Representation Theoretic Realization of Exact Categories

Haruhisa Enomoto

Graduate School of Mathematics, Nagoya University

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Outline

1. Introduction
2. Three Morita-type Results
   - Exact Category with a Progenerator
   - + Injective Cogenerator
   - + Higher Kernels
3. Applications
   - $(\operatorname{mod}\Lambda)/[\operatorname{Sub} M]$ as Torsionfree-class
   - Classification of CM-finite IG algebras

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Characterization of Module Categories

Theorem (Morita Theorem)

The following are equivalent for a category $\mathcal{E}$ and a field k.

1. $\mathcal{E} \simeq \operatorname{mod} \Lambda$ *for some f.d. k-algebra Λ.*
2. $\mathcal{E}$ *is an abelian Hom-finite k-category with a projective generator P and an injective cogenerator I.*

Proof.

(1) $\Rightarrow$ (2): $P := \Lambda$ and $I := D\Lambda = \operatorname{Hom}_k(\Lambda, k)$.
(2) $\Rightarrow$ (1): $\Lambda := \operatorname{End}_{\mathcal{E}}(P)$ and Consider the functor

$$\mathcal{E}(P, -) : \mathcal{E} \to \operatorname{mod} \Lambda.$$

This is equivalence. □

Motivation

Problem

Find a similar condition for an exact category $\mathcal{E}$.

1. (R): $\mathcal{E} \simeq$ (Representation theoretic category)
2. (C): $\mathcal{E}$ is an exact cat. with (Categorical property)

i.e. Morita type theorem for exact categories.

Assumption:
k: field. All algebras are f.d. over k and All Categories are:

- skeletally small
- additive Hom-finite k-categories
- idempotent complete.

Outline

(R): Extension-Closed Subcategory

Λ: f.d. algebra. $\operatorname{mod}\Lambda$: the category of f.d. right Λ-modules.

Definition

$\mathcal{E} \subset \operatorname{mod}\Lambda$: subcategory is extension-closed $:\Leftrightarrow$
for every exact sequence in $\operatorname{mod}\Lambda$

$$0 \to L \to M \to N \to 0,$$

$L \in \mathcal{E}$ and $N \in \mathcal{E}$ implies $M \in \mathcal{E}$.

Example

For $U \in \operatorname{mod}\Lambda$,

$${}^{\perp}U := \{M \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(M, U) = 0 \text{ for all } i > 0\}$$

(R): Resolving Subcategory

Definition (Auslander-Bridger 1969)

$\mathcal{E} \subset \operatorname{mod}\Lambda$: subcategory is resolving :$\Leftrightarrow$

1. $\mathcal{E}$ is ext-closed in $\operatorname{mod}\Lambda$.
2. All projective Λ-modules are in $\mathcal{E}$.
3. For every $M \in \mathcal{E}$, there is an exact sequence in $\operatorname{mod}\Lambda$

$$0 \to \Omega M \to P \to M \to 0$$

where P is projective Λ-module and $\Omega M \in \mathcal{E}$.

Example

For $U \in \operatorname{mod}\Lambda$, the subcat. ${}^{\perp}U$ is always resolving.

(C): Exact Category

Definition (Quillen 1973)

An exact category consists of the pair $(\mathcal{E}, F)$, where $\mathcal{E}$ is the category and F is the class of short exact sequences in $\mathcal{E}$ satisfying some conditions.

Remark

- *Any ext-closed subcat.* $\mathcal{E} \subset \operatorname{mod} \Lambda$ *is an exact category, whose s.e.s. are those in* $\operatorname{mod} \Lambda$ *with all terms in* $\mathcal{E}$.
- *Conversely, any exact cat. is an ext-closed subcat. of some big abelian category (Gabriel-Quillen).*

1st Morita-type Theorem

Proposition (folklore?)

For an exact category $\mathcal{E}$, TFAE.

1. *(C): $\mathcal{E}$ has a projective generator P, i.e. P is projective in $\mathcal{E}$ and for every $M \in \mathcal{E}$, there is a s.e.s. in $\mathcal{E}$*

$$0 \to \Omega M \to P^n \to M \to 0.$$

2. *(R): $\mathcal{E} \simeq$ (resolving subcat) of* mod Λ *for some f.d. alg. Λ.*

Categorical Property	Rep. Theoretic Property
Exact Cat with Progen.	Resolving subcat of mod Λ.

(R): Wakamatsu-tilting Module

For $W \in \operatorname{mod}\Lambda$, ${}^{\perp}W := \{M \in \operatorname{mod}\Lambda \mid \operatorname{Ext}_{\Lambda}^{>0}(M, W) = 0\}$.
$\mathsf{X}_W \subset {}^{\perp}W \subset \operatorname{mod}\Lambda$ consists of $X \in {}^{\perp}W$
such that there is an exact sequence

$$0 \to X \to W^{a_0} \xrightarrow{f^1} W^{a_1} \xrightarrow{f^2} W^{a_2} \to \cdots .$$

with $\operatorname{Im} f^i \in {}^{\perp}W$.
(W behaves like injective cogenerator in X_W)

Definition (Wakamatsu 1988)

$W \in \operatorname{mod}\Lambda$ is called Wakamatsu-tilting (or semi-dualizing) :⇔

1. $\operatorname{Ext}_{\Lambda}^{>0}(W, W) = 0$, i.e. $W \in \mathsf{X}_W$.
2. $\Lambda_{\Lambda} \in \mathsf{X}_W$.

(R): Properties of Wakamatsu-tilting Module

Proposition

For $W \in \operatorname{mod} \Lambda$*: Wakamatsu-tilting,*

- $\mathcal{X}_W \subset \operatorname{mod} \Lambda$ *is resolving, hence exact category.*
- $\mathcal{X}_W$ *has a progenerator* Λ *and an injective cogenerator* W*.*

Example

- Λ_Λ is always Wakamatsu-tilting.
  $\Rightarrow \mathcal{X}_\Lambda =: \operatorname{GP} \Lambda$, the cat. of Gorenstein projective modules.
  This is Frobenius exact category
  (:$\Leftrightarrow$ progenerator and inj. cogen. exist and coincide).
- Tilting and Cotilting modules are Wakamatsu-tilting.

2nd Morita-type Theorem

Theorem

For an exact category $\mathcal{E}$, TFAE.

1. *(C): $\mathcal{E}$ has a progenerator P and an injective cogenerator I.*
2. *(R): $\mathcal{E} \simeq$ (resolving-coresolving subcat) of X_W for some f.d. algebra Λ and Wakamatsu-tilting Λ-module W.*

Exact Cat. with...	Rep. Theoretic Property
Progenerator	Resolving subcat of mod Λ.
Progen. and Inj. cogen.	Resolving-Coresolving subcat of X_W for Wak. tilting W.

(R): Cotilting Module

Definition (Miyashita 1986)

$U \in \operatorname{mod}\Lambda$ is *n-cotilting* for $n \geq 0$:$\Leftrightarrow$

1. $\operatorname{Ext}_\Lambda^{>0}(U, U) = 0$.
2. $\operatorname{id} U_\Lambda \leq n$.
3. There exists an exact sequence in $\operatorname{mod}\Lambda$

$$0 \to U^{a_n} \to \cdots \to U^{a_1} \to U^{a_0} \to D\Lambda \to 0,$$

where $D\Lambda = \operatorname{Hom}_k(\Lambda, k)$.

Remark

Cotilting module U is always Wakamatsu-tilting and $\mathsf{X}_U = {}^\perp U$.

(R): Properties of Cotilting Module

Proposition

For U: a cotilting Λ-module, ${}^{\perp}U (= \mathsf{X}_U)$ is an exact category with a projective generator Λ and an injective cogenerator U.

Example

- 0-cotilting module = inj. cogen. of $\operatorname{mod}\Lambda$ = $D\Lambda$.
- Cotilting module = dual of (Miyashita) tilting module.
- Λ_Λ is cotilting $\Leftrightarrow$ Λ is Iwanaga-Gorenstein, i.e. $\operatorname{id}\Lambda_\Lambda = \operatorname{id}{}_\Lambda\Lambda < \infty$.
  In this case, ${}^{\perp}\Lambda =: \operatorname{CM}\Lambda$, the category of Cohen-Macaulay Λ-modules, which is Frobenius category.

(C): n-Kernels for $n \geq 1$

$\mathcal{E}$: category.

Definition (Jasso 2016)

$\mathcal{E}$ has *n-kernels* $:\Leftrightarrow$ for every $M \to N$ in $\mathcal{E}$, there is a cpx. in $\mathcal{E}$

$$0 \to K_n \to \cdots \to K_1 \to M \to N$$

such that for every $X \in \mathcal{E}$,

$$0 \to \mathcal{E}(X, K_n) \to \cdots \to \mathcal{E}(X, K_1) \to \mathcal{E}(X, M) \to \mathcal{E}(X, N)$$

is exact.

n-cotilting $\leftrightarrow$ $(n-1)$-kernel,

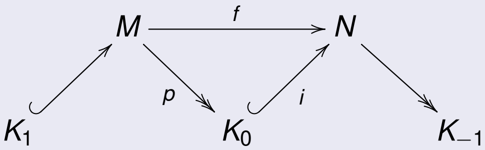
so we need 0-kernel and (-1)-kernel!

(C): n-Kernels for $n = 0, -1$

$\mathcal{E}$: exact category.
$Y \to Z$ in $\mathcal{E}$ is admissible epi
:$\Leftrightarrow$ there is a short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{E}$.

Definition (E)

- $\mathcal{E}$ has 0-kernel :$\Leftrightarrow$ every $f : M \to N$ can be written as $f = i \circ p$ with p: adm. epi and i:mono.
- $\mathcal{E}$ has (-1)-kernel :$\Leftrightarrow$ every $f : M \to N$ can be written as $f = i \circ p$ with p: adm. epi and i: adm. mono.



(C): Properties of Higher Kernels

$\mathcal{E}$: exact category.

The lower kernel $\mathcal{E}$ has, the better $\mathcal{E}$ behaves!

Proposition

- *$\mathcal{E}$ has (-1)-kernel $\Leftrightarrow$ $\mathcal{E}$ is abelian with usual exact str.*
- *$\mathcal{E}$ has n-kernel $\Rightarrow$ $\mathcal{E}$ has m-kernels for all $m \geq n$.*

Remark

For $n \geq 1$, $\mathcal{E}$ has n-kernel
$\Leftrightarrow$ $\mathcal{E}$ has the global dimension $\leq n+1$ as an algebra $\mathcal{E}$.

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Main Theorem

Theorem (E)

Let $\mathcal{E}$ be an exact cat and $n \geq 0$. The following are equivalent:

1. (C): $\mathcal{E}$ has projective generator P, injective cogenerator I and $(n-1)$-kernel.
2. (R): $\mathcal{E} \simeq {}^{\perp}U \subset \operatorname{mod}\Lambda$ for some n-cotilting module U over some f.d. algebra Λ.

Example

For $n = 0$, "$\mathcal{E}$ has (-1)-kernel $\Leftrightarrow$ $\mathcal{E}$ is abelian" and ${}^{\perp}$(0-cotilting) $= \operatorname{mod}\Lambda$. Thus

1. (C): $\mathcal{E}$ has progen, inj cogen and abelian.
2. (R): $\mathcal{E} \simeq \operatorname{mod}\Lambda$ for some alg Λ.

This equivalence is our first observation.

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Main Theorem

Exact Cat. with...	Rep. Theoretic Property
Progenerator	Resolving subcat of mod Λ.
+ Inj. Cogen.	Resolving-Coresolving subcat of X_W for Wak. tilting W.
+ $(n-1)$-Kernel	${}^{\perp}U$ for n-cotilting U.

Corollary (KIWY)

Let $\mathcal{E}$ be an exact cat and $n \geq 0$. The following are equivalent:

1. *(C): $\mathcal{E}$ is Frobenius and has $(n-1)$-kernel.*
2. *(R): $\mathcal{E} \simeq \mathrm{CM}\,\Lambda$ for some n-Iwanaga-Gorenstein algebra Λ.*

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For $M \in \operatorname{mod}\Lambda$ over f.d. algebra Λ,
$\operatorname{Sub} M \subset \operatorname{mod}\Lambda$: subcat consisting of submodules of M^n.
$(\operatorname{mod}\Lambda)/[\operatorname{Sub} M]$: the ideal quotient.

Corollary (E)

There is another algebra Γ and 1-cotilting $U \in \operatorname{mod}\Gamma$ s.t.

$$(\operatorname{mod}\Lambda)/[\operatorname{Sub} M] \simeq {}^{\perp}U (= \operatorname{Sub} U) \subset \operatorname{mod}\Gamma.$$

Proof.

Construct an exact str. on $(\operatorname{mod}\Lambda)/[\operatorname{Sub} M]$ with 0-kernel. □

Remark

If M is simple, then $[\operatorname{Sub} M] = [M]$.
This is a generalization of APR tilting.

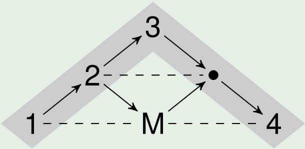
Example

Λ: the path algebra of $a \leftarrow b \leftarrow c$, M: simple at b.

mod Λ:

Sub M consists only of M.

$(\operatorname{mod}\Lambda)/[\operatorname{Sub} M]$: shaded.



$$(\operatorname{mod}\Lambda)/[\operatorname{Sub} M] \simeq {}^{\perp}U \subset \operatorname{mod}\Gamma.$$

$\Gamma := \operatorname{End}(1 \oplus 2 \oplus 3 \oplus 4)$

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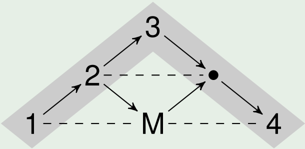

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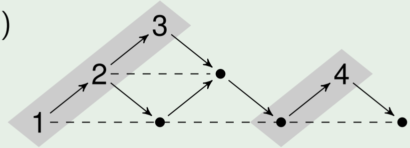

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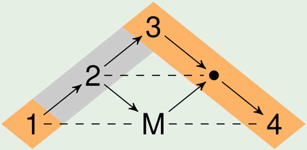

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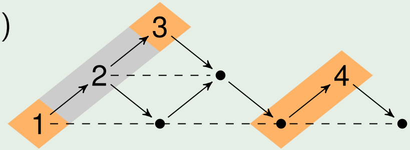

Classification of CM-finite IG algebras

Definition

- Λ is Iwanaga-Gorenstein $:\Leftrightarrow \operatorname{id} \Lambda_\Lambda = \operatorname{id} {}_\Lambda\Lambda < \infty$.
  ($\Leftrightarrow \Lambda_\Lambda$ is cotilting module.)
- $\operatorname{CM} \Lambda := {}^\perp\Lambda := \{M \in \operatorname{mod} \Lambda | \operatorname{Ext}_\Lambda^{>0}(M, \Lambda) = 0\}$
  (This is Frobenius category with progen. $= \Lambda$).
- Λ is CM-finite if $\operatorname{CM} \Lambda$ has finitely many indecomposables.

Example

- Algebra with finite global dimension Λ
  ($\operatorname{CM} \Lambda = \operatorname{proj} \Lambda$, the cat. of f.g. proj Λ-modules).
- Representation-finite self-injective algebra Λ
  ($\operatorname{CM} \Lambda = \operatorname{mod} \Lambda$).

Corollary (E, in preparation)

We can 'classify' all CM-finite Iwanaga-Gorenstein algebras.

Proof.

- There's a bijection {CM-finite IG alg. Λ} $\leftrightarrow$ {alg. Γ with fin. gl.dim. and Frobenius exact str. on proj Γ}.
- $\Lambda \mapsto$ its CM-Auslander alg Γ, then proj Γ $\simeq$ CM Λ.
  $\Gamma \mapsto$ End of progenerator of proj Γ.
- We cannot classify algebra with finite global dimension, BUT we can classify Frobenius exact str. on proj Γ by

$$\{ \text{ Set of stable } \tau\text{-orbits of proj}\,\Gamma \ \}.$$

□

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We can 'classify' all CM-finite Iwanaga-Gorenstein algebras.

Proof.

- There's a bijection {CM-finite IG alg. Λ} $\leftrightarrow$ {alg. Γ with fin. gl.dim. and Frobenius exact str. on $\operatorname{proj} \Gamma$}.
- $\Lambda \mapsto$ its CM-Auslander alg Γ, then $\operatorname{proj} \Gamma \simeq \operatorname{CM} \Lambda$.
  $\Gamma \mapsto$ End of progenerator of $\operatorname{proj} \Gamma$.
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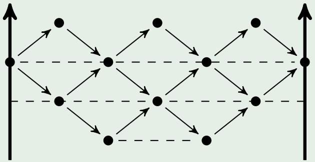
Example

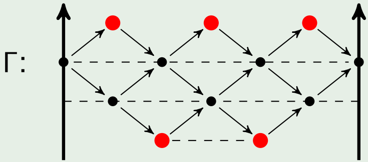
Γ:



with commutativity and zero relation.
Thus proj Γ has 2 stable τ-orbits.
$\Rightarrow$ proj Γ has 4 Frobenius exact structure.

Corresponding CM-finite IG Λ is the End of Red vertices, projective object in this exact structure.

Example



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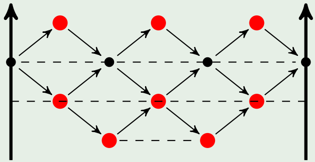

with commutativity and zero relation.
Thus $\operatorname{proj}\Gamma$ has 2 stable τ-orbits.
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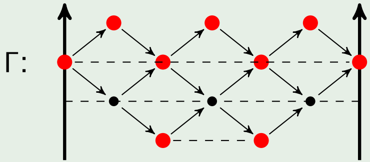
Corresponding CM-finite IG Λ is the End of Red vertices, projective object in this exact structure.

Example



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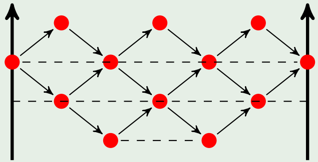

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