}$

$\Longleftrightarrow$ ———— $\quad \updownarrow {}^{\perp}(-)$

${}^{\perp}F(B) \ge \mathcal{U}$

$\kappa T(B) = {}^{\perp}B = {}^{\perp}F(B)$

□

Cor

$$\text{j-brick}(\{\text{wide itvs in } L\}) \simeq \text{wide}\, \Lambda$$

———— ICE ———— $\simeq$ ICE $\Lambda$

§ τ-tilting fin. case
( τ-t-f )

wide Λ

Suppose $\Lambda$ : τ-t-f
$( \Leftrightarrow \# \text{tors}\, \Lambda < \infty )$

## Def [BTZ]

$$\kappa : \mathrm{tors}\Lambda \xrightarrow{\sim} \mathrm{tors}\Lambda$$

$$T \longmapsto \bigcap_{i=1}^{\ell} {}^{\perp}B_i$$

(*) all arrow : $B_1 \rightleftarrows B_\ell$

$$\left( T \to B_1, \dots, T \to B_\ell \longmapsto B_1 \to \kappa T, \dots, B_\ell \to \kappa T \right)$$

## Fact [A, MS] $\Lambda$ : $\tau$-t. f

$$\mathrm{tors}\Lambda \xrightarrow{\sim} \mathrm{wide}\Lambda \; : \text{bij.}$$

$$T \longmapsto \mathrm{Filt}(B_1, \dots, B_\ell)$$

## Thm

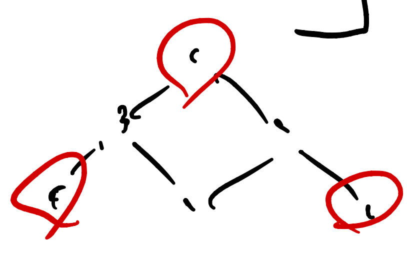
Define $\leq_w$ on $L := \mathrm{tors}\Lambda$

by $a \leq_w b \iff a \leq b$ and $\kappa(a) \geq \kappa(b)$

Then $(L, \leq_w) \simeq \mathrm{wide}\Lambda$

as posets. ∎

○ : not tors heart



## Simplicial cpx

$\Delta(\Lambda)$ : simp. cpx.

- vtx $\begin{cases} (M, 0) & M : \text{indec } \tau\text{-rigid} \\ (0, P) & P : \text{— proj.} \end{cases}$
- facet : summands of $\tau$-tilt. pair.

$\Big( \cong \{$simp. cpx of 2-silt cpx by [DIJ]$\}$
[AIR] $\Big)$

## Thm $\Lambda$ : $\tau$-t-f.

$\rightsquigarrow \Delta(\Lambda)$ can be recovered from the poset $L := \mathrm{tors}\Lambda$ ∎

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$\mathcal{C}$ : torsion heart

$\overset{?}{\iff}$ Image, Ext-closed &

induced exact str = maximal quasi-abelian