TACHIKAWA'S SECOND CONJECTURE IMPLIES THE AUSLANDER–REITEN CONJECTURE

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ABSTRACT. We prove that Tachikawa's second conjecture implies the Auslander–Reiten conjecture for artin algebras over a commutative artinian ring. The proof uses the two-fold trivial extension of an algebra. Together with known implications, it follows that the Auslander–Reiten conjecture, the generalized Nakayama conjecture, the Auslander–Gorenstein conjecture, the Nakayama conjecture, the Gorenstein-projective conjecture, and Tachikawa's second conjecture are all equivalent, and that Tachikawa's second conjecture implies his first.

1. INTRODUCTION

Auslander and Reiten proposed the following conjecture in their study of the generalized Nakayama conjecture [2].

Conjecture 1.1 (Auslander–Reiten [2]). *Let Λ be an artin algebra, and let X be a finitely generated right Λ-module. If*

$$\operatorname{Ext}^i_\Lambda(X, X \oplus \Lambda) = 0 \qquad \text{for every } i > 0, \tag{1.1}$$

then X is projective.

If an artin algebra Λ is self-injective, then Λ_Λ is injective, so $\operatorname{Ext}^i_\Lambda(X, \Lambda) = 0$ for every X and every $i > 0$. For self-injective algebras, Conjecture 1.1 therefore becomes the following conjecture of Tachikawa, known as his second conjecture.

Conjecture 1.2 (Tachikawa [7, §8]). *Let Λ be a self-injective artin algebra, and let X be a finitely generated right Λ-module. If*

$$\operatorname{Ext}^i_\Lambda(X, X) = 0 \qquad \text{for every } i > 0,$$

then X is projective.

We say that an artin algebra Λ satisfies the *Auslander–Reiten conjecture* (ARC) if every finitely generated right Λ-module X satisfying (1.1) is projective, and, when Λ is self-injective, that Λ satisfies *Tachikawa's second conjecture* (TC2) if every finitely generated right Λ-module X with $\operatorname{Ext}^i_\Lambda(X, X) = 0$ for every $i > 0$ is projective. Thus Conjectures 1.1 and 1.2 assert that every artin algebra satisfies (ARC) and that every self-injective artin algebra satisfies (TC2). Our main result is the following.

Theorem A (= Corollary 3.2). *Let R be a commutative artinian ring. If every self-injective artin R-algebra satisfies* (TC2)*, then every artin R-algebra satisfies* (ARC).

The proof uses the two-fold trivial extension

$$T_2(\Lambda) = \begin{pmatrix} \Lambda & D\Lambda \\ D\Lambda & \Lambda \end{pmatrix}, \qquad D\Lambda = \operatorname{Hom}_R(\Lambda, E),$$

of an artin R-algebra Λ, where E is an injective envelope of $R/\operatorname{rad} R$. This is the algebra of 2×2 matrices with these entries in which the product of two off-diagonal entries is zero. It is self-injective, and we show that if $T_2(\Lambda)$ satisfies (TC2), then Λ satisfies (ARC) (Theorem 3.1).

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We also consider the following classical homological conjectures. For an artin algebra Λ, let

$$0 \longrightarrow \Lambda_\Lambda \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots \tag{1.2}$$

be the minimal injective resolution of the right regular module. We say that Λ satisfies

- the *generalized Nakayama conjecture* (GNC) [2] if every indecomposable injective right Λ-module is a direct summand of I^i for some $i \geq 0$;
- the *Auslander–Gorenstein conjecture* (AGC) [3, §5] if the following holds: if $\operatorname{pd}_\Lambda I^i \leq i$ for every $i \geq 0$, then Λ_Λ and ${}_\Lambda\Lambda$ have finite injective dimension;
- the *Nakayama conjecture* (NC) [6] if the following holds: if I^i is projective for every $i \geq 0$, then Λ is self-injective;
- the *Gorenstein-projective conjecture* (GPC) [5] if every finitely generated Gorenstein-projective right Λ-module X with $\operatorname{Ext}^i_\Lambda(X, X) = 0$ for every $i > 0$ is projective.

Each of these conjectures asserts that every artin algebra satisfies the corresponding condition.

Theorem A completes the following cycle of implications, where $\Rightarrow$ denotes an implication for each individual algebra (each self-injective algebra when (TC2) is involved), and $\Rrightarrow$ denotes an implication between the assertions that every artin R-algebra (every self-injective artin R-algebra in the case of (TC2)) satisfies the given condition. The implications other than Theorem A are known and are recalled in Proposition 3.3.

$$\begin{tikzcd}
\text{(GNC)} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow, triple] & \text{(AGC)} \arrow[r, Rightarrow] & \text{(NC)} \arrow[d, Rrightarrow] \\
\text{(ARC)} \arrow[r, Rightarrow] & \text{(GPC)} \arrow[r, Rightarrow] & \text{(TC2)} \arrow[ll, Rrightarrow, bend left, "\text{Theorem A}"]
\end{tikzcd} \tag{1.3}$$

Corollary B (= Corollary 3.4)**.** *Let R be a commutative artinian ring. The following statements are equivalent.*

(1) *Every self-injective artin R-algebra satisfies* (TC2).
(2) *Every artin R-algebra satisfies* (ARC).
(3) *Every artin R-algebra satisfies* (GNC).
(4) *Every artin R-algebra satisfies* (AGC).
(5) *Every artin R-algebra satisfies* (NC).
(6) *Every artin R-algebra satisfies* (GPC).

We say that an artin algebra Λ satisfies *Tachikawa's first conjecture* (TC1) [7, §8] if the following holds: if $\operatorname{Ext}^i_\Lambda(D\Lambda, \Lambda) = 0$ for every $i > 0$, then Λ is self-injective; the conjecture asserts that every artin algebra satisfies (TC1). An algebra satisfying (ARC) also satisfies (TC1), and so Theorem A shows that Tachikawa's second conjecture implies his first: if every self-injective artin R-algebra satisfies (TC2), then every artin R-algebra satisfies (TC1) (Corollary 3.5).

Organization. In Section 2, Lemma 2.1 shows that induction along an idempotent reflects projectivity, and Lemma 2.3 shows that $T_2(\Lambda)$ is self-injective. The main output of that section is Proposition 2.4, which identifies Ext groups over $T_2(\Lambda)$ of induced modules with Ext groups over Λ for modules X with $\operatorname{Ext}^i_\Lambda(X, \Lambda) = 0$ for every $i > 0$. Section 3 proves Theorem 3.1 and Theorem A, recalls the known implications, and deduces Corollary B and the consequence for (TC1).

Conventions and notation. Throughout, R is a commutative artinian ring and E is an injective envelope of $R/\operatorname{rad} R$. An *artin R-algebra* is an associative unital R-algebra which is finitely generated as an R-module. All algebras are artin R-algebras, and all modules are finitely generated right modules unless otherwise specified. We write $\operatorname{mod}\Lambda$ for the category of finitely generated right Λ-modules. We write $D = \operatorname{Hom}_R(-, E)$; it is an exact duality between finitely generated right and left Λ-modules, with $DDM \cong M$ naturally. We give $D\Lambda$ the Λ-bimodule structure

$$(a\varphi b)(c) = \varphi(bca) \qquad (a, b, c \in \Lambda,\ \varphi \in D\Lambda).$$

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2. Induction to the two-fold trivial extension

The proof of Theorem A compares modules over Λ with modules over an algebra Γ containing an idempotent e with $e\Gamma e = \Lambda$, by means of the functor F in (2.1) below. Lemma 2.1 gives the properties of F used in the proof, for an arbitrary idempotent; its Ext comparison assumes a Tor vanishing condition. The rest of the section treats the case $\Gamma = T_2(\Lambda)$, for which Proposition 2.4 derives that condition from $\operatorname{Ext}^i_\Lambda(X, \Lambda) = 0$ for $i > 0$.

Let Γ be an algebra, let $e \in \Gamma$ be an idempotent, and put $\Lambda = e\Gamma e$, an algebra with identity e. Then $e\Gamma$ is a Λ-Γ-bimodule, and we consider the functors

$$F = - \otimes_\Lambda e\Gamma \colon \operatorname{mod}\Lambda \longrightarrow \operatorname{mod}\Gamma, \qquad H = (-)e \colon \operatorname{mod}\Gamma \longrightarrow \operatorname{mod}\Lambda. \tag{2.1}$$

We call F *induction along* e. Since $\operatorname{Hom}_\Gamma(e\Gamma, Z) \cong Ze$ for every $Z \in \operatorname{mod}\Gamma$, tensor–Hom adjunction gives isomorphisms

$$\operatorname{Hom}_\Gamma(FY, Z) \cong \operatorname{Hom}_\Lambda(Y, Ze) \tag{2.2}$$

natural in $Y \in \operatorname{mod}\Lambda$ and $Z \in \operatorname{mod}\Gamma$, so F is left adjoint to H. For every $Y \in \operatorname{mod}\Lambda$, since $e\Gamma e = \Lambda$,

$$(FY)e = Y \otimes_\Lambda e\Gamma e = Y \otimes_\Lambda \Lambda \cong Y,$$

and under this isomorphism the unit $Y \to (FY)e$, $y \mapsto y \otimes e$, of the adjunction is the identity. In particular, the unit is an isomorphism.

Lemma 2.1. *In the setting of* (2.1)*, the following hold.*

1. *The functor F is fully faithful. A module $Y \in \operatorname{mod}\Lambda$ is projective if and only if FY is projective.*
2. *Let $X \in \operatorname{mod}\Lambda$ satisfy $\operatorname{Tor}^\Lambda_i(X, e\Gamma) = 0$ for every $i > 0$. Then for every $Z \in \operatorname{mod}\Gamma$ and every $n \geq 0$, there is an isomorphism*

$$\operatorname{Ext}^n_\Gamma(FX, Z) \cong \operatorname{Ext}^n_\Lambda(X, Ze).$$

Proof. (1): A left adjoint is fully faithful if and only if the unit of the adjunction is an isomorphism, so F is fully faithful. Since $F\Lambda = e\Gamma$ is a direct summand of Γ_Γ, the functor F preserves projective modules. Conversely, suppose that FY is projective. Choose an epimorphism $\pi\colon \Lambda^m \to Y$. Since F is right exact, $F\pi\colon (e\Gamma)^m \to FY$ is an epimorphism, so there is $s\colon FY \to (e\Gamma)^m$ with $F\pi \circ s = 1_{FY}$. Since F is fully faithful, $s = Fs'$ for some $s'\colon Y \to \Lambda^m$, and $F(\pi \circ s') = F\pi \circ s = 1_{FY}$ gives $\pi \circ s' = 1_Y$. Hence Y is a direct summand of Λ^m, and so Y is projective.

(2): Choose a projective resolution $P_\bullet \to X$ in $\operatorname{mod}\Lambda$. The homology of $P_\bullet \otimes_\Lambda e\Gamma$ is FX in degree 0 and $\operatorname{Tor}^\Lambda_i(X, e\Gamma) = 0$ in degree $i > 0$. Hence

$$\cdots \longrightarrow FP_2 \longrightarrow FP_1 \longrightarrow FP_0 \longrightarrow FX \longrightarrow 0$$

is exact. Its terms FP_j are projective by (1), so $FP_\bullet$ is a projective resolution of FX. The isomorphisms (2.2) are natural in Y, so they give an isomorphism of cochain complexes

$$\operatorname{Hom}_\Gamma(FP_\bullet, Z) \cong \operatorname{Hom}_\Lambda(P_\bullet, Ze).$$

Taking cohomology in degree n gives $\operatorname{Ext}^n_\Gamma(FX, Z) \cong \operatorname{Ext}^n_\Lambda(X, Ze)$. □

We now construct, for an arbitrary algebra Λ, a self-injective algebra Γ and an idempotent e with $e\Gamma e = \Lambda$.

Definition 2.2. Let Λ be an artin R-algebra. The *two-fold trivial extension* of Λ is the R-module

$$T_2(\Lambda) = \begin{pmatrix} \Lambda & D\Lambda \\ D\Lambda & \Lambda \end{pmatrix} \tag{2.3}$$

with multiplication

$$\begin{pmatrix} a & \varphi \\ \psi & b \end{pmatrix} \begin{pmatrix} a' & \varphi' \\ \psi' & b' \end{pmatrix} = \begin{pmatrix} aa' & a\varphi' + \varphi b' \\ \psi a' + b\psi' & bb' \end{pmatrix}, \tag{2.4}$$

where the products in the off-diagonal entries are the bimodule actions on $D\Lambda$.

For finite-dimensional algebras over a field, $T_2(\Lambda)$ is the case $r = 2$ of the r-fold trivial extension; see, for example, [4, §2.2].

The off-diagonal matrices form a submodule I, which is a $(\Lambda \times \Lambda)$-bimodule via

$$(a, b) \begin{pmatrix} 0 & \varphi \\ \psi & 0 \end{pmatrix} (a', b') = \begin{pmatrix} 0 & a\varphi b' \\ b\psi a' & 0 \end{pmatrix}.$$

By (2.4), $T_2(\Lambda)$ is the trivial extension of $\Lambda \times \Lambda$ by I, in which products of two elements of I are zero. Hence $T_2(\Lambda)$ is an associative R-algebra with identity $\operatorname{diag}(1, 1)$, and it is finitely generated as an R-module since Λ and $D\Lambda$ are. For $e = \operatorname{diag}(1, 0)$, we have $eT_2(\Lambda)e = \Lambda$, and

$$eT_2(\Lambda) = \begin{pmatrix} \Lambda & D\Lambda \end{pmatrix} \cong \Lambda \oplus D\Lambda \qquad \text{as left } \Lambda\text{-modules.} \tag{2.5}$$

Lemma 2.3. *The artin R-algebra $T_2(\Lambda)$ is self-injective.*

Proof. Put $\Gamma = T_2(\Lambda)$. We show that $\Gamma_\Gamma \cong D({}_\Gamma\Gamma)$. Define an R-linear map $\ell\colon \Gamma \to E$ by

$$\ell \begin{pmatrix} a & \varphi \\ \psi & b \end{pmatrix} = \varphi(1) + \psi(1).$$

For $x = \begin{pmatrix} a & \varphi \\ \psi & b \end{pmatrix}$ and $y = \begin{pmatrix} a' & \varphi' \\ \psi' & b' \end{pmatrix}$ in Γ, the multiplication (2.4) and the bimodule structure on $D\Lambda$ give

$$\ell(xy) = \varphi'(a) + \varphi(b') + \psi(a') + \psi'(b).$$

Suppose that $\ell(xy) = 0$ for every $y \in \Gamma$. Taking y with only φ', only b', only a', or only ψ' nonzero shows that $\varphi'(a) = 0$ for every $\varphi' \in D\Lambda$, $\varphi = 0$, $\psi = 0$, and $\psi'(b) = 0$ for every $\psi' \in D\Lambda$. Since E is an injective cogenerator of R-modules, the first and last conditions give $a = 0$ and $b = 0$. Hence the right Γ-module homomorphism

$$\Gamma_\Gamma \longrightarrow D({}_\Gamma\Gamma), \qquad x \longmapsto \big(y \longmapsto \ell(xy)\big)$$

is injective, where $(hr)(y) = h(ry)$ for $h \in D\Gamma$ and $r, y \in \Gamma$. Since D preserves the length of R-modules, both sides have the same length as R-modules, so this map is an isomorphism. The right Γ-module $D({}_\Gamma\Gamma)$ is injective, since D sends the projective left module ${}_\Gamma\Gamma$ to an injective right module. Therefore Γ_Γ is injective. $\square$

For $\Gamma = T_2(\Lambda)$, the next proposition shows that the hypothesis $\operatorname{Ext}^i_\Lambda(X, \Lambda) = 0$ for $i > 0$ implies the Tor vanishing of Lemma 2.1(2), and applies that lemma with $Z = FY$.

Proposition 2.4. *Let $\Gamma = T_2(\Lambda)$ and $e = \operatorname{diag}(1, 0)$, and let F be as in (2.1). Let $X \in \operatorname{mod}\Lambda$ satisfy $\operatorname{Ext}^i_\Lambda(X, \Lambda) = 0$ for every $i > 0$. Then for every $Y \in \operatorname{mod}\Lambda$ and every $n \geq 0$,*

$$\operatorname{Ext}^n_\Gamma(FX, FY) \cong \operatorname{Ext}^n_\Lambda(X, Y).$$

Proof. Choose a projective resolution $P_\bullet \to X$ in $\operatorname{mod}\Lambda$ by finitely generated projective modules. Tensor–Hom adjunction and the isomorphism $DD\Lambda \cong \Lambda$ of right Λ-modules give an isomorphism of cochain complexes

$$D(P_\bullet \otimes_\Lambda D\Lambda) \cong \operatorname{Hom}_\Lambda(P_\bullet, DD\Lambda) \cong \operatorname{Hom}_\Lambda(P_\bullet, \Lambda).$$

Since D is exact, taking cohomology gives

$$D \operatorname{Tor}^\Lambda_i(X, D\Lambda) \cong \operatorname{Ext}^i_\Lambda(X, \Lambda) = 0 \qquad (i > 0),$$

and hence $\operatorname{Tor}^\Lambda_i(X, D\Lambda) = 0$ for $i > 0$. As $\operatorname{Tor}^\Lambda_i(X, \Lambda) = 0$ for $i > 0$, the decomposition (2.5) gives $\operatorname{Tor}^\Lambda_i(X, e\Gamma) = 0$ for every $i > 0$. Lemma 2.1(2) with $Z = FY$ gives $\operatorname{Ext}^n_\Gamma(FX, FY) \cong \operatorname{Ext}^n_\Lambda(X, (FY)e)$, and $(FY)e \cong Y$ by the unit isomorphism stated before Lemma 2.1. $\square$

3. Proof of Theorem A and the equivalence of the conjectures

This section first proves, for an individual algebra, that (TC2) for its two-fold trivial extension implies (ARC), and deduces Theorem A. It then recalls the known implications among the conditions defined in the Introduction, deduces Corollary B, and proves the consequence for (TC1).

Theorem 3.1. *Let Λ be an artin algebra. If $T_2(\Lambda)$ satisfies* (TC2)*, then Λ satisfies* (ARC).

Proof. Let $X \in \operatorname{mod}\Lambda$ satisfy (1.1). Put $\Gamma = T_2(\Lambda)$ and $e = \operatorname{diag}(1,0)$, and let $F = - \otimes_\Lambda e\Gamma$. By Lemma 2.3, Γ is self-injective. Since $\operatorname{Ext}^i_\Lambda(X,\Lambda) = 0$ for $i > 0$, Proposition 2.4 with $Y = X$ gives

$$\operatorname{Ext}^i_\Gamma(FX, FX) \cong \operatorname{Ext}^i_\Lambda(X,X) = 0 \qquad (i > 0).$$

Since Γ satisfies (TC2), the module FX is projective. By Lemma 2.1(1), X is projective. $\square$

Corollary 3.2. *Let R be a commutative artinian ring. If every self-injective artin R-algebra satisfies* (TC2)*, then every artin R-algebra satisfies* (ARC).

Proof. Let Λ be an artin R-algebra. By Lemma 2.3, $T_2(\Lambda)$ is a self-injective artin R-algebra, so it satisfies (TC2) by hypothesis. Theorem 3.1 shows that Λ satisfies (ARC). $\square$

Theorem 3.1 passes from Λ to the different algebra $T_2(\Lambda)$, which is why the arrow labelled Theorem A in (1.3) is an implication between assertions about all algebras.

To deduce Corollary B, we recall the known implications in (1.3).

Proposition 3.3. *Let R be a commutative artinian ring.*

(1) *Every artin R-algebra satisfies* (ARC) *if and only if every artin R-algebra satisfies* (GNC).
(2) *Let Λ be an artin R-algebra. If Λ satisfies* (GNC)*, then Λ satisfies* (AGC)*. If Λ satisfies* (AGC)*, then Λ satisfies* (NC).
(3) *Suppose that every artin R-algebra satisfies* (NC)*. Then every self-injective artin R-algebra satisfies* (TC2).
(4) *Let Λ be an artin R-algebra. If Λ satisfies* (ARC)*, then Λ satisfies* (GPC).

Proof. (1): See [2, Theorem 1.1], which states this with (ARC) replaced by the condition that every generator M with $\operatorname{Ext}^i(M,M) = 0$ for every $i > 0$ is projective. The two conditions agree for each algebra [1, pp. 283–284].

(2): See [8, §3.3].

(3): By [2, Corollary 1.2 and the remark following it], the hypothesis implies that every generator-cogenerator M over an artin R-algebra with $\operatorname{Ext}^i(M,M) = 0$ for every $i > 0$ is projective. If Λ is self-injective and $\operatorname{Ext}^i_\Lambda(X,X) = 0$ for every $i > 0$, apply this to $M = \Lambda \oplus X$. It is a generator, and it is a cogenerator because every indecomposable injective right module over the self-injective algebra Λ is projective and hence a direct summand of Λ_Λ. Since Λ_Λ is projective and injective, $\operatorname{Ext}^i_\Lambda(M,M) \cong \operatorname{Ext}^i_\Lambda(X,X) = 0$ for $i > 0$. Hence M is projective, and so is its direct summand X.

(4): This is noted in [5]. Every finitely generated Gorenstein-projective module X satisfies $\operatorname{Ext}^i_\Lambda(X,\Lambda) = 0$ for $i > 0$. If moreover $\operatorname{Ext}^i_\Lambda(X,X) = 0$ for $i > 0$, then X satisfies (1.1). $\square$

Corollary 3.4. *Let R be a commutative artinian ring. The following statements are equivalent.*

(1) *Every self-injective artin R-algebra satisfies* (TC2).
(2) *Every artin R-algebra satisfies* (ARC).
(3) *Every artin R-algebra satisfies* (GNC).
(4) *Every artin R-algebra satisfies* (AGC).
(5) *Every artin R-algebra satisfies* (NC).
(6) *Every artin R-algebra satisfies* (GPC).

Proof. Corollary 3.2 gives (1) $\Rightarrow$ (2). Proposition 3.3(1) gives (2) $\Rightarrow$ (3), Proposition 3.3(2) applied to each algebra gives (3) $\Rightarrow$ (4) $\Rightarrow$ (5), and Proposition 3.3(3) gives (5) $\Rightarrow$ (1). Proposition 3.3(4) applied to each algebra gives (2) $\Rightarrow$ (6). Finally, (6) $\Rightarrow$ (1) holds because a self-injective artin algebra satisfying (GPC) satisfies (TC2): every finitely generated module over it is Gorenstein-projective. $\square$

Finally, Corollary 3.2 shows that Tachikawa's second conjecture implies his first.

Corollary 3.5. *Let R be a commutative artinian ring. If every self-injective artin R-algebra satisfies* (TC2)*, then every artin R-algebra satisfies* (TC1).

Proof. Let Λ be an artin R-algebra with $\mathrm{Ext}^i_\Lambda(D\Lambda, \Lambda) = 0$ for every $i > 0$. The right Λ-module $D\Lambda$ is injective, so $X = D\Lambda$ satisfies (1.1). By Corollary 3.2, Λ satisfies (ARC), so $D\Lambda$ is projective. Every indecomposable injective right Λ-module is a direct summand of $D\Lambda$, and is therefore projective. The numbers of isomorphism classes of indecomposable injective and of indecomposable projective right Λ-modules are both equal to the number of isomorphism classes of simple modules, so every indecomposable projective right Λ-module is injective. Hence Λ is self-injective. $\square$

References

[1] M. Auslander, S. Ding, and Ø. Solberg, *Liftings and weak liftings of modules*, J. Algebra **156** (1993), no. 2, 273–317.

[2] M. Auslander and I. Reiten, *On a generalized version of the Nakayama conjecture*, Proc. Amer. Math. Soc. **52** (1975), 69–74.

[3] M. Auslander and I. Reiten, *k-Gorenstein algebras and syzygy modules*, J. Pure Appl. Algebra **92** (1994), no. 1, 1–27.

[4] A. Chan, E. Darpö, O. Iyama, and R. Marczinzik, *Periodic trivial extension algebras and fractionally Calabi–Yau algebras*, Ann. Sci. Éc. Norm. Supér. (4) **58** (2025), no. 2, 463–510.

[5] R. Luo and Z. Huang, *When are torsionless modules projective?*, J. Algebra **320** (2008), no. 5, 2156–2164.

[6] T. Nakayama, *On algebras with complete homology*, Abh. Math. Sem. Univ. Hamburg **22** (1958), 300–307.

[7] H. Tachikawa, *Quasi-Frobenius rings and generalizations: QF-3 and QF-1 rings*, Lecture Notes in Mathematics, vol. 351, Springer-Verlag, Berlin, 1973. Notes by C. M. Ringel.

[8] X. Tang and Z. Huang, *Higher differential objects in additive categories*, J. Algebra **549** (2020), 128–164.

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