

MULTIPLICATIVE BASES AND DEFORMATIONS OF REPRESENTATION-FINITE ALGEBRAS

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ABSTRACT. Bautista, Gabriel, Roiter, and Salmerón proved that every finite-dimensional representation-finite algebra over an algebraically closed field admits a multiplicative basis and, outside characteristic two, is standard, meaning that its category of indecomposable modules is presented by the Auslander–Reiten quiver with mesh relations. We give a new, self-contained proof of both theorems by viewing the algebra as a filtered deformation of its standard form, which we construct from the radical layers of the module category and equip with a multiplicative basis. Outside characteristic two, this deformation is trivial, which yields both theorems at once. In characteristic two, we embed the algebra into an associative family of deformations of the standard form obtained by modifying the multiplication table only on pairs of basis elements whose product in the standard form is zero. After rescaling, the original algebra is isomorphic to a member of this family in which each modified product is either zero or a single basis element, so the multiplicative basis persists.

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1. INTRODUCTION

A finite-dimensional algebra is *representation-finite* if it has only finitely many isomorphism classes of indecomposable finite-dimensional modules. Although this condition is module-theoretic, it imposes strong combinatorial restrictions on the multiplication of the algebra itself: over an algebraically closed field, every representation-finite algebra admits a *multiplicative basis*, a basis in which the product of any two elements is either zero or another basis element (Theorem A below). Consequently, the entire algebra multiplication is encoded by a finite combinatorial table without scalar structure constants.

A companion structural theorem governs the category of indecomposable modules. A basic representation-finite algebra is called *standard* if one can choose representatives of the irreducible morphisms so that every morphism between indecomposable modules is a linear combination of paths in these representatives, and all relations among such paths follow from the quadratic mesh relations induced by almost split sequences. Outside characteristic two, every basic representation-finite algebra over an algebraically closed field is standard (Theorem B below). This structure theory was established in the landmark work of Bautista, Gabriel, Roiter, and Salmerón [2], abbreviated BGRS below, building on the construction of the standard form by Bretscher and Gabriel [8].

Because of the intricacy of the original combinatorial case analysis, finding a more transparent proof has remained a longstanding problem. Bongartz described the BGRS paper as “beautiful but hard to read” [6, Section 1] and surveyed subsequent simplifications and extensions; more recently, Ringel remarked that “any publication which provides help, provides improvement would be strongly appreciated” [20, Section 3].

In this paper we give an alternative, self-contained proof of both classical theorems. The central idea is to pass first to the radical grading of the module category to construct the standard form of the algebra, and then to analyze the original algebra as a filtered deformation of its standard form. All arguments required beyond basic Auslander–Reiten theory and elementary homological algebra are developed in full.

1.1. The classical results revisited

We begin with the existence theorem for multiplicative bases compatible with primitive idempotents and powers of the Jacobson radical.

Definition 1.1. A basis $\mathcal{B}$ of a finite-dimensional algebra A is *multiplicative* if $bc \in \mathcal{B} \cup \{0\}$ for all $b, c \in \mathcal{B}$. It is *normed* if it contains a complete set of primitive orthogonal idempotents and $\mathcal{B} \cap (\text{rad } A)^n$ is a basis of $(\text{rad } A)^n$ for every $n \geq 1$.

Bautista, Gabriel, Roiter, and Salmerón proved that every representation-finite algebra over an algebraically closed field admits a normed multiplicative basis [2, Sections 1.2 and 1.12].

Theorem A (= Theorem 11.7). Every finite-dimensional representation-finite algebra over an algebraically closed field has a normed multiplicative basis.

Using matrix units to account for the multiplicities of indecomposable projective summands, Lemma 2.1 reduces Theorem A immediately to basic algebras. We therefore focus on basic algebras in what follows.

Let A be a basic finite-dimensional representation-finite algebra over an algebraically closed field k , let $\text{ind}(\text{mod } A)$ be a skeletal category of indecomposable finite-dimensional right A -modules, and let Γ_A be its Auslander–Reiten quiver. Writing compositions of morphisms in path order from left to right, the components of an almost split sequence

$$0 \longrightarrow \tau X \xrightarrow{(a_i)_i} \bigoplus_i Y_i \xrightarrow{(b_i)_i} X \longrightarrow 0$$

satisfy $\sum_i a_i b_i = 0$. Let $\mathcal{J}$ denote the categorical Jacobson radical of $\text{ind}(\text{mod } A)$. Choosing a basis of each space $\mathcal{J}(X, Y)/\mathcal{J}^2(X, Y)$ of irreducible morphisms—which is at most one-dimensional when A is representation-finite (Lemma 2.2)—and expressing the components a_i and b_i modulo $\mathcal{J}^2$ in these bases yields a quadratic mesh relation at each nonprojective vertex X . The *mesh category* $k(\Gamma_A)$ has the vertices of Γ_A as objects and paths in Γ_A modulo these quadratic mesh relations as morphisms. Up to isomorphism fixing the objects, $k(\Gamma_A)$ is independent of the chosen bases, and its mesh coefficients can all be rescaled to one (Lemma 4.4).

Definition 1.2. A basic finite-dimensional representation-finite algebra A over an algebraically closed field k is *standard* if there is a k -linear equivalence

$$k(\Gamma_A) \simeq \text{ind}(\text{mod } A)$$

sending each vertex of Γ_A to the corresponding indecomposable module.

Theorem B (= Corollary 5.5). Every basic finite-dimensional representation-finite algebra over an algebraically closed field of characteristic different from two is standard.

In characteristic two, standardness can fail [6, Section 3.7], even though Theorem A remains valid. To treat all characteristics uniformly, we construct in Section 2 a basic representation-finite algebra A_0 whose category of indecomposable modules is equivalent to $k(\Gamma_A)$: this is the *standard form* of A , introduced classically by Bretscher and Gabriel [8]. Specifically, the powers $\mathcal{J}^n$ of the categorical radical filter the morphism spaces $\text{Hom}_A(P_y, P_x)$ between indecomposable projective modules by allowing factorizations through arbitrary indecomposable modules, not merely projective ones. This categorical filtration can be strictly finer than the ordinary radical filtration of A , and its associated graded algebra is A_0 . Thus A is naturally a positive filtered deformation of A_0 , and A is standard if and only if $A \cong A_0$ (Corollary 4.15). Outside characteristic two, we show in Section 5 that any higher-degree error terms in the mesh relations can be eliminated degree by degree by adjusting the chosen irreducible morphisms; hence the filtered deformation is trivial and Theorem B follows.

1.2. The construction and its relation to earlier proofs

Bretscher and Gabriel constructed the standard form as the opposite endomorphism algebra of the projective vertices inside the mesh category [8, Section 3.1]. We recover this description directly by showing that the associated graded category $\text{gr}_{\mathcal{J}} \text{ind}(\text{mod } A)$ is isomorphic to $k(\Gamma_A)$ and that evaluation at the projective objects gives an equivalence $\text{add } k(\Gamma_A) \simeq \text{mod } A_0$ (Propositions 2.6 and 2.10; compare Iyama’s graded realization theorems for τ -categories [12, Theorems 5.2 and 9.2]). In Section 4, we construct a normed multiplicative basis of A_0 by lifting the mesh category to the universal cover of the Bongartz–Gabriel mesh complex [7, Section 4], extending the basis one projective vertex at a time as in [8, Section 2.6], and descending along the free deck group.

In characteristic two, A need not be isomorphic to A_0 . Instead, we keep the multiplicative basis of A_0 and study deformations of A_0 obtained by modifying the multiplication table only on pairs of basis elements whose product in A_0 is zero, setting each such modified product equal to a scalar multiple of another basis element. Using a Bockstein connecting map from coefficients modulo four, we show that cocycles of this form span the positive-degree Hochschild cohomology spaces governing filtered deformations of A_0 (Corollary 10.5). The local factorization restrictions developed in Section 8 then imply that these zero-supported cocycles fit together into a global associative family of algebras over a polynomial ring (Proposition 11.3). By finite filtered normalization and diagonal rescaling along the weight lattice (Theorem 7.7 and Corollary 7.8), A is isomorphic to a fibre of this family whose parameters all take values in $\{0, 1\}$. In any such binary fibre, the basis of A_0 remains a normed multiplicative basis.

Bautista, Gabriel, Roiter, and Salmerón likewise obtained the nonstandard characteristic-two algebras by replacing specified zero products in the standard multiplication table with nonzero basis elements [2, Sections 9.2–9.7]. In our approach, the admissible modifications are organized cohomologically by Hochschild two-cocycles and their underlying derivation-lifting equations. Along the way, we establish two general deformation-theoretic tools of independent interest: a module-lifting theorem for positive filtered deformations based on internal-degree bounds for Ext groups (Lemma 6.3 and Proposition 6.5), and an elementary construction of associative multi-parameter families from compatible multiplication tables (Lemma 11.1). Table 1 summarizes the correspondence between the classical steps and the present proof.

Task	Classical constructions	Present organization
Construct the standard form	Take the endomorphism algebra of the projective vertices of the mesh category [8]; alternatively, build an algebra from the multiplication table of basis elements up to scalars [2].	Take the projective corner of the radical grading of $\text{ind}(\text{mod } A)$, then normalize the scalar structure constants on a universal cover.
Prove standardness ($\text{char } k \neq 2$)	Compare paths with equal values [2], or correct the mesh relations on stable components step by step [19].	Eliminate higher-degree error terms in the mesh relations degree by degree by modifying the chosen irreducible morphisms.
Treat characteristic two	Replace specified zero products by nonzero basis elements in explicit normal forms [2].	Represent positive-degree first-order deformations by zero-supported cocycles via $\mathbb{Z}/4$ -lifts of derivations, and realize A as a binary fibre of the resulting associative family.
Restrict local configurations	Exhibit representation-infinite subcategories and analyze contours locally [2]; see also the shorter mild-contour proofs in [5].	Bound trace spaces on τ -periodic components of Γ_A , and restrict factorizations of basis elements in A_0 via bimodule retractions onto representation-infinite incidence and bound-quiver algebras.

TABLE 1. Comparison between the classical proofs and the present approach.

The combinatorial case analysis that remains is concentrated in two places: Appendices A and B analyze the linearized mesh derivative on τ -periodic components of Γ_A , while Section 8 and Appendix C restrict how basis elements of A_0 can factor through one another. In the periodic mesh calculations we bound spaces of paths modulo cyclic rotation—recovering special cases of Schedler’s description of zeroth Hochschild homology of preprojective algebras [21, Theorem 13.1.1]—as well as twisted rotation under tree automorphisms. The degree-by-degree correction of irreducible morphisms follows Riedtmann’s method for stable blocks [19].

1.3. Reading routes

Readers interested only in the standard form or in the theorems outside characteristic two need only the first part of the paper. The table below indicates the sections required for each main result.

Reading goal	Route and conclusion
Construct the standard form	Sections 2–4, culminating in Corollary 4.15.
Prove Theorems A and B outside characteristic two	Continue through Section 5 to Corollary 5.5 (with Lemma 2.1 covering nonbasic algebras). Appendix A proves Propositions 4.1 and 5.3, whose statements suffice for the main text.
Prove Theorem A in characteristic two	After Sections 2–4, read Subsections 5.1 and 5.2 together with Sections 6–11, ending at Theorems 11.7 and 11.8. Section 12 establishes further properties of module categories and polynomial parameter spaces.

Organization. Part I (Sections 2–5) constructs the standard form A_0 , equips it with a normed multiplicative basis, and completes the proof of Theorems A and B outside characteristic two. Part II (Sections 6–7) turns to characteristic two: it relates positive filtered deformations of A_0 to the residual mesh cokernels on periodic components of Γ_A and reduces Theorem A to constructing zero-supported Hochschild cocycles (Proposition 7.9). Part III (Sections 8–12) constructs those cocycles by lifting derivation equations from $\mathbb{F}_2$ to $\mathbb{Z}$ and $\mathbb{Z}/4$ via induction on the number of basis elements, builds the resulting associative family of deformations to complete the proof of Theorem A, and develops additional consequences for module categories and mesh parameters in Section 12. Appendix A contains the periodic mesh calculations used in Part I, while Appendices B and C contain the characteristic-two mesh calculations and local ray exclusions.

Conventions and notation. Throughout the paper, k denotes an algebraically closed field unless explicitly stated otherwise. All algebras and modules are finite-dimensional, and modules are right modules. We assume familiarity with standard Auslander–Reiten theory, Morita equivalence [1], and basic homological algebra. We write $D_k = \text{Hom}_k(-, k)$ for standard vector-space duality and τX for the starting term of the almost split sequence ending at a nonprojective indecomposable module X .

For a basic algebra A with a chosen complete set of primitive orthogonal idempotents $\{e_x\}$, we set $A(x, y) = e_x A e_y$ and $P_x = e_x A$. Products of paths and morphisms are written in path order from left to right, whereas $\circ$ always denotes ordinary function composition: thus for morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, their path-order product is $fg = g \circ f$. An arrow $a : x \rightarrow y$ of the quiver of A lies in $A(x, y) = e_x A e_y$. Under the canonical identification $A(x, y) \cong \text{Hom}_A(P_y, P_x)$ sending a to left multiplication by a , endpoints are reversed, so the algebra product ab of $a \in A(x, y)$ and $b \in A(y, z)$ corresponds to the composite $a \circ b : P_z \rightarrow P_x$. Because endomorphism algebras are formed using the path-order product $fg = g \circ f$, we have $A \cong \text{End}_A(A)^{\text{op}}$ and, similarly, $A_0 = \text{End}_{\mathcal{C}}(\bigoplus_x P_x)^{\text{op}}$ in Construction 2.8.

For an additive category $\mathcal{C}$, we write $\text{add } \mathcal{C}$ for its closure under finite direct sums and direct summands. Any k -linear category $\mathcal{C}$ with finitely many objects and finite-dimensional morphism spaces is identified with its category algebra $\bigoplus_{i,j} \mathcal{C}(i, j)$, where the product of $f \in \mathcal{C}(i, j)$ and $g \in \mathcal{C}(j, l)$ is $fg = g \circ f$ and composites of noncomposable morphisms are zero; thus $\mathcal{C}(i, j)$ is the (i, j) -corner of the category algebra. We write $\mathcal{J}$ for the categorical Jacobson radical of a category of indecomposable modules and $\text{rad } A$ for the Jacobson radical of an algebra A .

Use of AI. The proof was developed by GPT-6-Astra under the author’s direction. The manuscript was written with GPT-6-Astra, followed by Claude Opus 5.5 and Gemini 4 Argon. GPT-6-Astra and Claude Opus 5.5 were used for formalization. The author directed the research and revisions and is responsible for the mathematical content and the final manuscript.

Part I. The standard form and the proof outside characteristic two

In this part we construct the standard form A_0 of a basic representation-finite algebra A in arbitrary characteristic, equip A_0 with a normed multiplicative basis, and prove Theorems A and B when $\text{char } k \neq 2$.

2. REDUCTION TO BASIC ALGEBRAS AND GRADED RECONSTRUCTION

We begin by reducing the existence of a normed multiplicative basis to the case of basic algebras. We then pass from the category of indecomposable modules to its radical grading and reconstruct the standard form A_0 from the graded projective objects. Both constructions, together with the normalization of A_0 in the next two sections, apply in arbitrary characteristic.

2.1. Multiplicative bases and matrix units

Using matrix units to account for the multiplicities of indecomposable projective summands, a normed multiplicative basis of a basic algebra lifts immediately to any Morita-equivalent finite-dimensional algebra.

Lemma 2.1. *Let A be a finite-dimensional algebra over an arbitrary field, and let B_0 be its basic algebra. If B_0 has a normed multiplicative basis, then so does A .*

Proof. Let $\{e_i\}$ be the complete set of primitive orthogonal idempotents contained in the given normed multiplicative basis of B_0 , and let n_i be the multiplicity of the corresponding indecomposable projective module in the right regular module A_A . Every element b of the basis of B_0 lies in a unique corner $e_i B_0 e_j$: indeed, in the decomposition $b = \sum_{i,j} e_i b e_j$, each nonzero summand belongs to the basis by multiplicativity, and since basis elements in distinct corners are linearly independent, only one summand can be nonzero.

Under Morita equivalence, the regular module A_A corresponds to the B_0 -module $\bigoplus_i (e_i B_0)^{n_i}$. Taking the opposite of its endomorphism algebra under our left-to-right composition convention identifies A with the block matrix algebra whose (i, j) -block is $\text{Mat}_{n_i \times n_j}(e_i B_0 e_j)$. For each basis element $b \in e_i B_0 e_j$, let $E_{i\alpha, j\beta}(b)$ ($1 \leq \alpha \leq n_i$, $1 \leq \beta \leq n_j$) denote the matrix having b in entry (α, β) of the (i, j) -block and zero elsewhere. These matrices form a basis of A and multiply according to the rule

$$E_{i\alpha, j\beta}(b) E_{l\gamma, m\delta}(c) = \begin{cases} E_{i\alpha, m\delta}(bc), & \text{if } (j, \beta) = (l, \gamma), \\ 0, & \text{otherwise.} \end{cases}$$

Since the given basis of B_0 is multiplicative, each product is either zero or another basis element. Moreover, the diagonal elements $E_{i\alpha, i\alpha}(e_i)$ form a complete set of primitive orthogonal idempotents of A , and for each $n \geq 1$ the (i, j) -block of $(\text{rad } A)^n$ is $\text{Mat}_{n_i \times n_j}(e_i (\text{rad } B_0)^n e_j)$. Thus the constructed basis intersects each radical power $(\text{rad } A)^n$ in a basis. $\square$

From now on we may therefore assume that A is basic, invoking Lemma 2.1 at the conclusion of each characteristic branch to recover the general case.

2.2. Radical layers and mesh relations

We filter the morphisms in $\text{ind}(\text{mod } A)$ by powers of the categorical radical, that is, by sums of composites of nonisomorphisms between indecomposable modules. Because intermediate modules in such a composite need not be projective, the induced filtration on morphisms between indecomposable projective modules is in general finer than the ordinary radical filtration of A .

Let A be a basic representation-finite algebra over k , and let $\mathcal{J}$ denote the categorical Jacobson radical of $\text{ind}(\text{mod } A)$. Since A is representation-finite, the direct sum of a representative set of indecomposable modules is a finite-dimensional additive generator;

its endomorphism algebra is finite-dimensional, and $\mathcal{J}$ consists of the matrix entries of its Jacobson radical, so $\mathcal{J}$ is nilpotent. We define the *radical grading* of $\text{ind}(\text{mod } A)$ to be the graded category $\mathcal{C}$ with the same objects as $\text{ind}(\text{mod } A)$ and morphism spaces

$$\mathcal{C}(X, Y) = \bigoplus_{n \geq 0} \mathcal{J}^n(X, Y) / \mathcal{J}^{n+1}(X, Y),$$

writing $\mathcal{C}_n(X, Y) = \mathcal{J}^n(X, Y) / \mathcal{J}^{n+1}(X, Y)$ for the homogeneous component of degree n . For a morphism $f \in \mathcal{J}^n(X, Y) \setminus \mathcal{J}^{n+1}(X, Y)$, its *initial form* is its residue class in $\mathcal{C}_n(X, Y)$. Composition in $\mathcal{C}$ is induced by composition in $\text{ind}(\text{mod } A)$. Since the endomorphism ring of each indecomposable module is local with residue field k , we have $\mathcal{C}_0(X, X) = k \cdot 1_X$ and $\mathcal{C}_0(X, Y) = 0$ for $X \neq Y$.

Let us recall how the Auslander–Reiten quiver Γ_A and its mesh relations arise in this setting. The vertices of Γ_A are the objects of $\text{ind}(\text{mod } A)$, and the arrows $X \rightarrow Y$ index a chosen basis of the space $\text{Irr}(X, Y) = \mathcal{J}(X, Y) / \mathcal{J}^2(X, Y)$ of irreducible morphisms. For each nonprojective indecomposable module Y , choose an almost split sequence

$$0 \longrightarrow \tau Y \longrightarrow E_Y \longrightarrow Y \longrightarrow 0.$$

Each indecomposable direct summand X of E_Y contributes a component map $X \rightarrow Y$ and a component map $\tau Y \rightarrow X$; modulo $\mathcal{J}^2$, the former components form a basis of $\text{Irr}(X, Y)$ and the latter a basis of $\text{Irr}(\tau Y, X)$ indexed by the multiplicity of X in E_Y . The next well-known lemma ensures that each multiplicity is at most one, so that these components differ from the chosen arrows only by nonzero scalars.

Lemma 2.2. *Let A be a basic representation-finite algebra over an algebraically closed field. Its Auslander–Reiten quiver Γ_A has no loops and at most one arrow between each ordered pair of vertices.*

Proof. Consider any irreducible morphism $f : X \rightarrow Y$ between indecomposable modules and factor f through its image. By irreducibility, either the canonical surjection $X \twoheadrightarrow \text{im } f$ is a split monomorphism or the inclusion $\text{im } f \hookrightarrow Y$ is a split epimorphism; thus one of the two factors is an isomorphism, and f is either a proper monomorphism or a proper epimorphism. In particular, $\dim X \neq \dim Y$, so Γ_A has no loops.

Suppose for a contradiction that there are at least two arrows $X \rightarrow Y$ in Γ_A . If $\dim X > \dim Y$, any irreducible morphism $X \rightarrow Y$ is an epimorphism, so Y cannot be projective (otherwise the epimorphism would split). The middle term of the almost split sequence ending at Y then contains $X^{\oplus 2}$ as a direct summand, which gives

$$\dim \tau Y \geq 2 \dim X - \dim Y > \dim X,$$

and the first map of the sequence provides at least two arrows $\tau Y \rightarrow X$. Iterating this step produces indecomposable modules of strictly increasing dimensions, contradicting the assumption that A is representation-finite. If $\dim X < \dim Y$, the dual argument applied to the almost split sequence starting at X yields the same contradiction. $\square$

Thanks to Lemma 2.2, we can formulate the mesh category of Γ_A using scalar coefficients rather than matrices.

Definition 2.3. A *translation quiver* is a quiver Q equipped with a bijection τ from a subset of vertices, called *nonprojective*, onto a subset of vertices, called *noninjective*, together with a bijection σ from the set of arrows ending at nonprojective vertices onto the set of arrows starting at noninjective vertices, such that $\sigma a : \tau x \rightarrow y$ for every arrow $a : y \rightarrow x$ with x nonprojective. We call σa the *mate* of a . Vertices outside the domain (respectively, image) of τ are called *projective* (respectively, *injective*). When both x and y are nonprojective, we write $\tau a = \sigma^2 a : \tau y \rightarrow \tau x$. The translation quiver Q is *stable* if every vertex is both nonprojective and noninjective.

Definition 2.4. Let Q be a translation quiver with at most one arrow between each ordered pair of vertices. Given a family $c = (c_{x,a})$ of nonzero scalars $c_{x,a} \in k^*$ indexed by nonprojective vertices x and arrows $a : y \rightarrow x$ ending at x , the associated *scalar mesh category* is the path-length graded quotient category

$$C(Q, c) = kQ / \langle r_x : x \text{ nonprojective} \rangle, \quad r_x = \sum_{a: y \rightarrow x} c_{x,a}(\sigma a)a,$$

where kQ denotes the k -linear path category of Q .

Multiplying a mesh relation r_x by a nonzero scalar leaves the relation ideal unchanged, while rescaling the arrows of Q rescales the coefficients $c_{x,a}$ and induces a graded isomorphism between the corresponding scalar mesh categories.

For $Q = \Gamma_A$, choose a basis element in each one-dimensional space $\text{Irr}(X, Y)$ to serve as the unique arrow $a : X \rightarrow Y$. For each nonprojective indecomposable module Y , the components of the first and second maps of the chosen almost split sequence ending at Y take the form $u_a(\sigma a)$ and $v_a a$ modulo $\mathcal{J}^2$, with $u_a, v_a \in k^*$. Since the composite of the two maps in the almost split sequence is zero, passing to degree two in $\mathcal{C}$ yields the mesh relation

$$r_Y = \sum_{a: X \rightarrow Y} c_{Y,a}(\sigma a)a, \quad c_{Y,a} = u_a v_a. \quad (1)$$

We write $k(\Gamma_A)$ for the scalar mesh category $C(\Gamma_A, c)$ defined by these coefficients.

Up to isomorphism fixing the objects, $k(\Gamma_A)$ is independent of both the chosen arrow representatives and the chosen almost split sequences. Indeed, any isomorphism between two almost split sequences ending at Y acts by scalars $\xi, \eta \in k^*$ on the indecomposable end terms τY and Y . Because the indecomposable summands of the middle term are pairwise nonisomorphic by Lemma 2.2, the automorphism of the middle term acts by scalars $\lambda_a \in k^*$ on those summands modulo radical maps, whose composites with the maps of the sequence vanish modulo $\mathcal{J}^2$. The coefficients therefore transform according to

$$u'_a = \lambda_a u_a \xi^{-1}, \quad v'_a = \eta v_a \lambda_a^{-1}, \quad c'_{Y,a} = \eta \xi^{-1} c_{Y,a},$$

which simply rescales the entire relation r_Y by $\eta \xi^{-1}$. Moreover, Lemma 4.4 will show that the arrows can always be rescaled so that $c_{Y,a} = 1$ for all Y and a .

In Sections 4 and 5 we will restrict a scalar mesh category $C(Q, c)$ to a subset S of vertices that is closed under τ and τ^{-1} wherever defined (such as the set of τ -periodic vertices). Let Q_S be the full translation subquiver on S . Quotienting $C(Q, c)$ by the ideal generated by the identity morphisms 1_x for $x \notin S$ and restricting to the objects in S gives a canonical isomorphism

$$(C(Q, c) / \langle 1_x : x \notin S \rangle) |_S \cong C(Q_S, c|_S). \quad (2)$$

Indeed, in the path category kQ , killing 1_x for $x \notin S$ annihilates precisely those paths that pass through a vertex outside S . Under this quotient, a mesh relation r_x with $x \notin S$ maps to zero, while a mesh relation r_x with $x \in S$ retains precisely the terms $(\sigma a)a$ whose middle vertex lies in S , which is the mesh relation of Q_S at x . In particular, if $C(Q, c)$ is finite-dimensional, then so is $C(Q_S, c|_S)$.

2.3. Exactness on radical layers

We next show that almost split sequences remain exact after passing to the associated graded category $\mathcal{C}$. This exactness is the key to identifying $\mathcal{C}$ with the mesh category $k(\Gamma_A)$ in Proposition 2.6(3) and to reconstructing the module category $\text{mod } A_0$ in Proposition 2.10. Our argument adapts Iyama's complementary-monomorphism technique for τ -categories [12, Sections 3.3, 3.6 and 4.2] directly to module categories.

Lemma 2.5. *Let A be an Artin algebra, let $\mathcal{J}$ be the categorical radical of its category of finite-length modules, and let $a : U \rightarrow Y$ be a radical monomorphism. Suppose that $a + h$*

is a monomorphism for every $h \in \mathcal{J}^2(U, Y)$. Then for every finite-length module W and every $n \geq 1$,

$$a \circ z \in \mathcal{J}^n(W, Y) \iff z \in \mathcal{J}^{n-1}(W, U).$$

Furthermore, any finite direct sum of left almost split monomorphisms satisfies the hypothesis that all such $\mathcal{J}^2$ -perturbations remain monomorphisms.

Proof. Step 1: Almost split kernels and the perturbation property. For any finite direct sum Y of indecomposable modules, let $p : E_Y \rightarrow Y$ be the direct sum of the minimal right almost split morphisms into its indecomposable summands (using the radical inclusion $\text{rad } P \hookrightarrow P$ at projective summands). By the lifting property of right almost split maps, every radical morphism into Y factors through p . For $n \geq 0$, writing a morphism in $\mathcal{J}^{n+1}(W, Y)$ as a sum of composites of $n + 1$ radical morphisms and factoring the last radical morphism through p shows that

$$p \circ \mathcal{J}^n(W, E_Y) = \mathcal{J}^{n+1}(W, Y) \quad (n \geq 0). \quad (3)$$

At each nonprojective indecomposable summand of Y , the kernel of p is the left almost split morphism of the corresponding almost split sequence, while at each projective summand the inclusion $\text{rad } P \hookrightarrow P$ has zero kernel. Consequently, the kernel inclusion $i : T \rightarrow E_Y$ of p is a finite direct sum of left almost split monomorphisms.

More generally, let $i = \bigoplus_j i_j : T \rightarrow E$ be any finite direct sum of left almost split monomorphisms $i_j : T_j \rightarrow E_j$. Any morphism in $\mathcal{J}^2(T, Z)$ is a sum of composites $v \circ u$ with u and v radical. Since the restriction of u to each T_j is radical, it factors through i_j as $u_j \circ i_j$, and $v \circ u_j$ remains radical. Thus

$$\mathcal{J}^2(T, Z) = \{q \circ i : q \in \mathcal{J}(E, Z)\}.$$

Taking $Z = E$, any $h \in \mathcal{J}^2(T, E)$ has the form $h = q \circ i$ with $q \in \mathcal{J}(E, E)$. Since $1 + q$ is an automorphism of E , the perturbation $i + h = (1 + q) \circ i$ is again a monomorphism.

Step 2: Splitting off the given monomorphism. Returning to the given monomorphism $a : U \rightarrow Y$, since a is radical and $p : E_Y \rightarrow Y$ is right almost split, we can write $a = p \circ u$ for some $u : U \rightarrow E_Y$. By performing row and column operations to split off any invertible components of u between indecomposable summands, we obtain decompositions $U = U'' \oplus U'$ and $E_Y = U'' \oplus E'$ under which $u = 1_{U''} \oplus u'$ with $u' \in \mathcal{J}(U', E')$. Since p is radical, the restriction $a|_{U'} = p|_{E'} \circ u'$ lies in $\mathcal{J}^2(U', Y)$. Setting $h = -a|_{U'} \circ \text{pr}_{U'} \in \mathcal{J}^2(U, Y)$, we find that $a + h$ vanishes on U' . Since $a + h$ is a monomorphism by hypothesis, we must have $U' = 0$, so u is a split monomorphism. Replacing p and i by $p \circ \alpha^{-1}$ and $\alpha \circ i$ for a suitable automorphism α of E_Y preserves (3) and the factorization property of i , and allows us to write

$$E_Y = V \oplus U, \quad p = (f, a), \quad i = \begin{pmatrix} b \\ c \end{pmatrix}.$$

Step 3: The complementary monomorphism. We claim that the first component $b : T \rightarrow V$ of i is also a radical monomorphism satisfying the $\mathcal{J}^2$ -perturbation hypothesis. First, b and c are radical because i is radical. If $x \in \ker b$, then $i(x) = (0, c(x))$ and $0 = p(i(x)) = a(c(x))$; since a is injective, $c(x) = 0$ and hence $i(x) = 0$, so $x = 0$. Next, given any $h \in \mathcal{J}^2(T, V)$, Step 1 lets us write $h = q \circ i$ for a radical map $q = (q_V, q_U) : V \oplus U \rightarrow V$. The upper triangular matrix

$$L = \begin{pmatrix} 1 + q_V & q_U \\ 0 & 1 \end{pmatrix}$$

is an automorphism of $V \oplus U$, and the kernel inclusion of $p \circ L^{-1}$ is $L \circ i = \begin{pmatrix} b+h \\ c \end{pmatrix}$. On the other hand, the restriction of $p \circ L^{-1}$ to U is $a - f \circ (1 + q_V)^{-1} \circ q_U$, which differs from a by a morphism in $\mathcal{J}^2(U, Y)$ and is therefore a monomorphism by hypothesis. As just shown for p and b , the injectivity of the restriction of $p \circ L^{-1}$ to U implies that the first component $b + h$ of its kernel inclusion is a monomorphism.

Step 4: Induction on the radical degree. We now prove by induction on $n \geq 1$, simultaneously for all radical monomorphisms a satisfying the perturbation hypothesis, that $a \circ z \in \mathcal{J}^n(W, Y)$ if and only if $z \in \mathcal{J}^{n-1}(W, U)$. Since a is radical, the “if” direction is immediate, as is the base case $n = 1$. For the induction step, suppose $a \circ z \in \mathcal{J}^{n+1}(W, Y)$ with $n \geq 1$. By (3), there exist $v \in \mathcal{J}^n(W, V)$ and $u \in \mathcal{J}^n(W, U)$ such that $a \circ z = p \circ \begin{pmatrix} v \\ u \end{pmatrix} = f \circ v + a \circ u$. Thus $\begin{pmatrix} v \\ u-z \end{pmatrix}$ lies in the kernel of p , so $\begin{pmatrix} v \\ u-z \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix} \circ s$ for some $s : W \rightarrow T$. In particular, $b \circ s = v \in \mathcal{J}^n(W, V)$. Applying the induction hypothesis to the complementary monomorphism b yields $s \in \mathcal{J}^{n-1}(W, T)$. Since c is radical, it follows that $u - z = c \circ s \in \mathcal{J}^n(W, U)$, and hence $z \in \mathcal{J}^n(W, U)$, completing the induction. $\square$

Applying Lemma 2.5 and its dual to the maps in each almost split sequence yields the graded realization theorem below (compare Iyama [12, Theorems 5.2 and 9.2] in the setting of τ -categories).

Proposition 2.6. *Let $\mathcal{C} = \text{gr}_{\mathcal{J}} \text{ind}(\text{mod } A)$. Then the following statements hold:*

- (1) *Let $0 \rightarrow \tau Y \rightarrow E_Y \rightarrow Y \rightarrow 0$ be an almost split sequence in $\text{mod } A$, and let $\tau Y \xrightarrow{\bar{i}} E_Y \xrightarrow{\bar{p}} Y$ be the sequence of initial forms of its morphisms in $\text{add } \mathcal{C}$. We call $\bar{p}$ the right mesh map at Y . For every object W of $\mathcal{C}$, the induced sequences*

$$0 \rightarrow \mathcal{C}(W, \tau Y) \rightarrow \mathcal{C}(W, E_Y) \rightarrow \mathcal{C}(W, Y), \quad 0 \rightarrow \mathcal{C}(Y, W) \rightarrow \mathcal{C}(E_Y, W) \rightarrow \mathcal{C}(\tau Y, W)$$
are exact; in particular, $\bar{i}$ is a monomorphism and $\bar{p}$ is an epimorphism in $\text{add } \mathcal{C}$.
- (2) *For each indecomposable projective module P , the initial forms of the components of the inclusion $\text{rad } P \hookrightarrow P$ define a monomorphism in $\text{add } \mathcal{C}$. Dually, for each indecomposable injective module I , the initial forms of the components of $I \twoheadrightarrow I/\text{soc } I$ define an epimorphism in $\text{add } \mathcal{C}$.*
- (3) *The chosen arrow representatives in $\mathcal{C}_1$ induce an isomorphism of graded categories $k(\Gamma_A) \xrightarrow{\sim} \mathcal{C}$, where $k(\Gamma_A)$ has the scalar mesh relations (1).*

Proof. Let $i : \tau Y \rightarrow E_Y$ and $p : E_Y \rightarrow Y$ be the maps in the almost split sequence ending at Y . By Lemma 2.5, i induces an injection $\mathcal{C}_{n-1}(W, \tau Y) \hookrightarrow \mathcal{C}_n(W, E_Y)$ for every $n \geq 1$. Moreover, if $w \in \mathcal{J}^n(W, E_Y)$ ($n \geq 1$) satisfies $p \circ w \in \mathcal{J}^{n+2}(W, Y)$, then (3) yields some $w' \in \mathcal{J}^{n+1}(W, E_Y)$ with $p \circ w' = p \circ w$. Since $p \circ (w - w') = 0$, we have $w - w' = i \circ z$ for some $z : W \rightarrow \tau Y$, and $i \circ z \in \mathcal{J}^n(W, E_Y)$ implies $z \in \mathcal{J}^{n-1}(W, \tau Y)$ by Lemma 2.5. In degree $n = 0$, any $w \in \text{Hom}_A(W, E_Y)$ with $p \circ w \in \mathcal{J}^2(W, Y)$ differs by an element of $\mathcal{J}(W, E_Y)$ from a morphism killed by p , which factors through the radical map i and therefore also lies in $\mathcal{J}(W, E_Y)$; thus $\bar{p}$ is injective on $\mathcal{C}_0(W, E_Y)$. Since $p \circ i = 0$, this proves exactness of the first sequence in (1).

For an indecomposable projective module P , let $p : \text{rad } P \hookrightarrow P$ be the inclusion. If $z \in \mathcal{J}^n(W, \text{rad } P)$ ($n \geq 0$) satisfies $p \circ z \in \mathcal{J}^{n+2}(W, P)$, then (3) gives $z' \in \mathcal{J}^{n+1}(W, \text{rad } P)$ with $p \circ z' = p \circ z$, and injectivity of p forces $z = z' \in \mathcal{J}^{n+1}(W, \text{rad } P)$. Thus the initial form of p is a monomorphism in $\text{add } \mathcal{C}$. Applying the dual arguments in the opposite module category establishes exactness of the second sequence in (1) and the statement for injective modules in (2).

Finally, sending each arrow of Γ_A to its chosen class in $\mathcal{C}_1$ defines a graded functor $k\Gamma_A \rightarrow \mathcal{C}$. By (1), the degree-two initial form of the relation $p \circ i = 0$ is precisely the mesh relation r_Y , so this functor descends to $k(\Gamma_A) \rightarrow \mathcal{C}$. It is full because each radical power $\mathcal{J}^n$ is generated by products of n morphisms in $\mathcal{J}$. To see that it is faithful, we show by induction on $n \geq 0$ that any homogeneous combination F of paths of length n from W to Y whose image vanishes in $\mathcal{C}_n(W, Y)$ belongs to the mesh ideal; the cases $n = 0, 1$ hold by definition of Γ_A . For $n \geq 2$, group F by its final arrow as $F = \sum_a F_a a$, where $a : X_a \rightarrow Y$. If Y is nonprojective, the components of $\bar{p} : E_Y \rightarrow Y$ are $v_a a$, so the tuple $(F_a v_a^{-1})_a$ maps to zero under $\bar{p} : \mathcal{C}_{n-1}(W, E_Y) \rightarrow \mathcal{C}_n(W, Y)$. By the exactness proved in (1), there is a path combination H of length $n - 2$ from W to τY whose image under $\bar{i}$

equals the image of $(F_a v_a^{-1})_a$ in $\mathcal{C}_{n-1}(W, E_Y)$. Since the a -component of $\bar{v}$ is $u_a(\sigma a)$ and $u_a v_a = c_{Y,a}$, each difference $F_a - H c_{Y,a}(\sigma a)$ has length $n - 1$ and maps to zero in $\mathcal{C}$. By the induction hypothesis, each such difference lies in the mesh ideal, and hence so does

$$F = H r_Y + \sum_a (F_a - H c_{Y,a}(\sigma a)) a.$$

If Y is projective, then $\bar{p}$ is a monomorphism by (2), so each F_a maps to zero in $\mathcal{C}_{n-1}(W, X_a)$ and lies in the mesh ideal by induction. $\square$

We also record the cokernels of the right mesh maps in $\mathcal{C}$, which provide minimal projective resolutions for the simple functors on $\mathcal{C}$.

Lemma 2.7. *For every nonprojective object X of $\mathcal{C}$, the image of the right mesh map $\bar{p} : \mathcal{C}(-, E_X) \rightarrow \mathcal{C}(-, X)$ is the radical $\bigoplus_{n \geq 1} \mathcal{C}_n(-, X)$ of the representable functor $\mathcal{C}(-, X)$, and its cokernel is the simple functor $\mathcal{C}_0(-, X)$. The same conclusion holds at each projective object P for the map $\mathcal{C}(-, \text{rad } P) \rightarrow \mathcal{C}(-, P)$ of Proposition 2.6(2).*

Proof. Let $p : E_X \rightarrow X$ be the minimal right almost split map in $\text{mod } A$ (or the inclusion $\text{rad } X \hookrightarrow X$ when X is projective). By (3), $p \circ \mathcal{J}^n(W, E_X) = \mathcal{J}^{n+1}(W, X)$ for every $n \geq 0$ and every object W . Passing to the associated graded spaces shows that the image of $\bar{p}$ on $\mathcal{C}(W, E_X)$ is precisely the positive-degree ideal $\bigoplus_{n \geq 1} \mathcal{C}_n(W, X)$. Since this ideal is nilpotent and the degree-zero quotient has $\mathcal{C}_0(X, X) = k$ and $\mathcal{C}_0(W, X) = 0$ for $W \neq X$, the positive-degree part is the radical of $\mathcal{C}(-, X)$ and its cokernel is the simple functor $\mathcal{C}_0(-, X)$. $\square$

2.4. Evaluation on projective objects

Following Bretscher and Gabriel [8, Section 3.1], we now define the standard form A_0 as the opposite endomorphism algebra of the projective objects in $\mathcal{C}$, and we show using the graded exactness properties above that evaluating objects of $\text{add } \mathcal{C}$ on the projectives gives an equivalence $\text{add } \mathcal{C} \simeq \text{mod } A_0$.

Construction 2.8. Let $\mathcal{D}$ be the full subcategory of $\mathcal{C}$ on the objects $P_x = e_x A$ corresponding to the indecomposable projective A -modules. We define the graded algebra $A_0 = \bigoplus_{x,y} A_0(x, y)$ by

$$A_0(x, y) = \mathcal{D}(P_y, P_x) = \bigoplus_{n \geq 0} \mathcal{J}^n(P_y, P_x) / \mathcal{J}^{n+1}(P_y, P_x),$$

with multiplication $bc = b \circ c \in \mathcal{D}(P_z, P_x)$ for $b \in \mathcal{D}(P_y, P_x)$ and $c \in \mathcal{D}(P_z, P_y)$ (so that c is applied first, then b). Equivalently, under our path-order convention for endomorphism algebras, $A_0 = \text{End}_{\mathcal{C}}(\bigoplus_x P_x)^{\text{op}}$. The *projective evaluation functor* sends each object $X \in \text{add } \mathcal{C}$ to the right A_0 -module $\bigoplus_x \mathcal{C}(P_x, X)$, on which $b \in A_0(x, y)$ acts on $m \in \mathcal{C}(P_x, X)$ by $mb = m \circ b \in \mathcal{C}(P_y, X)$.

To prove that projective evaluation is an equivalence of categories, we use a standard homological realization criterion from the Auslander correspondence (compare [11, Theorem 3.3]).

Lemma 2.9. *Let Λ be a finite-dimensional algebra over a field, let $e \in \Lambda$ be an idempotent, and set $\Lambda_e = e\Lambda e$. Suppose that*

$$\text{Hom}_{\Lambda}(T, \Lambda) = \text{Ext}_{\Lambda}^1(T, \Lambda) = 0 \quad \text{for every simple } \Lambda\text{-module } T \text{ with } Te = 0.$$

Then the functor $P \mapsto Pe$ is fully faithful on $\text{proj } \Lambda$. If, in addition, $\text{gl. dim } \Lambda \leq 2$ and every finite-dimensional injective Λ_e -module is isomorphic to Qe for some $Q \in \text{proj } \Lambda$, then $P \mapsto Pe$ is an equivalence of categories $\text{proj } \Lambda \xrightarrow{\sim} \text{mod } \Lambda_e$, and in particular Λ_e is representation-finite.

Proof. By induction on composition length using the long exact sequence for Ext , the assumed vanishings hold for every finite-dimensional Λ -module T satisfying $Te = 0$. For any right Λ -module M , the canonical multiplication map $(Me) \otimes_{\Lambda_e} e\Lambda \rightarrow M$ becomes an isomorphism upon multiplying on the right by e , so its kernel and cokernel are annihilated by e . For any $Q \in \text{proj } \Lambda$, applying $\text{Hom}_\Lambda(-, Q)$ and using the tensor–Hom adjunction therefore yields canonical isomorphisms

$$\text{Hom}_\Lambda(M, Q) \xrightarrow{\sim} \text{Hom}_\Lambda((Me) \otimes_{\Lambda_e} e\Lambda, Q) \xrightarrow{\sim} \text{Hom}_{\Lambda_e}(Me, Qe)$$

whose composite sends a homomorphism $f : M \rightarrow Q$ to its restriction $f|_{Me} : Me \rightarrow Qe$. Taking $M \in \text{proj } \Lambda$ shows that $P \mapsto Pe$ is fully faithful on $\text{proj } \Lambda$.

Now suppose $\text{gl. dim } \Lambda \leq 2$ and every injective Λ_e -module lies in the essential image of $\text{proj } \Lambda$. Given any $X \in \text{mod } \Lambda_e$, choose the beginning of an injective coresolution $0 \rightarrow X \rightarrow I^0 \rightarrow I^1$ in $\text{mod } \Lambda_e$. By full faithfulness and the assumption on injectives, we may lift $I^0 \rightarrow I^1$ to a homomorphism $f : Q^0 \rightarrow Q^1$ in $\text{proj } \Lambda$. Because $\text{gl. dim } \Lambda \leq 2$, the kernel $\ker f$ is a second syzygy over Λ and is therefore projective. Since the functor $M \mapsto Me$ is exact, we obtain $X \cong (\ker f)e$ with $\ker f \in \text{proj } \Lambda$, proving essential surjectivity. Finally, since $\text{proj } \Lambda$ has only finitely many indecomposable objects up to isomorphism, the equivalence implies that Λ_e is representation-finite. $\square$

We now verify the hypotheses of Lemma 2.9 for $\text{add } \mathcal{C}$ and A_0 .

Proposition 2.10. *Let A be a basic representation-finite k -algebra, and set $\mathcal{C} = \text{gr}_{\mathcal{J}} \text{ind}(\text{mod } A)$ and A_0 as in Construction 2.8.*

- (1) *The projective evaluation functor induces an equivalence of categories $\text{add } \mathcal{C} \xrightarrow{\sim} \text{mod } A_0$.*
- (2) *The algebra A_0 is basic and representation-finite.*
- (3) *If $X_0 \in \text{mod } A_0$ denotes the evaluation of an object $X \in \mathcal{C}$, then*

$$\dim_k X_0 e_x = \dim_k X e_x$$

for every primitive idempotent e_x .

Proof. Because the radical filtration on $\text{Hom}_A(X, Y)$ is finite, passing to the associated graded vector space preserves total dimension:

$$\dim_k \mathcal{C}(X, Y) = \sum_{n \geq 0} \dim_k \mathcal{J}^n(X, Y) / \mathcal{J}^{n+1}(X, Y) = \dim_k \text{Hom}_A(X, Y). \quad (4)$$

In particular, $\dim_k X_0 e_x = \dim_k \mathcal{C}(P_x, X) = \dim_k \text{Hom}_A(P_x, X) = \dim_k X e_x$, which proves (3).

To prove (1) and (2), let $G = \bigoplus_{X \in \text{ob } \mathcal{C}} X$ and set $\Lambda = \text{End}_{\text{add } \mathcal{C}}(G)^{\text{op}}$. The Yoneda functor $X \mapsto R_X = \mathcal{C}(G, X)$ is an equivalence $\text{add } \mathcal{C} \xrightarrow{\sim} \text{proj } \Lambda$, and as X ranges over $\text{ob } \mathcal{C}$, the modules R_X form a complete set of pairwise nonisomorphic indecomposable projective Λ -modules with simple tops $L_X = \mathcal{C}_0(G, X)$. Let $e \in \Lambda$ be the sum of the idempotents corresponding to the projective objects P_x . Then $e\Lambda e = A_0$, and under the Yoneda equivalence, projective evaluation on $\text{add } \mathcal{C}$ coincides with the functor $\text{ev}(M) = Me$ on $\text{proj } \Lambda$.

Global dimension and full faithfulness. For each nonprojective object $X \in \mathcal{C}$, Proposition 2.6(1) and Lemma 2.7 yield a projective resolution in $\text{mod } \Lambda$:

$$0 \longrightarrow R_{\tau X} \longrightarrow \bigoplus_{Y \rightarrow X} R_Y \longrightarrow R_X \longrightarrow L_X \longrightarrow 0.$$

For each projective object P_x , the same results yield a projective resolution $0 \rightarrow R_{\text{rad } P_x} \rightarrow R_{P_x} \rightarrow L_{P_x} \rightarrow 0$ of length at most one. Consequently, every simple Λ -module has projective dimension at most two, so $\text{gl. dim } \Lambda \leq 2$. Moreover, when X is nonprojective, applying $\text{Hom}_\Lambda(-, \Lambda)$ to the resolution of L_X gives by Yoneda's lemma the sequence

$0 \rightarrow \mathcal{C}(X, G) \rightarrow \mathcal{C}(E_X, G) \rightarrow \mathcal{C}(\tau X, G)$, which is exact by Proposition 2.6(1). Thus $\text{Hom}_\Lambda(L_X, \Lambda) = \text{Ext}_\Lambda^1(L_X, \Lambda) = 0$ for every simple module L_X annihilated by e , and Lemma 2.9 shows that ev is fully faithful on $\text{proj } \Lambda$.

Essential surjectivity. Let $S_x = P_x / \text{rad } P_x$ and $I_x = D_k(Ae_x)$ denote the simple and indecomposable injective A -modules at vertex x . By (4), the evaluation $S_x^0 = \text{ev}(R_{S_x})$ satisfies $\dim_k S_x^0 e_y = \dim_k \text{Hom}_A(P_y, S_x) = \delta_{xy}$, so S_x^0 is the simple A_0 -module at x . Set $N_x = \text{ev}(R_{I_x})$. Since ev is fully faithful on $\text{proj } \Lambda$, we have

$$\dim_k \text{Hom}_{A_0}(S_y^0, N_x) = \dim_k \mathcal{C}(S_y, I_x) = \dim_k \text{Hom}_A(S_y, I_x) = \delta_{xy},$$

which shows that $\text{soc } N_x \cong S_x^0$. On the other hand, (4) gives

$$\begin{aligned} \dim_k N_x e_y &= \dim_k \mathcal{C}(P_y, I_x) = \dim_k I_x e_y = \dim_k e_y A e_x \\ &= \dim_k \mathcal{C}(P_x, P_y) = \dim_k e_y A_0 e_x = \dim_k D_k(A_0 e_x) e_y. \end{aligned}$$

Since N_x has simple socle S_x^0 , it embeds into the injective hull $D_k(A_0 e_x)$, and equality of dimensions implies $N_x \cong D_k(A_0 e_x)$. Thus every indecomposable injective A_0 -module lies in the image of ev . Lemma 2.9 now implies that $\text{add } \mathcal{C} \xrightarrow{\sim} \text{mod } A_0$ is an equivalence. Since the objects P_x are pairwise nonisomorphic in $\mathcal{C}$, the algebra A_0 is basic as well as representation-finite. $\square$

Corollary 2.11. *The Auslander–Reiten quivers of A_0 and A are isomorphic as translation quivers: $\Gamma_{A_0} \cong \Gamma_A$.*

Proof. By Proposition 2.10(1), we may compute Γ_{A_0} directly inside $\text{add } \mathcal{C}$. Since $\mathcal{C}$ is generated in degree one, the radical of $\text{add } \mathcal{C}$ is its positive-degree ideal and the radical square is the ideal of degree at least two; hence the arrows of Γ_A form bases of the spaces of irreducible morphisms in $\text{add } \mathcal{C}$. By Lemma 2.7, each right mesh map $\bar{p} : E_X \rightarrow X$ is right almost split in $\text{add } \mathcal{C}$. Because the indecomposable summands of E_X are pairwise nonisomorphic by Lemma 2.2, any endomorphism of E_X is diagonal modulo the radical, and if it commutes with $\bar{p}$, the linear independence of the arrow components modulo the radical square forces its diagonal entries to be 1. Thus every such endomorphism is an automorphism, which means that $\bar{p}$ is right minimal. The mesh sequences in $\text{add } \mathcal{C}$ are therefore the almost split sequences of $\text{mod } A_0$, so the translation of Γ_{A_0} agrees with that of Γ_A . $\square$

We conclude this section with a simple example illustrating how the categorical filtration $\mathcal{J}^n(P_y, P_x)$ used to define A_0 can shift the degrees of generators of $\text{rad } A$.

Example 2.12. Let $A = k[t]/(t^2)$ be the algebra of dual numbers. Up to isomorphism, its indecomposable modules are $P = A$ and $S = A/(t)$, connected by the canonical projection $\pi : P \rightarrow S$ and the inclusion $\iota : S \hookrightarrow P$ sending $1 + (t)$ to t . In path order, the multiplication-by- t endomorphism of P factors as

$$P \xrightarrow{\pi} S \xrightarrow{\iota} P, \quad \pi \iota = t, \quad \iota \pi = 0.$$

Consequently, $\mathcal{J}$ is spanned by π , ι , and t , while $\mathcal{J}^2$ is spanned by t and $\mathcal{J}^3 = 0$. On the projective endomorphism ring $A = \text{End}_A(P)^{\text{op}}$, the categorical radical filtration is therefore

$$A \supset \mathcal{J}(P, P) = kt = \mathcal{J}^2(P, P) \supset \mathcal{J}^3(P, P) = 0,$$

whereas $(\text{rad } A)^2 = 0$. The associated graded algebra A_0 is again isomorphic to $k[t]/(t^2)$, but its radical generator t sits in categorical degree two, reflecting its factorization through the nonprojective simple module S .

3. RAYS OF THE ALGEBRA AND OF ITS STANDARD FORM

To compare the original algebra A with its standard form A_0 , we study the orbits of elements of $A(x, y)$ under the left and right actions of the unit groups of the local endomorphism rings $R_x = e_x A e_x$ and $R_y = e_y A e_y$. These orbits are the *rays* of A (Definition 3.2). In this section we show that the nonzero rays of A correspond bijectively via initial forms to the nonzero homogeneous components $A_0(x, y)_d$, each of which is one-dimensional, and that the homogeneous lines of A_0 form an associative category with zero under ray multiplication.

3.1. Endpoint subbimodules

For each primitive idempotent e_x of A , set $R_x = e_x A e_x$. The classical subbimodule structure of $A(x, y)$ over R_x and R_y follows from the finiteness of the ideal lattice of a representation-finite algebra (see Jans [13, Corollary 1.3 and Theorem 2.1] and Bongartz [6, Section 2.3]).

Lemma 3.1. *Let A be a basic representation-finite algebra over k .*

- (1) *The algebra A has only finitely many two-sided ideals, and every idempotent corner eAe is representation-finite (these two properties hold without assuming A basic).*
- (2) *For each primitive idempotent e_x , there is an isomorphism $R_x \cong k[t]/(t^{N_x})$ for some $N_x \geq 1$.*
- (3) *For each pair of vertices x, y , the R_x – R_y -subbimodules of $A(x, y)$ form a finite chain with one-dimensional successive quotients. In particular, for every nonzero element $f \in A(x, y)$, either $R_x f R_y = R_x f$ or $R_x f R_y = f R_y$.*

Proof. Each two-sided ideal I of A is the annihilator of the cyclic right module A/I . Because A has only finitely many isomorphism classes of modules of dimension at most $\dim_k A$, it has only finitely many two-sided ideals. For any idempotent $e \in A$ and $C = eAe$, the induction functor $V \mapsto V \otimes_C eA$ is fully faithful because the unit of its adjunction with the restriction functor $(-)_e$ is the canonical isomorphism

$$V \xrightarrow{\sim} (V \otimes_C eA)e \cong V \otimes_C eAe.$$

Since induction preserves finite dimension, indecomposability, and isomorphism classes, C is also representation-finite. This proves (1).

For a primitive idempotent $e = e_x$, every ideal I of the local algebra $C = eAe$ equals $e(AIA)e$, so C has only finitely many ideals. Every linear subspace of $\text{rad } C/(\text{rad } C)^2$ is the image of an ideal containing $(\text{rad } C)^2$; since k is infinite, a space of dimension at least two would contain infinitely many lines and thus produce infinitely many ideals of C . Hence $\dim_k \text{rad } C/(\text{rad } C)^2 \leq 1$. By Nakayama's lemma, any lift $t \in \text{rad } C$ of a spanning element of $\text{rad } C/(\text{rad } C)^2$ generates all powers of $\text{rad } C$. Since $\text{rad } C$ is nilpotent, t generates C as a k -algebra, giving $C \cong k[t]/(t^N)$ for some $N \geq 1$, which proves (2).

Similarly, any R_x – R_y -subbimodule $N \subseteq A(x, y)$ satisfies $e_x(ANA)e_y = N$, so $A(x, y)$ has only finitely many R_x – R_y -subbimodules. We may view $A(x, y)$ as a module over the commutative local algebra $T = R_x \otimes_k R_y^{\text{op}}$, which has residue field k . If $A(x, y)$ contained two incomparable T -submodules, their sum modulo their intersection would contain a direct sum of two nonzero submodules and hence a two-dimensional semisimple submodule $k^{\oplus 2}$ in its socle; pulling back the infinitely many lines in $k^{\oplus 2}$ would yield infinitely many T -submodules of $A(x, y)$, a contradiction. Thus the R_x – R_y -subbimodules of $A(x, y)$ form a chain, and its successive quotients are simple T -modules, hence one-dimensional.

Finally, for any nonzero $f \in A(x, y)$, the cyclic T -submodule $Tf = R_x f R_y$ is isomorphic to $T/\text{ann}_T(f)$. Because the ideals of the quotient ring $T/\text{ann}_T(f)$ also form a chain, its radical modulo its radical square has dimension at most one and is therefore spanned by the image of the radical generator of either R_x or R_y . By Nakayama's lemma, the

corresponding endpoint algebra surjects onto $T/\text{ann}_T(f)$, which means that $R_x f R_y = R_x f$ or $R_x f R_y = f R_y$. $\square$

Definition 3.2 ([2, Section 1]). For a nonzero element $f \in A(x, y)$, the *ray* of f is the orbit

$$[f] = \{u f v : u \in R_x^*, v \in R_y^*\}.$$

We set $U(f) = R_x f R_y$ and

$$R(f) = (\text{rad } R_x) f R_y + R_x f (\text{rad } R_y).$$

In each corner $A(x, y)$ we also include a zero ray 0. Since $U(f)$ depends only on the ray $r = [f]$, we often write $U_r = U(f)$.

By Lemma 3.1(3), the R_x - R_y -subbimodule $U(f)$ is uniserial with unique maximal proper subbimodule $R(f)$, and $U(f)$ is cyclic over either R_x or R_y . By Nakayama's lemma, the generators of $U(f)$ over that endpoint algebra are precisely the elements of $U(f) \setminus R(f)$, and any two such generators differ by a unit of that endpoint algebra. Thus

$$[f] = U(f) \setminus R(f). \quad (5)$$

Corollary 3.3. *Let A be a basic representation-finite algebra over k . Choose one representative from each nonzero ray of A , taking e_x as the representative of $[e_x]$. Then these representatives form a basis of A whose intersection with every two-sided ideal I of A is a basis of I .*

Proof. In each corner $A(x, y)$, the chosen representatives project to bases of the one-dimensional successive quotients $U(f)/R(f)$ of the finite chain of R_x - R_y -subbimodules. They therefore form a basis of $A(x, y)$ adapted to every subbimodule in the chain. Since $I \cap A(x, y) = e_x I e_y$ is an R_x - R_y -subbimodule, it belongs to this chain and is spanned by the representatives it contains. $\square$

To define multiplication of rays independently of the choice of representatives, one must quotient out contributions coming from the radical of the middle vertex.

Definition 3.4. For composable nonzero elements $f \in A(x, y)$ and $h \in A(y, z)$, define the subbimodule

$$M(f, h) = R_x f (\text{rad } R_y) h R_z.$$

The *ray product* of $[f]$ and $[h]$ is defined by

$$[f][h] = \begin{cases} [fh], & \text{if } fh \notin M(f, h), \\ 0, & \text{if } fh \in M(f, h), \end{cases} \quad (6)$$

and any product involving a zero ray is defined to be zero.

As the following example illustrates, a product fh of specific representatives can be nonzero in A even when its ray product $[f][h]$ is zero because $fh \in M(f, h)$.

Example 3.5. Let B be the bound quiver algebra given by

$$b \xrightarrow{\alpha} c \xrightarrow[\overset{q}{\curvearrowright}]{\beta} a \quad q^2 = 0, \quad \alpha\beta = 0.$$

This is a representation-finite string algebra with basis

$$e_b, e_c, e_a, \quad \alpha, q, \beta, \quad \alpha q, q\beta, \quad u = \alpha q\beta.$$

Each basis vector spans a distinct ray. Here $U_\alpha = \text{span}_k\{\alpha, \alpha q\}$ and $U_\beta = \text{span}_k\{\beta, q\beta\}$, while $M(\alpha, \beta) = ku$. Notice that α and $\alpha(1 + q)$ represent the same ray $[\alpha]$, yet $\alpha\beta = 0$ whereas $\alpha(1 + q)\beta = u \neq 0$. Because both products lie in $M(\alpha, \beta) = ku$, Definition 3.4 assigns $[\alpha][\beta] = 0$.

Lemma 3.6. *The ray product (6) is independent of the choice of representatives. In particular, if $[f][h] \neq 0$, then $f'h' \neq 0$ for all $f' \in [f]$ and $h' \in [h]$.*

Proof. Every unit of R_y has the form $\lambda(1 + n)$ with $\lambda \in k^*$ and $n \in \text{rad } R_y$. Changing the representatives f and h within their rays therefore replaces fh , up to action by outer units in $R_x^* \times R_y^*$, by $fh + fnh$ with $fnh \in M(f, h)$, while the subbimodule $M(f, h)$ itself is unchanged. If $fh \in M(f, h)$, then $fh + fnh \in M(f, h)$ as well. If $fh \notin M(f, h)$, then $U(fh) \not\subseteq M(f, h)$; since the subbimodules of $A(x, z)$ form a chain and $R(fh)$ is the unique maximal proper subbimodule of $U(fh)$, it follows that $M(f, h) \subseteq R(fh)$. Hence $fh + fnh \in U(fh) \setminus R(fh) = [fh]$, so the perturbed product is nonzero and defines the same ray $[fh]$. $\square$

We call the set of rays of A equipped with the operation (6) the *ray table* ray A . At this stage we do not yet claim that ray A is associative for an arbitrary A ; we first establish associativity for the graded standard form A_0 , and later transfer it to A in Proposition 8.3(2).

3.2. Homogeneous rays of the standard form

Consider the categorical radical filtration $F^n A(x, y) = \mathcal{J}^n(P_y, P_x)$ used in Construction 2.8 to define $A_0 = \text{gr}_F A$. Since each subspace $F^n A(x, y)$ is an R_x – R_y -subbimodule, multiplication by units in $R_x^* \times R_y^*$ preserves the filtration. Every nonzero ray r of A therefore has a well-defined categorical degree $\deg_{\mathcal{J}}(r)$, the unique integer d such that $r \subset F^d A \setminus F^{d+1} A$ (compare [8, Section 1.1]).

Lemma 3.7. *Let A be a basic representation-finite algebra over k . Each homogeneous component $A_0(x, y)_d$ has dimension at most one, and passing to initial forms induces a bijection between the nonzero rays of A in $A(x, y)$ and the nonzero homogeneous spaces $A_0(x, y)_d$. Moreover, if two rays r, r' in $A(x, y)$ satisfy $U_{r'} \subsetneq U_r$, then $\deg_{\mathcal{J}}(r') > \deg_{\mathcal{J}}(r)$.*

Proof. Let $A(x, y) = U_0 \supsetneq U_1 \supsetneq \cdots \supsetneq U_l = 0$ be the complete chain of R_x – R_y -subbimodules of $A(x, y)$. Its successive quotients U_i/U_{i+1} are one-dimensional, and $U_{i+1} = (\text{rad } R_x)U_i + U_i(\text{rad } R_y)$. Since every $F^d A(x, y)$ is an R_x – R_y -subbimodule, it must equal one of the U_i . Because $\text{rad } R_x \subseteq \mathcal{J}(P_x, P_x)$ and $\text{rad } R_y \subseteq \mathcal{J}(P_y, P_y)$, we have

$$F^d A(x, y) = U_i \implies U_{i+1} \subseteq F^{d+1} A(x, y) \subseteq U_i.$$

As d increases from 0, the filtration starts at $U_0 = A(x, y)$, ends at 0, and at each step either remains at U_i or drops by a single step to U_{i+1} . Consequently, each one-dimensional quotient U_i/U_{i+1} appears as $A_0(x, y)_d = F^d A(x, y)/F^{d+1} A(x, y)$ for a unique degree d . Since the nonzero rays of $A(x, y)$ are precisely the layers $U_i \setminus U_{i+1}$, all claims follow. $\square$

We now examine the intrinsic rays of the representation-finite graded algebra A_0 itself.

Proposition 3.8. *Let $P = \text{ray } A_0$ be the ray table of A_0 . Then:*

- (1) *each nonzero ray of A_0 contains the nonzero vectors of a unique homogeneous line $A_0(x, y)_d$, giving a bijection between the nonzero rays in $P(x, y)$ and the nonzero homogeneous lines $A_0(x, y)_d$;*
- (2) *for any two nonzero rays of A_0 , the product of their homogeneous lines in A_0 is nonzero if and only if their ray product in P is nonzero, in which case the product line represents that ray product;*
- (3) *P is an associative category with zero.*

Proof. By Proposition 2.10(2), A_0 is basic and representation-finite, so Lemma 3.1 applies to A_0 . For each $a \in k^*$, the grading automorphism of A_0 that scales degree d by a^d preserves every endpoint subbimodule of $A_0(x, y)$, because the subbimodules of $A_0(x, y)$ form a chain with at most one member of each dimension. Since k is infinite, invariance under these scaling automorphisms implies that every endpoint subbimodule is homogeneous. In

particular, its one-dimensional successive quotients are the nonzero homogeneous lines $A_0(x, y)_d$, and because the radical of each endpoint algebra of A_0 lies in positive degrees, each nonzero ray $U_i \setminus U_{i+1}$ contains the nonzero vectors of a unique homogeneous line $A_0(x, y)_d$. This proves (1).

For homogeneous elements $f \in A_0(x, y)_p$ and $g \in A_0(y, z)_q$, the product fg is homogeneous of degree $p + q$, whereas every element of $M(f, g) = R_x f(\text{rad } R_y) g R_z$ is supported in degrees strictly greater than $p + q$. Thus $fg \in M(f, g)$ if and only if $fg = 0$, which proves (2). Finally, (3) follows immediately from (2) and the associativity of multiplication in A_0 . $\square$

In Section 4 we will show that A_0 is isomorphic to the linearization of $P = \text{ray } A_0$, and in Theorem 11.8(3) we will also apply our constructions to algebras defined directly from combinatorial ray categories. We record the precise terminology for both settings here.

Definition 3.9. We adopt the following conventions:

- (1) When the ray table $\text{ray } A$ of a basic algebra A is associative, we call $\text{ray } A$ the *ray category* of A : its objects are the vertices indexing the primitive idempotents of A , and its composition is the ray product of Definition 3.4.
- (2) Given a finite skeletal category with zero Q , its *linearization* kQ is the k -algebra with basis consisting of the nonzero morphisms of Q and multiplication extending the composition in Q (with zero compositions in Q giving 0 in kQ). Note that here Q is already a category with zero, rather than a quiver whose path algebra is being formed.
- (3) A finite skeletal category with zero Q is called a *finite ray category* if the identity morphisms of Q form a complete set of primitive orthogonal idempotents in kQ , the ray product (6) on kQ is independent of the choice of representatives, and the map $r \mapsto [r]$ identifies Q with $\text{ray}(kQ)$.

When $\text{ray } A$ is associative, we write $k \text{ray } A$ for its linearization. For $P = \text{ray } A_0$, choose a nonzero homogeneous representative $b_r \in A_0$ for each nonzero ray $r \in P$, taking $b_{1_x} = e_x$ for the identity rays. By Proposition 3.8, there exist nonzero structure constants $\kappa(r, s) \in k^*$ such that

$$b_r b_s = \begin{cases} \kappa(r, s) b_{rs}, & \text{if } rs \neq 0, \\ 0, & \text{if } rs = 0, \end{cases} \quad (7)$$

and associativity of A_0 is equivalent to the two-cocycle identity $\kappa(r, s)\kappa(rs, t) = \kappa(s, t)\kappa(r, st)$ for all composable triples with $rst \neq 0$.

4. SCALAR NORMALIZATION ON A UNIVERSAL COVER

Starting from the homogeneous ray representatives $b_r \in A_0$ and their structure constants $\kappa(r, s) \in k^*$ in (7), we wish to find nonzero scalars $a_r \in k^*$ such that the rescaled elements $a_r b_r$ multiply with structure constants equal to 1, that is,

$$a_r a_s \kappa(r, s) = a_{rs} \quad \text{whenever } rs \neq 0.$$

Because distinct pairs of rays can have the same nonzero product, these equations are coupled across all corners of A_0 . Following Bongartz and Gabriel [7, Section 4] and Bretscher and Gabriel [8, Sections 2.6 and 3.1–3.2], we solve them by constructing a two-dimensional cell complex K from the Auslander–Reiten quiver Γ_A and its meshes, and lifting the mesh category $\mathcal{C}$ to the universal cover $\tilde{K}$. On the universal cover, the category of lifted projectives is directed and thin, allowing us to construct a multiplicative basis one vertex at a time. Because every multiplicative scalar weight on the cover is a ratio of object values, and the fundamental group $G = \pi_1(K)$ is free (Lemma 4.2), we can then rescale the lifted basis to make it G -invariant and descend it to A_0 .

4.1. A cell complex of meshes and its periodic components

Since a basic algebra is a direct product of connected blocks, we may work with one connected block at a time and assume throughout this section that A is connected; then Γ_A is connected by [1, Theorem VII.2.1]. Write $\Gamma = \Gamma_A$, and identify $\mathcal{C}$ with $\text{ind}(\text{mod } A_0)$ via Proposition 2.10. To form the cell complex K of [7, Section 4.1], start with the underlying undirected graph of Γ , adjoin a translation edge $t_X : \tau X \rightarrow X$ for every nonprojective vertex X , and for every arrow $a : Y \rightarrow X$ with X nonprojective, attach a two-dimensional triangle cell whose boundary is the closed walk $(\sigma a)t_X^{-1}$ from τX through Y and X back to τX . Here a *walk* in K means a path of edges traversed in either orientation; each triangle cell identifies the two-edge walk $(\sigma a)a$ up to homotopy with the translation edge t_X (rather than equating the individual summands of the mesh relation in the linear category).

Let Γ_{per} denote the full subquiver of Γ on the vertices that are periodic under τ , and call its connected components the *periodic components* of Γ . Because τ and τ^{-1} are everywhere defined on Γ_{per} and the mate bijection σ restricts to arrows between periodic vertices, Γ_{per} is a stable translation quiver. By (2), its scalar mesh category is the object quotient of $\mathcal{C}$ obtained by killing all nonperiodic vertices, and in particular is finite-dimensional. Riedtmann's structure theorem [18, Sections 1.5–1.6 and 2.1] describes such stable components as quotients of repetition quivers $\mathbb{Z}\Delta$, where for an oriented tree Δ the quiver $\mathbb{Z}\Delta$ has vertices $(x, j) \in \Delta_0 \times \mathbb{Z}$, arrows $(x, j) \rightarrow (y, j)$ and $(y, j) \rightarrow (x, j+1)$ for each arrow $x \rightarrow y$ of Δ , and translation $\tau(x, j) = (x, j-1)$. We prove the following self-contained form in Appendix A.1.

Proposition 4.1. *Let Q be a finite stable translation quiver with at most one arrow between each ordered pair of vertices. Fix a coefficient system c and suppose that the scalar mesh category $C(Q, c)$ of Definition 2.4 is finite-dimensional.*

- (1) *Every connected component of Q containing at least one arrow is isomorphic to $\mathbb{Z}\Delta/\Theta$, where Δ is a finite simply-laced Dynkin tree and Θ is an infinite cyclic group of automorphisms of $\mathbb{Z}\Delta$ acting freely on vertices and commuting with τ .*
- (2) *If Δ has type D_n , E_6 , E_7 , E_8 , or A_{2n+1} , root Δ at its unique branch vertex (for types D and E) or at its middle vertex (for type A_{2n+1}), and orient all edges of Δ away from the root. Then Θ is generated by $\tau^s \rho$ for some integer $s \geq 1$, where ρ is the automorphism $(v, j) \mapsto (\rho v, j)$ of $\mathbb{Z}\Delta$ induced by a root-fixing automorphism ρ of Δ . If $\Delta = A_{2n}$, then Θ is generated by η^r for some integer $r \geq 1$, where η is an automorphism of $\mathbb{Z}\Delta$ satisfying $\eta^2 = \tau^{-1}$ that interchanges the two halves of Δ .*

Geometrically, the mesh triangles lying over each edge of Δ form an infinite strip in $\mathbb{Z}\Delta$, so the subcomplex of K on a periodic component is the mapping torus of the tree Δ under the monodromy automorphism generating Θ . Using this description, we can deformation-retract each periodic component onto a circle and K itself onto a finite graph.

Lemma 4.2 ([7, Theorem 4.2 and its corollary]). *Let K be the mesh cell complex of the connected Auslander–Reiten quiver Γ_A defined in Subsection 4.1. Then K is homotopy equivalent to a finite graph, and consequently its fundamental group $G = \pi_1(K)$ is free.*

Proof. The partial mate bijection σ partitions the arrows of Γ_A into finite chains and finite cycles. Since $\sigma^2 = \tau$, an arrow lies in a cycle of σ if and only if both of its endpoints are τ -periodic; thus the cyclic σ -orbits are precisely the arrows of Γ_{per} . At the end of any finite σ -chain of length at least two, the terminal arrow belongs to only one mesh triangle of K . We may therefore collapse that free arrow together with its unique triangle, and repeat the collapse inductively along each finite σ -chain until only a single quiver edge per chain and the translation edges t_X remain outside the periodic subcomplexes.

Next, consider a periodic component $\mathbb{Z}\Delta/\Theta$, modeled as in Proposition 4.1. When Δ has a central root vertex fixed by ρ , we inductively prune the outermost leaves of Δ in

complete ρ -orbits. Specifically, for a leaf v of Δ with parent u , the arrows between their τ -orbits in $\mathbb{Z}\Delta$ are

$$a_j : u_j \longrightarrow v_j, \quad b_j : v_j \longrightarrow u_{j+1},$$

and the incident mesh triangles have boundaries $a_j b_j t_{u_{j+1}}^{-1}$ and $b_j a_{j+1} t_{v_{j+1}}^{-1}$. Because v is a leaf, the translation edge $t_{v_{j+1}}$ belongs only to the second triangle; collapsing $t_{v_{j+1}}$ across that triangle leaves the edge b_j incident only to the first triangle, and collapsing b_j across the first triangle leaves only the edges $a_j : u_j \rightarrow v_j$, which then contract onto the parent vertices u_j . Performing these elementary collapses simultaneously across a ρ -orbit of leaves is compatible with the Θ -action, and repeating the pruning down to the root contracts the periodic component onto the circle formed by the translation edges over the root.

When $\Delta = A_{2n}$, its center is the middle edge rather than a vertex. Pruning all other edges as above reduces the component to the periodic strip over the central edge. Subdividing this strip along its midline into two half-strips gives a deformation retraction onto the midline that is equivariant with respect to both τ and the involution η exchanging the two endpoints of the central edge; hence the quotient again contracts onto a circle. Likewise, any periodic component consisting of a single τ -orbit with no arrows is already a circle of translation edges t_X .

Finally, after contracting all periodic subcomplexes to circles and collapsing all non-periodic triangles as in the first paragraph, only one-dimensional cells remain, so K deformation-retracts onto a finite graph, whose fundamental group is free. $\square$

Reversing the elementary triangle collapses of Lemma 4.2 also shows that any closed edge weight on the periodic subcomplexes extends to all of K . Here an *edge weight* on K with values in an abelian group assigns opposite values to the two orientations of each edge, and it is *closed* if the sum of its values around the boundary of every triangle cell is zero.

Lemma 4.3. *Every closed edge weight on the union of the periodic subcomplexes of K extends to a closed edge weight on K .*

Proof. As in the first paragraph of the proof of Lemma 4.2, we may collapse all triangles outside the periodic subcomplexes one by one along finite σ -chains. Assign weight zero to all nonperiodic edges that remain after these collapses, and then restore the collapsed triangles in reverse order: since each restored triangle has a unique free edge that was removed at that step, its weight is uniquely determined by requiring the boundary sum around the triangle to vanish. This extends the given periodic weight to a closed edge weight on all of K . $\square$

As an immediate topological consequence of Lemma 4.2, we can normalize all scalar coefficients in the mesh relations of $k(\Gamma_A)$ to 1.

Lemma 4.4. *In every characteristic, any system of nonzero mesh coefficients $c_{X,a}$ on Γ_A can be normalized to 1 by rescaling the arrows and the mesh relations.*

Proof. Working on one connected component of Γ_A at a time, view the family of nonzero scalars $c_{X,a} \in k^*$ as a multiplicative cellular 2-cochain on K assigning $c_{X,a}$ to the triangle cell with boundary $(\sigma a) a t_X^{-1}$. A multiplicative 1-cochain ϕ on K assigns a nonzero scalar $\phi(e) \in k^*$ to each edge e of K , and its coboundary on the triangle (X, a) is

$$(d\phi)(X, a) = \phi(\sigma a) \phi(a) \phi(t_X)^{-1}.$$

Rescaling each arrow a of Γ_A by $\phi(a)$ and dividing each mesh relation r_X by $\phi(t_X)$ multiplies the coefficient $c_{X,a}$ by $(d\phi)(X, a)$ without changing the mesh ideal. Since K has no 3-cells and is homotopy equivalent to a graph by Lemma 4.2, we have $H^2(K; k^*) = 1$, so there exists a 1-cochain ϕ with $d\phi = c^{-1}$. This rescaling normalizes all mesh coefficients to 1 and induces an object-fixing isomorphism on $k(\Gamma_A)$. $\square$

4.2. The universal cover and the lifted mesh category

By Proposition 2.6(3), we identify $\mathcal{C}$ with the scalar mesh category $k(\Gamma_A)$ equipped with the coefficients (1). Let $\tilde{K}$ be the universal cover of K , whose vertices are homotopy classes of walks in K starting at a fixed basepoint, and on which the fundamental group $G = \pi_1(K)$ acts freely by deck transformations $\tilde{Y} \mapsto g\tilde{Y}$. Lifting the arrows and translation edges of Γ to $\tilde{K}$ produces a translation quiver $\tilde{\Gamma}$. Because each two-arrow path $(\sigma a)a$ in a mesh at X is homotopic in K to the translation edge t_X , all terms of a mesh relation r_X lift to paths in $\tilde{\Gamma}$ sharing the same source $\tau\tilde{X}$ and target $\tilde{X}$. Imposing the lifted relations $r_{\tilde{X}}$ with the same scalar coefficients $c_{X,a}$ defines the lifted scalar mesh category $\tilde{\mathcal{C}}$.

Any *contextual mesh relation*—that is, a relation of the form ur_Yv with u and v composable paths in Γ —lifts uniquely to $\tilde{\Gamma}$ once a lift of its starting vertex is chosen, and all of its terms share the same lifted endpoints. Consequently, both the path space and the mesh relation ideal decompose as direct sums over lifted endpoints, so pushing paths down to Γ induces a canonical linear isomorphism for any chosen lifts $\tilde{X}$ of X and $\tilde{Y}$ of Y :

$$\bigoplus_{g \in G} \tilde{\mathcal{C}}(\tilde{X}, g\tilde{Y}) \xrightarrow{\sim} \mathcal{C}(X, Y). \quad (8)$$

In particular, the uniform bound on the length of nonzero paths in the finite-dimensional category $\mathcal{C}$ also holds in $\tilde{\mathcal{C}}$. Note also that the lifted quiver $\tilde{\Gamma}$ is connected: in a module category every almost split sequence has a nonzero middle term, so every translation edge t_X in a walk on $\tilde{K}$ can be replaced by a two-arrow path $(\sigma a)a$ in $\tilde{\Gamma}$.

Assigning length 1 to each quiver arrow and length 2 to each translation edge t_X preserves length across every triangle cell of K . Because closed walks in the simply connected cover $\tilde{K}$ are null-homotopic, this edge length integrates on $\tilde{K}$ to an integer-valued *height function* on the vertices of $\tilde{\Gamma}$ that increases by 1 along every arrow. In particular, $\tilde{\Gamma}$ is directed, every object of $\tilde{\mathcal{C}}$ has trivial endomorphism ring k , and only finitely many vertices can be reached by directed paths between any two vertices of prescribed heights.

4.3. Finite subcategories and evaluation of lifted objects

Let $\mathcal{L}$ denote the opposite of the full subcategory of $\tilde{\mathcal{C}}$ on the lifted projective vertices. Taking the opposite reverses morphism directions so that they match those of $A_0(x, y) = \mathcal{C}(P_y, P_x)$: choosing one reference lift $\tilde{x}$ for each projective vertex P_x , the covering decomposition (8) gives a G -grading on A_0 ,

$$A_0(x, y) = \bigoplus_{g \in G} A_0(x, y)_g, \quad A_0(x, y)_g = \mathcal{L}(\tilde{x}, g\tilde{y}) = \tilde{\mathcal{C}}(P_{g\tilde{y}}, P_{\tilde{x}}),$$

which we refer to as the *deck grading* to distinguish it from the categorical integer grading of Section 2. Since each homogeneous component $A_0(x, y)_d$ of the categorical grading is at most one-dimensional by Lemma 3.7 and the two gradings are compatible, each nonzero homogeneous ray line $A_0(x, y)_d$ is concentrated in a single deck degree $g \in G$ and therefore lifts uniquely to a nonzero morphism space $\mathcal{L}(\tilde{x}, g\tilde{y})$ once the source lift $\tilde{x}$ is fixed.

More generally, both $\mathcal{L}$ and the finite categories constructed from integer weights in Section 8 are instances of the following standard covering construction for group-graded algebras.

Definition 4.5. Let B be an algebra graded by a group $\mathcal{G}$, with a complete set of primitive orthogonal idempotents $\{e_x\}$ homogeneous of identity degree. The *graded covering category* of B has objects (x, g) with $g \in \mathcal{G}$, morphism spaces

$$\mathrm{Hom}((x, g), (y, h)) = B(x, y)_{g^{-1}h},$$

and composition induced by the multiplication of B (for an additively written group $\mathcal{G}$, the degree $g^{-1}h$ is written $h - g$).

Because the deck group G acts freely on $\tilde{K}$ and the chosen lifts $\tilde{x}$ form a transversal for the G -orbits of lifted projectives, sending $g\tilde{x}$ to (x, g) identifies $\mathcal{L}$ with the graded covering category of A_0 for its deck grading:

$$\mathcal{L}(g\tilde{x}, h\tilde{y}) = \mathcal{L}(\tilde{x}, g^{-1}h\tilde{y}) = A_0(x, y)_{g^{-1}h}.$$

To transfer finite representation type from A_0 to finite full subcategories of $\mathcal{L}$ (and later to subquotients in Sections 8 and 9), we use the standard induction–restriction argument for split bimodule retractions (compare the cleaving-functor method in [2, Sections 3.2 and 3.8]).

Lemma 4.6. *Let $C \subseteq B$ be a unital subalgebra of a finite-dimensional algebra B over any field, let $q : C \twoheadrightarrow D$ be a quotient algebra, and suppose that q extends to a C -bimodule homomorphism $\pi : B \rightarrow D$. If B is representation-finite, then so is D .*

Proof. View any right D -module M as a right C -module via q . The canonical C -module homomorphism $M \rightarrow (M \otimes_C B)|_C$ sending m to $m \otimes 1$ admits the C -linear retraction $m \otimes b \mapsto m\pi(b)$. Thus M is a direct summand of the restriction to C of the B -module $M \otimes_C B$. Since B has only finitely many indecomposable modules up to isomorphism, only finitely many indecomposable C -modules can arise as direct summands of their restrictions, and since inflation along $q : C \twoheadrightarrow D$ is fully faithful, D is representation-finite. $\square$

Lemma 4.7. *Let B be a representation-finite algebra graded by a group $\mathcal{G}$, with primitive idempotents in the identity degree. Then every finite full subcategory of the graded covering category of B is representation-finite.*

Proof. Let H be the full subcategory on finitely many objects $(x_1, g_1), \dots, (x_m, g_m)$, so that $H(i, j) = B(x_i, x_j)_{g_i^{-1}g_j}$, and embed H into the ungraded category B' on the same object indices with morphism spaces $B'(i, j) = B(x_i, x_j)$. Because B' is Morita equivalent to the idempotent corner of B supported on the distinct vertices among $x_1, \dots, x_m$, it is representation-finite by Lemma 3.1(1). Moreover, the linear projection $\pi : B' \rightarrow H$ that projects each corner $B(x_i, x_j)$ onto its homogeneous component of degree $g_i^{-1}g_j$ is an H -bimodule retraction: for $p \in H(i, j)$, $b \in B'(j, l)$, and $q \in H(l, r)$, the product with the homogeneous component of b of degree $g_j^{-1}g_l$ is the unique component of pbq of degree $(g_i^{-1}g_j)(g_j^{-1}g_l)(g_l^{-1}g_r) = g_i^{-1}g_r$, so $\pi(pbq) = p\pi(b)q$. Applying Lemma 4.6 with $C = D = H$ shows that H is representation-finite. $\square$

Lemma 4.8. *The category $\mathcal{L}$ is connected and thin (meaning that $\dim_k \mathcal{L}(u, v) \leq 1$ for all objects u, v). For each object $\tilde{x}$ of $\mathcal{L}$, write $P_{\tilde{x}}$ for the corresponding lifted projective vertex in $\tilde{\mathcal{C}}$, and for each object $\tilde{X} \in \tilde{\mathcal{C}}$ define the $\mathcal{L}$ -module $V_{\tilde{X}}$ by*

$$V_{\tilde{X}}(\tilde{x}) = \tilde{\mathcal{C}}(P_{\tilde{x}}, \tilde{X}).$$

Then each $V_{\tilde{X}}$ is a finite-dimensional indecomposable $\mathcal{L}$ -module, and the evaluation functor $\tilde{X} \mapsto V_{\tilde{X}}$ on $\tilde{\mathcal{C}}$ is fully faithful.

Proof. Because the negative of height strictly increases along nonidentity morphisms of $\mathcal{L}$, every object of $\mathcal{L}$ has endomorphism ring k , and in any two-object full subcategory $\{u, v\}$ of $\mathcal{L}$ with $\mathcal{L}(u, v) \neq 0$ we have $\mathcal{L}(v, u) = 0$. Every linear subspace of $\mathcal{L}(u, v)$ is then a two-sided ideal of the two-object subcategory. Since this subcategory is representation-finite by Lemma 4.7, it has only finitely many ideals by Lemma 3.1(1), which forces $\dim_k \mathcal{L}(u, v) \leq 1$. Thus $\mathcal{L}$ is thin.

For any $\tilde{X} \in \tilde{\mathcal{C}}$, the uniform bound on nonzero path lengths in $\tilde{\mathcal{C}}$ implies that $V_{\tilde{X}}$ is supported on finitely many objects of $\mathcal{L}$. If $X \in \mathcal{C}$ denotes the projection of $\tilde{X}$, the covering isomorphism (8) gives, for each reference projective lift $\tilde{x}$,

$$\mathcal{C}(P_x, X) = \bigoplus_{g \in G} \tilde{\mathcal{C}}(P_{g\tilde{x}}, \tilde{X}) = \bigoplus_{g \in G} V_{\tilde{X}}(g\tilde{x}).$$

Summing over x identifies the evaluated A_0 -module $\text{ev}(X)$ of Proposition 2.10 with the direct sum of the spaces $V_{\tilde{X}}(\tilde{z})$ over all objects $\tilde{z} \in \mathcal{L}$. Any homomorphism $\varphi : V_{\tilde{X}} \rightarrow V_{\tilde{Y}}$ of $\mathcal{L}$ -modules acts componentwise on these spaces and commutes with the action of each deck-homogeneous element of A_0 , so it defines an A_0 -module homomorphism $\text{ev}(X) \rightarrow \text{ev}(Y)$ preserving each individual summand indexed by $\tilde{z}$. By Proposition 2.10(1), this A_0 -module map has the form $\text{ev}(f)$ for a unique $f \in \mathcal{C}(X, Y)$. Using (8), decompose $f = \sum_{g \in G} \bar{f}_g$, where $\bar{f}_g \in \mathcal{C}(X, Y)$ is the projection of $f_g \in \tilde{\mathcal{C}}(\tilde{X}, g\tilde{Y})$. Because composition with f_g maps $V_{\tilde{X}}(\tilde{z})$ into $\tilde{\mathcal{C}}(P_{\tilde{z}}, g\tilde{Y}) = V_{\tilde{Y}}(g^{-1}\tilde{z})$ and G acts freely, the component $\text{ev}(\bar{f}_g)$ for $g \neq 1$ maps the summand at $\tilde{z}$ into the distinct summand at $g^{-1}\tilde{z}$. Since $\varphi = \text{ev}(f)$ preserves each summand $\tilde{z}$, we must have $\text{ev}(\bar{f}_g) = 0$ for all $g \neq 1$; faithfulness of ev and (8) then give $\bar{f}_g = 0$ for $g \neq 1$, so $f = \bar{f}_1$ and φ is induced by $f_1 \in \tilde{\mathcal{C}}(\tilde{X}, \tilde{Y})$. This proves fullness, and the same decomposition shows faithfulness. In particular, $\text{End}_{\mathcal{L}}(V_{\tilde{X}}) \cong \tilde{\mathcal{C}}(\tilde{X}, \tilde{X}) = k$, so each $V_{\tilde{X}}$ is indecomposable.

Finally, since each $V_{\tilde{X}}$ is indecomposable, its support lies in a single connected component of $\mathcal{L}$. For every arrow $\tilde{X} \rightarrow \tilde{Y}$ of $\tilde{\Gamma}$, the induced nonzero homomorphism $V_{\tilde{X}} \rightarrow V_{\tilde{Y}}$ forces the supports of $V_{\tilde{X}}$ and $V_{\tilde{Y}}$ to lie in the same connected component of $\mathcal{L}$. Because $\tilde{\Gamma}$ is connected and every object $\tilde{x} \in \mathcal{L}$ belongs to the support of $V_{P_{\tilde{x}}}$, it follows that $\mathcal{L}$ is connected. $\square$

4.4. Extending a basis by one object

We will construct a multiplicative basis of $\mathcal{L}$ inductively along an exhaustion by finite full subcategories, adding one source or sink at a time. When we adjoin a source s to a finite directed thin category, the objects y receiving a nonzero morphism from s form a finite poset, and normalizing the new products amounts to trivializing a multiplicative 1-cocycle on the incidence category of that poset (compare Bretscher and Gabriel [8, Section 2.6]). Recall that the *incidence category* of a finite poset R has the elements of R as objects and a single basis morphism $x \rightarrow y$ whenever $x \leq y$, with the composite of any two comparable morphisms equal to the basis morphism between their endpoints; an *additive weight* on R with values in an abelian group M is a function $w(x, y) \in M$ on comparable pairs $x \leq y$ satisfying $w(x, y) + w(y, z) = w(x, z)$ for $x \leq y \leq z$.

Lemma 4.9. *If the incidence algebra of a finite poset R is representation-finite, then every additive weight w on R with values in an abelian group M has the form $w(x, y) = \zeta_y - \zeta_x$ for some family of object values $\zeta_x \in M$ (written multiplicatively as $w(x, y) = \zeta_y / \zeta_x$ when $M = k^*$).*

Proof. We proceed by induction on $|R|$. Choose a minimal element $s \in R$ and set $U = \{y \in R : s < y\} \subseteq R \setminus \{s\}$. We claim that distinct connected components of the subposet U lie in distinct connected components of $R \setminus \{s\}$. Suppose to the contrary that two connected components U_1, U_2 of U lie in the same connected component C of $R \setminus \{s\}$. For each scalar $\lambda \in k^*$, define a representation of R of dimension vector $(1, \dots, 1)$ by placing k at every vertex, the identity map 1_k on every relation inside $R \setminus \{s\}$ and from s to $U \setminus U_1$, and multiplication by λ on every relation from s to U_1 . Any isomorphism between two such representations must act by a constant scalar on the connected component C and therefore cannot change the ratio of the maps from s into U_1 and U_2 . Varying $\lambda \in k^*$ thus yields infinitely many pairwise nonisomorphic indecomposable representations, contradicting the hypothesis that R is representation-finite.

Because the incidence algebra of $R \setminus \{s\}$ is an idempotent corner of that of R , it is representation-finite by Lemma 3.1(1). By the induction hypothesis, after subtracting a coboundary $\zeta_y - \zeta_x$ on $R \setminus \{s\}$, we may assume that $w(x, y) = 0$ for all $x \leq y$ in $R \setminus \{s\}$. For any $y \leq z$ in U , the additivity condition $w(s, y) + w(y, z) = w(s, z)$ then gives $w(s, y) = w(s, z)$, so $w(s, -)$ is constant on each connected component of U . Since distinct

connected components of U lie in distinct connected components of $R \setminus \{s\}$, we may add a constant to ζ on each connected component of $R \setminus \{s\}$ to make $w(s, y) = \zeta_y - \zeta_s$ for all $y \in U$ without altering the zero weights on $R \setminus \{s\}$. $\square$

To apply Lemma 4.9 when adjoining a source s , we note that the functor $y \mapsto T(s, y)$ on a finite directed thin category T defines a thin module whose faithful support is a poset. We will reuse this observation in Sections 7 and 8.

Lemma 4.10. *Let T be a finite directed category, and let M be a covariant or contravariant k -linear functor from T to vector spaces satisfying $\dim_k M(y) \leq 1$ for every object y . Let R be the category obtained from T by passing to the quotient by the annihilator ideal $\text{ann}_T(M)$ and discarding the objects y with $M(y) = 0$. Then R is isomorphic to the incidence category of a finite poset, and if T is representation-finite, then so is R .*

Proof. Fix a nonzero basis vector in each one-dimensional space $M(y)$ for $y \in R$. Because R acts faithfully on M , every morphism space $R(y, z)$ acts faithfully on the one-dimensional spaces $M(y)$ and $M(z)$ and therefore has dimension at most one, spanned when nonzero by the unique morphism sending the chosen basis vector to the chosen basis vector. Since T is directed, the relation $R(y, z) \neq 0$ defines a partial order on the objects of R , and these normalized morphisms identify R with the incidence category of that poset. Finally, if T is representation-finite, then the full subcategory of T on the objects of R is an idempotent corner of T and hence representation-finite by Lemma 3.1(1); since R is a quotient of that corner, R is representation-finite as well. $\square$

Lemma 4.11. *Let T be a finite directed thin representation-finite category, let s be a source of T , and set $T' = T \setminus \{s\}$ and $M = \text{rad } T(s, -)$. Let R be the representation-finite poset incidence category obtained from T' and M as in Lemma 4.10.*

- (1) *Every multiplicative basis of T' extends to a multiplicative basis of T .*
- (2) *Let $\text{Scal}(T)$ denote the abelian group of multiplicative k^* -valued weights on the nonzero nonidentity morphism lines of T , that is, assignments $d(r) \in k^*$ satisfying $d(rr') = d(r)d(r')$ whenever $rr' \neq 0$. Then restriction to T' is a split surjection of abelian groups*

$$\text{Scal}(T) \longrightarrow \text{Scal}(T')$$

whose kernel is isomorphic to $(k^*)^{\pi_0(R)}$.

Both assertions hold dually when s is a sink of T .

Proof. Fix a multiplicative basis $\{b_{yz}\}$ of T' , and choose an arbitrary nonzero vector $f_y \in T(s, y)$ for each object y of R . Whenever a basis element $b_{yz} \in T'(y, z)$ acts nontrivially on M , so that $T(s, y)b_{yz} = T(s, z) \neq 0$, we have $f_y b_{yz} = \mu_{yz} f_z$ for a nonzero scalar $\mu_{yz} \in k^*$. Associativity of composition in T implies that $\mu_{yz}\mu_{zw} = \mu_{yw}$ for all comparable triples $y \leq z \leq w$ in R . By Lemma 4.9, there exist scalars $\zeta_y \in k^*$ such that $\mu_{yz} = \zeta_z/\zeta_y$. Replacing each f_y by $\zeta_y f_y$ gives $(\zeta_y f_y)b_{yz} = \zeta_z f_z$, extending the multiplicative basis of T' to T . This proves (1).

For (2), an element $d \in \text{Scal}(T')$ extends to $d \in \text{Scal}(T)$ if and only if its values $d(sy) \in k^*$ at the objects $y \in R$ satisfy $d(sy)d(b_{yz}) = d(sz)$ whenever $y \leq z$ in R . Choosing a base object y_D in each connected component D of R and a spanning tree of walks in D from y_D to every $y \in D$, Lemma 4.9 shows that we obtain a well-defined homomorphic section $\text{Scal}(T') \rightarrow \text{Scal}(T)$ by setting $d(sy_D) = 1$ and propagating $d(sy)$ along the walks in D via products and inverses of the values $d(b_{yz})$. Any two extensions of the same weight on T' differ by a constant scalar on each connected component $D \in \pi_0(R)$, so the kernel of the restriction map is $(k^*)^{\pi_0(R)}$. $\square$

We now apply Lemma 4.11 along an exhaustion of $\mathcal{L}$. Recall that a full subcategory of $\mathcal{L}$ is *convex* if it contains every object lying on a chain of nonzero morphisms between any two of its objects.

Lemma 4.12. *There exists an increasing sequence $\mathcal{L}_1 \subset \mathcal{L}_2 \subset \cdots$ of finite convex full subcategories of $\mathcal{L}$ whose union is $\mathcal{L}$, such that each $\mathcal{L}_{m+1}$ is obtained from $\mathcal{L}_m$ by adjoining a single source or sink. Every finite full subcategory of $\mathcal{L}$ is directed, thin, and representation-finite. Consequently, $\mathcal{L}$ admits a multiplicative basis, and the group $\text{Scal}(\mathcal{L})$ of multiplicative k^* -valued weights on $\mathcal{L}$ is an abelian divisible group.*

Proof. Because $\mathcal{L}$ is connected and locally finite, it has countably many objects, and because the integer height strictly decreases along every nonidentity morphism of $\mathcal{L}$, there are only finitely many directed paths between any two objects. The convex hull in $\mathcal{L}$ of any finite set of objects is therefore finite. Enumerating the objects of $\mathcal{L}$ and taking the convex hulls of initial segments yields a cofinal sequence of finite convex subcategories. Whenever $C \subsetneq C'$ is a proper inclusion of finite convex subcategories, reachability along nonzero morphisms defines a partial order on C' . If $C' \setminus C$ contains an element lying below some element of C , any maximal such element of $C' \setminus C$ is a source in the union of C with that element, and adjoining it preserves convexity; otherwise, any minimal element of $C' \setminus C$ retains convexity and adjoins a sink. Refining the cofinal sequence in this way produces the desired one-object exhaustion $\mathcal{L}_1 \subset \mathcal{L}_2 \subset \cdots$.

Every finite full subcategory of $\mathcal{L}$ is directed by height, thin by Lemma 4.8, and representation-finite by Lemma 4.7. Applying Lemma 4.11(1) at each step $\mathcal{L}_m \subset \mathcal{L}_{m+1}$ therefore builds a compatible sequence of multiplicative bases whose union is a multiplicative basis of $\mathcal{L}$. Likewise, by Lemma 4.11(2), the inverse system of weight groups $\text{Scal}(\mathcal{L}_m)$ consists of split surjections whose kernels are products of copies of the divisible group k^* (since k is algebraically closed). Hence the inverse limit $\text{Scal}(\mathcal{L}) = \varprojlim \text{Scal}(\mathcal{L}_m)$ is isomorphic to a direct product of copies of k^* and is therefore a divisible abelian group. $\square$

4.5. Scalar automorphisms and descent

To make the multiplicative basis of $\mathcal{L}$ invariant under the action of the free deck group $G = \pi_1(K)$, we first show that every multiplicative scalar weight on $\mathcal{L}$ is a coboundary of object values.

Lemma 4.13. *Every multiplicative scalar weight $d \in \text{Scal}(\mathcal{L})$ is a ratio of object values: there exist nonzero scalars $\zeta_x \in k^*$ for $x \in \text{ob } \mathcal{L}$ such that $d(r) = \zeta_y / \zeta_x$ for every nonzero morphism line $r \in \mathcal{L}(x, y)$.*

Proof. Each weight $d \in \text{Scal}(\mathcal{L})$ acts on $\mathcal{L}$ by an object-fixing automorphism α_d that scales each nonzero morphism line r by $d(r)$. Precomposing $\mathcal{L}$ -module actions with α_d^{-1} defines an action of the group $\text{Scal}(\mathcal{L})$ on the set of isomorphism classes of finite-dimensional indecomposable $\mathcal{L}$ -modules. For any fixed finite subset $S \subset \text{ob } \mathcal{L}$, the full subcategory of $\mathcal{L}$ on S is representation-finite by Lemma 4.12, so there are only finitely many isomorphism classes of indecomposable $\mathcal{L}$ -modules supported in S . Because $\text{Scal}(\mathcal{L})$ is a divisible abelian group by Lemma 4.12, it has no nontrivial finite quotient groups and must therefore act trivially on this finite set of isomorphism classes. In particular, for every object $\tilde{X} \in \tilde{\mathcal{C}}$, the evaluated module $V_{\tilde{X}}$ of Lemma 4.8 satisfies $V_{\tilde{X}} \cong T(V_{\tilde{X}})$, where T denotes pullback along α_d^{-1} .

Choose an $\mathcal{L}$ -module isomorphism $\theta_{\tilde{X}} : V_{\tilde{X}} \xrightarrow{\sim} T(V_{\tilde{X}})$ for each $\tilde{X} \in \tilde{\mathcal{C}}$, taking $\theta_{P_{\tilde{x}}}(v) = \alpha_d^{-1}(v)$ on the representable modules $V_{P_{\tilde{x}}} = \mathcal{L}(\tilde{x}, -)$. By the full faithfulness of $\tilde{X} \mapsto V_{\tilde{X}}$ (Lemma 4.8), transporting morphisms via $f \mapsto \theta_{\tilde{Y}}^{-1} \circ T(f) \circ \theta_{\tilde{X}}$ defines an object-fixing automorphism Φ of $\tilde{\mathcal{C}}$ whose restriction to the opposite subcategory $\mathcal{L}$ of lifted projectives is precisely α_d .

Because the endpoints of each arrow $a : \tilde{U} \rightarrow \tilde{V}$ of $\tilde{\Gamma}$ differ in height by 1, the space $\tilde{\mathcal{C}}(\tilde{U}, \tilde{V})$ is one-dimensional and spanned by a , so $\Phi(a) = w(a)a$ for some $w(a) \in k^*$. Since Φ preserves the mesh relation $r_{\tilde{X}}$ at each nonprojective vertex $\tilde{X}$, it scales $r_{\tilde{X}}$ by some

$\nu_{\tilde{X}} \in k^*$, which gives

$$w(\sigma a)w(a) = \nu_{\tilde{X}}$$

for every arrow a ending at $\tilde{X}$. Assigning $\nu_{\tilde{X}}$ to each translation edge $t_{\tilde{X}}$ makes w a closed multiplicative edge weight on the simply connected complex $\tilde{K}$. Integrating w along walks from a basepoint produces vertex scalars $\zeta_{\tilde{X}} \in k^*$ such that $w(a) = \zeta_{\tilde{V}}/\zeta_{\tilde{U}}$ for every arrow $a : \tilde{U} \rightarrow \tilde{V}$, and hence Φ acts on every morphism in $\tilde{\mathcal{C}}(\tilde{X}, \tilde{Y})$ by multiplication by $\zeta_{\tilde{Y}}/\zeta_{\tilde{X}}$. Since $\mathcal{L}(\tilde{x}, \tilde{y}) = \tilde{\mathcal{C}}(P_{\tilde{y}}, P_{\tilde{x}})$ and Φ acts on $\mathcal{L}(\tilde{x}, \tilde{y})$ by $d(\tilde{x}, \tilde{y})$, we obtain $d(\tilde{x}, \tilde{y}) = \zeta_{P_{\tilde{x}}}/\zeta_{P_{\tilde{y}}}$, which has the required form on setting $\zeta_{\tilde{x}} = \zeta_{P_{\tilde{x}}}^{-1}$. $\square$

We can now descend the multiplicative basis from $\mathcal{L}$ to A_0 (compare Bretscher and Gabriel [8, Sections 3.1–3.2]).

Proposition 4.14. *Let A_0 be the standard form of A constructed in Construction 2.8, and let $P = \text{ray } A_0$. Then there is a graded algebra isomorphism $A_0 \cong kP$ preserving the primitive idempotents and the homogeneous ray lines. In particular, A_0 admits a normed multiplicative basis.*

Proof. Let $\{b_{xy}\}$ be the multiplicative basis of $\mathcal{L}$ provided by Lemma 4.12, where $b_{xy} \in \mathcal{L}(x, y)$ when $\mathcal{L}(x, y) \neq 0$. For each generator g of the free deck group G , the deck action on the basis defines a multiplicative weight $d_g \in \text{Scal}(\mathcal{L})$ by

$$g(b_{xy}) = d_g(x, y)b_{gx, gy}.$$

By Lemma 4.13, for each free generator g there exist scalars $a_g(x) \in k^*$ such that $d_g(x, y) = a_g(y)/a_g(x)$. Because G is a free group acting freely on $\text{ob } \mathcal{L}$, each G -orbit of objects is a Cayley tree for the free generators of G , so we can solve the equations $f(gx) = a_g(x)f(x)$ simultaneously for all free generators g and all objects $x \in \text{ob } \mathcal{L}$. Rescaling the basis of $\mathcal{L}$ to

$$b'_{xy} = \frac{f(y)}{f(x)}b_{xy}$$

preserves multiplicativity and makes the basis G -invariant: $g(b'_{xy}) = b'_{gx, gy}$.

As shown in Subsection 4.3, each nonzero homogeneous ray line of A_0 lies in a single deck degree and therefore lifts uniquely to a nonzero morphism space $\mathcal{L}(x, y)$ once a lift x of its source is chosen; by G -invariance, the image of b'_{xy} in A_0 is independent of the choice of lift x . Since nonzero products of homogeneous ray lines in A_0 lift to nonzero compositions in $\mathcal{L}$, the descended vectors form a homogeneous multiplicative basis of A_0 realizing the isomorphism $A_0 \cong kP$, and Corollary 3.3 shows that this basis is normed. $\square$

Corollary 4.15. *For $P = \text{ray } A_0$, the linearization $kP \cong A_0$ is representation-finite and satisfies $\text{add } k(\Gamma_A) \simeq \text{mod } kP$. Moreover, A is standard in the sense of Definition 1.2 if and only if $A \cong A_0$.*

Proof. The first statement combines Propositions 2.6(3), 2.10, and 4.14. If A is standard, then the object-preserving equivalence $k(\Gamma_A) \simeq \text{ind}(\text{mod } A)$ restricts on the projective vertices to an isomorphism $A_0 \cong A$. Conversely, any algebra isomorphism $A \cong A_0$ induces an equivalence $F : \text{ind}(\text{mod } A) \xrightarrow{\sim} \text{ind}(\text{mod } A_0) \simeq \mathcal{C}$, possibly permuting the vertices of Γ_A . Since $\mathcal{C} \cong k(\Gamma_A)$ is generated in degree one with homogeneous relations, its associated graded category $\text{gr}_{\mathcal{J}} \mathcal{C}$ is canonically isomorphic to $\mathcal{C}$ itself, and the induced graded equivalence $\text{gr}_{\mathcal{J}} F : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ effects the same vertex permutation as F . Composing F with $(\text{gr}_{\mathcal{J}} F)^{-1}$ therefore produces an equivalence $\text{ind}(\text{mod } A) \xrightarrow{\sim} \mathcal{C} \cong k(\Gamma_A)$ fixing every vertex. $\square$

5. STANDARDNESS OUTSIDE CHARACTERISTIC TWO

We now complete the proof of Theorems A and B when $\text{char } k \neq 2$. By Corollary 4.15, it suffices to prove that $A \cong A_0$, or equivalently that $\text{ind}(\text{mod } A) \cong \mathcal{C}$. In the graded category $\mathcal{C} = \text{gr}_{\mathcal{J}} \text{ind}(\text{mod } A)$, the chosen arrow representatives satisfy the quadratic mesh relations (1) exactly, whereas any lifts of those arrows to $\text{ind}(\text{mod } A)$ satisfy the mesh relations modulo $\mathcal{J}^3$. If the linear derivative map sending degree- m perturbations of the arrows to their degree- $(m+1)$ mesh errors is surjective for every $m \geq 2$, we can inductively adjust the chosen irreducible morphisms degree by degree until the mesh relations hold on the nose in $\text{ind}(\text{mod } A)$.

In Subsection 5.1 we show that, in any characteristic, the cokernel of this linearized mesh derivative localizes to the τ -periodic components of Γ_A . In Appendix A we prove that the localized cokernel vanishes whenever $\text{char } k \neq 2$, provided the mesh category satisfies a natural necessary condition for finite representation type that we call the *brick hypothesis*: every nonempty object quotient contains an object with endomorphism ring k . Subsection 5.2 verifies that $\mathcal{C}$ satisfies this hypothesis (which rules out the small-period components of types E_6 , E_7 , and E_8 that would otherwise have nonzero cokernels in characteristics 3 and 5), and Subsection 5.3 carries out the successive arrow corrections.

5.1. The derivative and periodic vertices

Throughout this subsection, let Q be an arbitrary finite translation quiver with at most one arrow between each ordered pair of vertices, and let $C = C(Q, c)$ be the scalar mesh category of Definition 2.4 for a fixed system of nonzero coefficients $c = (c_{x,a})$ (we reserve the calligraphic letter $\mathcal{C}$ for the graded module category $\text{gr}_{\mathcal{J}} \text{ind}(\text{mod } A)$).

For each integer $m \geq 2$, define the homogeneous spaces of degree- m arrow perturbations and degree- $(m+1)$ mesh errors by

$$U_m(C) = \bigoplus_{a:u \rightarrow v} C_m(u, v), \quad V_{m+1}(C) = \bigoplus_{x \text{ nonprojective}} C_{m+1}(\tau x, x),$$

where C_m denotes the homogeneous component of path length m . Substituting $a \mapsto a + f_a$ with $f_a \in C_m(u, v)$ into the quadratic mesh relations $r_x = \sum_{a:y \rightarrow x} c_{x,a}(\sigma a)a$ and taking the linear term in f yields the *mesh derivative*

$$D_m : U_m(C) \longrightarrow V_{m+1}(C),$$

$$(D_m f)_x = \sum_{a:y \rightarrow x} c_{x,a}(f_{\sigma a}a + (\sigma a)f_a). \quad (9)$$

Thus $\text{coker } D_m(C)$ measures the degree- $(m+1)$ mesh errors that cannot be canceled by a degree- m modification of the arrows.

Let $T \subseteq Q_0$ be the set of τ -periodic vertices of Q , so that the full subquiver Q_T is a stable translation quiver, and let $C_T = C(Q_T, c|_T)$ be its scalar mesh category, which by (2) is the object quotient of C obtained by killing the identity morphisms outside T (compare Riedtmann's stable-block construction [19]). The following localization proposition is used both in Appendix A (for $\text{char } k \neq 2$) and in Section 6 (for $\text{char } k = 2$).

Proposition 5.1. *Let $C = C(Q, c)$ be a scalar mesh category on a finite translation quiver Q with at most one arrow between each ordered pair of vertices, and let C_T be its object quotient on the set T of τ -periodic vertices. For every $m \geq 2$, the quotient functor $C \twoheadrightarrow C_T$ induces an isomorphism*

$$\text{coker } D_m(C) \xrightarrow{\sim} \text{coker } D_m(C_T).$$

Proof. Before imposing the mesh relations of C , let $\tilde{U}_m$ and $\tilde{V}_{m+1}$ be the free vector spaces spanned by literal paths in Q , with basis (b, p) for $\tilde{U}_m$ (where $b : u \rightarrow v$ is an arrow and $p : u \rightarrow v$ is a path of length m) and basis (x, w) for $\tilde{V}_{m+1}$ (where x is a nonprojective

vertex and $w : \tau x \rightarrow x$ is a path of length $m + 1$). Formula (9) defines a lifted linear map $\tilde{D}_m : \tilde{U}_m \rightarrow \tilde{V}_{m+1}$.

In the matrix of $\tilde{D}_m$ with respect to these path bases, each column (b, p) has at most two nonzero entries: if v is nonprojective, the occurrence of b as the second arrow in r_v contributes $c_{v,b}(\sigma b)p$ to the row $(v, (\sigma b)p)$; if u is noninjective, writing $b = \sigma b'$ for $b' : v \rightarrow \tau^{-1}u$, the occurrence of b as the first arrow in $r_{\tau^{-1}u}$ contributes $c_{\tau^{-1}u,b'}pb'$ to the row $(\tau^{-1}u, pb')$. Conversely, each row (x, w) with $w : \tau x \rightarrow x$ of length $m + 1$ is incident to exactly two columns (counted with multiplicity), corresponding to the first and last arrows of w : writing $w = bp$ with first arrow $b = \sigma a : \tau x \rightarrow y$ gives the incidence with column (a, p) , while writing $w = pa$ with last arrow $a : y \rightarrow x$ gives the incidence with column $(\sigma a, p)$, which links (x, pa) to $(y, (\sigma^2 a)p)$ whenever y is nonprojective and has no second entry when y is projective.

Viewing the rows (x, w) as vertices and the two-entry columns as edges (with one-entry columns as half-edges at exposed endpoints), the connected components of this incidence graph are finite chains and finite cycles. Every chain component ends at a one-entry column, so Gaussian elimination from that exposed end places the entire chain component in $\text{im } \tilde{D}_m$. Along a two-entry column, moving from (x, pa) to $(y, (\sigma^2 a)p)$ rotates the last arrow a of w to the front and applies $\tau = \sigma^2$; iterating this rotation $d = m + 1$ times gives

$$\text{rot}^d(x, w) = (\tau x, \tau w).$$

Consequently, a row (x, w) lies on a cycle component if and only if every vertex visited by w admits arbitrarily many positive and negative powers of τ , which in the finite quiver Q means precisely that the path w lies entirely inside Q_T . Moreover, because a vertex is in T if and only if its τ -translate is in T , no column of $\tilde{D}_m$ links a path contained in Q_T to a path passing outside T . The free row space $\tilde{V}_{m+1}$ therefore decomposes as a direct sum of the subspace of paths contained in Q_T and the subspace of paths passing outside T ; the latter is a sum of chain components and lies in $\text{im } \tilde{D}_m$, so projection onto paths in Q_T induces an isomorphism $\text{coker } \tilde{D}_m(Q) \xrightarrow{\sim} \text{coker } \tilde{D}_m(Q_T)$.

Finally, let $R_V \subseteq \tilde{V}_{m+1}$ be the subspace spanned by the contextual mesh relations in $\tilde{V}_{m+1}$, so that

$$\text{coker } D_m(C) = \tilde{V}_{m+1} / (R_V + \text{im } \tilde{D}_m).$$

Because the projection onto paths contained in Q_T kills every contextual mesh relation passing outside T and sends the remaining terms to the contextual mesh relations of C_T , it induces the stated isomorphism $\text{coker } D_m(C) \xrightarrow{\sim} \text{coker } D_m(C_T)$. $\square$

5.2. The restriction imposed by finite representation type

We say that a k -linear category C satisfies the *brick hypothesis* if every nonempty object quotient of C contains a *brick*, that is, an object whose endomorphism ring is k . Here, for any subset Σ of objects of C , the object quotient $C/[\Sigma]$ denotes the quotient of C by the ideal of morphisms factoring through finite direct sums of objects in Σ , with the objects in Σ removed. Although the periodic quotient $\mathcal{C}_T$ of $\mathcal{C} = \text{gr}_{\mathcal{J}} \text{ind}(\text{mod } A)$ need not itself be the category of indecomposable modules of an algebra, both $\mathcal{C}$ and $\mathcal{C}_T$ inherit the brick hypothesis from $\text{mod } A_0$.

Lemma 5.2. *Let $\mathcal{D}$ be a full additive subcategory of a finite-length k -linear abelian category, closed under images and direct summands, with finite-dimensional Hom spaces and Krull–Schmidt decomposition. For any subset $\Sigma \subseteq \text{ob}(\text{ind } \mathcal{D})$, every nonempty object quotient $(\text{ind } \mathcal{D})/[\Sigma]$ contains an object whose endomorphism ring is a division algebra (and hence is equal to k when k is algebraically closed).*

Proof. Choose an object $X \in \text{ob}(\text{ind } \mathcal{D}) \setminus \Sigma$ of minimal composition length. Every radical endomorphism $f \in \text{rad End}(X)$ has a proper image $\text{im } f$ of strictly smaller composition

length. By hypothesis, $\text{im } f$ belongs to $\mathcal{D}$, and each of its indecomposable summands has length smaller than that of X and therefore belongs to Σ . Thus every radical endomorphism of X vanishes in $(\text{ind } \mathcal{D})/[\Sigma]$. On the other hand, 1_X cannot factor through a finite direct sum of objects in Σ by the Krull–Schmidt property, so the endomorphism ring of X in $(\text{ind } \mathcal{D})/[\Sigma]$ is the residue division ring of the local ring $\text{End}(X)$. $\square$

By Proposition 2.10(1), $\text{add } \mathcal{C} \simeq \text{mod } A_0$ is a finite-length abelian category, so Lemma 5.2 shows that $\mathcal{C}$ satisfies the brick hypothesis. Since every object quotient of $\mathcal{C}_T$ is also an object quotient of $\mathcal{C}$, the periodic mesh category $\mathcal{C}_T$ satisfies the brick hypothesis as well. Using Proposition 4.1 and Proposition 5.1, we prove the following vanishing theorem in Appendix A.2.

Proposition 5.3. *Let $C = C(Q, c)$ be a finite-dimensional scalar mesh category on a finite translation quiver Q with at most one arrow between each ordered pair of vertices. If $\text{char } k \neq 2$ and C satisfies the brick hypothesis, then the mesh derivative $D_m(C)$ is surjective for every $m \geq 2$.*

5.3. Successive correction of irreducible representatives

When D_m is surjective for all $m \geq 2$, a standard degree-by-degree perturbation argument (compare Riedtmann [19, Section 3.3]) shows that any filtered category whose associated graded category is C is already isomorphic to C .

Proposition 5.4. *Let $C = kQ/(r_1, \dots, r_t)$ be a finite-dimensional path-length graded algebra over an arbitrary field k , where Q is a finite quiver and each relation r_i is homogeneous of degree 2. For each $q \geq 3$, let D_{q-1} denote the linear map sending a tuple $(f_a)_a$ of homogeneous elements $f_a \in C_{q-1}$ parallel to the arrows a of Q to the tuple in C_q obtained by substituting $a \mapsto a + f_a$ into $(r_1, \dots, r_t)$ and retaining the linear terms in f . If D_{q-1} is surjective for every $q \geq 3$, then any finite-dimensional filtered algebra or category E equipped with a graded isomorphism $\text{gr } E \cong C$ is isomorphic to C via a filtered isomorphism inducing the given identification on $\text{gr } E$.*

Proof. Let $F^\bullet E$ be the finite decreasing filtration of E . Lift the vertex idempotents of C to a complete set of orthogonal idempotents of E (taking the identity morphisms when E is a category), and choose initial lifts in $F^1 E$ of the arrows a in the corresponding corners of E . Their quadratic relation errors r_i then lie in $F^3 E$.

Suppose by induction that we have chosen arrow lifts whose relation errors lie in $F^q E$ for some $q \geq 3$. Passing to $F^q E/F^{q+1} E \cong C_q$, the leading classes of these relation errors lie in the image of D_{q-1} by hypothesis. Subtracting a parallel correction of degree $q-1$ from each arrow lift cancels the degree- q part of each relation error, while the terms quadratic in the correction have degree at least $2(q-1) > q$; thus the new relation errors lie in $F^{q+1} E$. Because the filtration $F^\bullet E$ is finite, this process terminates after finitely many steps, yielding arrow lifts in E that satisfy the relations $r_1, \dots, r_t$ on the nose. The induced homomorphism $C \rightarrow E$ has associated graded map equal to the identity on C and is therefore an isomorphism. $\square$

Corollary 5.5. *Let A be a basic finite-dimensional representation-finite algebra over an algebraically closed field k with $\text{char } k \neq 2$.*

- (1) *There is an isomorphism $A \cong k \text{ ray } A$, and A is standard.*
- (2) *The algebra A admits a normed multiplicative basis consisting of one representative from each nonzero ray of A , with the primitive idempotents representing the identity rays.*

Proof. By Proposition 2.6(3) and Lemma 5.2, the graded category $\mathcal{C} = \text{gr}_{\mathcal{J}} \text{ind}(\text{mod } A)$ is a finite-dimensional scalar mesh category $k(\Gamma_A)$ satisfying the brick hypothesis. By Proposition 5.3, its mesh derivative $D_{q-1}(\mathcal{C})$ is surjective for every $q \geq 3$. Applying

Proposition 5.4 to $E = \text{ind}(\text{mod } A)$ with its radical filtration $F^n E = \mathcal{J}^n$ yields a filtered category isomorphism $\text{ind}(\text{mod } A) \cong \mathcal{C} \cong k(\Gamma_A)$ fixing the objects, which proves that A is standard. Restricting this isomorphism to the projective vertices and taking opposite algebras gives $A \cong A_0$, and combining this with Proposition 4.14 gives $A \cong A_0 \cong k \text{ ray } A_0 \cong k \text{ ray } A$. Finally, transporting the normed multiplicative ray basis of A_0 across this isomorphism yields (2). $\square$

Combining Corollary 5.5 with Lemma 2.1 completes the proof of Theorem A outside characteristic two, as well as Theorem B. All periodic mesh calculations required for this branch are given in Appendix A; none of the results of Parts II and III are needed when $\text{char } k \neq 2$.

Part II. Deformations in characteristic two

In this part we assume $\text{char } k = 2$. Viewing A as a positive filtered deformation of its standard form A_0 , we show that first-order deformations of A_0 are governed by residual mesh parameters on the τ -periodic components of Γ_A , and we reduce Theorem A to the construction of homogeneous Hochschild two-cocycles supported on zero products (Proposition 7.9).

6. RESIDUAL MESH PARAMETERS AND DEFORMATION OF MODULE CATEGORIES

From now on we assume that A is basic and $\text{char } k = 2$, retaining the graded category $\mathcal{C} = \text{gr}_{\mathcal{J}} \text{ind}(\text{mod } A)$ and its standard form A_0 from Part I. In characteristic two, the mesh derivative $D_m(\mathcal{C})$ need not be surjective, so higher-degree error terms in the mesh relations cannot in general be eliminated by adjusting the irreducible morphisms. We begin by describing the cokernel of $D_m(\mathcal{C})$ on the τ -periodic components of Γ_A . We then introduce positive filtered deformations of A_0 —of which A itself is the primary example—and establish internal-degree bounds on Ext groups that allow us to lift graded A_0 -modules and their morphisms to any positive filtered deformation. In Section 7 we use this module lifting to embed the positive-degree Hochschild cohomology of A_0 into the cokernel of the mesh derivative.

6.1. The remaining linear correction spaces

By Proposition 5.1, the cokernel of the mesh derivative $D_m(\mathcal{C})$ of (9) is isomorphic to $\text{coker } D_m(\mathcal{C}_T)$ on the stable subquiver of τ -periodic vertices. Evaluating $\text{coker } D_m$ on the periodic tree models of Proposition 4.1 in characteristic two yields the following classification, proved in Appendix B.1 (where periodic components of type A are shown to have vanishing cokernel in Appendix B.1.1).

Proposition 6.1. *Let $C = C(Q, c)$ be a finite-dimensional scalar mesh category on a finite translation quiver Q with at most one arrow between each ordered pair of vertices. Assume that $\text{char } k = 2$ and that C satisfies the brick hypothesis of Subsection 5.2. Call a periodic component of Q containing at least one arrow residual if $\bigoplus_{m \geq 2} \text{coker } D_m(C)$ has a nonzero class supported on that component, and let I index the residual components of Q .*

- (1) *The i th residual component ($i \in I$) is isomorphic to $\mathbb{Z}\Delta / \langle \tau^{s_i} \rho \rangle$, where Δ is a Dynkin tree of type D or E and $s_i \geq 1$, as in Proposition 4.1(2).*
- (2) *For each $i \in I$, there exist a vertex v_i of the i th residual component and a path $p_i : \tau v_i \rightarrow v_i$ of length $d_i = 2s_i + 2$ such that, viewing p_i as an element of $V_{d_i}(C)$ supported in the row of v_i and writing $[p_i]$ for its class in $\text{coker } D_{d_i-1}(C)$, we have*

$$\bigoplus_{m \geq 2} \text{coker } D_m(C) = \bigoplus_{i \in I} k[p_i].$$

In particular, $d_i = 2s_i + 2$ is the unique path degree in which the i th residual component has a nonzero cokernel class.

By Subsection 5.2, Proposition 6.1 applies directly to $C = \mathcal{C}$.

6.2. Degrees of extensions between normalized modules

To relate filtered deformations of A_0 to the mesh cokernel classes $[p_i]$, we will lift the graded indecomposable A_0 -modules to any positive filtered deformation B of A_0 .

Definition 6.2. A positive filtered deformation of a finite-dimensional nonnegatively graded algebra $\Lambda = \bigoplus_{d \geq 0} \Lambda_d$ is an algebra B equipped with a finite decreasing multiplicative filtration $F^\bullet B$ and a specified graded algebra isomorphism $\text{gr}_F B \cong \Lambda$.

In particular, the categorical radical filtration $F^n A(x, y) = \mathcal{J}^n(P_y, P_x)$ of Section 3 makes the original algebra A a positive filtered deformation of its standard form A_0 .

In classical deformation theory [9, 17], the obstructions to lifting a graded Λ -module action to a positive filtered deformation lie in Ext_Λ^2 in positive internal degrees, while the choices of module lifts and the obstructions to lifting morphisms lie in Ext_Λ^1 in positive internal degrees. Here, for graded right Λ -modules M and N , a homomorphism or extension has *internal degree* $t \in \mathbb{Z}$ if it is represented on a graded projective resolution of M by a cochain that shifts degrees by $+t$. Using the shift convention $(N\langle a \rangle)_q = N_{q-a}$ (so that a degree-zero map $M\langle a \rangle \rightarrow N\langle b \rangle$ has underlying internal degree $a - b$), we have

$$\text{Ext}_\Lambda^n(M, N)_t = \text{Ext}_{\text{gr-}\Lambda}^n(M, N\langle -t \rangle),$$

where $\text{Ext}_{\text{gr-}\Lambda}$ is computed in the category of graded right Λ -modules with degree-zero morphisms.

For $\Lambda = A_0$, enumerate the objects of $\mathcal{C}$ as $X_1, \dots, X_h$, and let $M_1, \dots, M_h$ be the corresponding graded indecomposable right A_0 -modules obtained by projective evaluation, graded by $(M_i)_{de_x} = \mathcal{C}_d(P_x, X_i)$. We refer to $M_1, \dots, M_h$ as the *degree-normalized* indecomposable A_0 -modules; note that the indecomposable projective A_0 -modules $e_x A_0$ carry their natural gradings under this normalization. Full faithfulness of projective evaluation (Proposition 2.10(1)) immediately gives

$$\text{Hom}_{A_0}(M_i, M_j)_r = 0 \quad (r < 0), \quad \text{Hom}_{A_0}(M_i, M_j)_0 = \begin{cases} k \cdot \text{id}_{M_i}, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \quad (10)$$

Lemma 6.3. *Let A_0 be the standard form of Construction 2.8, and let $M_1, \dots, M_h$ be the degree-normalized indecomposable A_0 -modules defined above.*

- (1) *Every finite-dimensional graded right A_0 -module is isomorphic to a direct sum of degree shifts of the modules M_i .*
- (2) *For all i, j and all $n \geq 1$,*

$$\text{Ext}_{A_0}^n(M_i, M_j)_t = 0 \quad \text{for all } t > -2n.$$

Proof. Let X be a finite-dimensional graded right A_0 -module that is indecomposable as a graded module. Decomposing X as an ungraded A_0 -module into copies of the indecomposable modules M_i , let $p_a : X \rightarrow M_{i_a}$ and $j_a : M_{i_a} \rightarrow X$ be the projection and inclusion maps of the summands, so that $1_X = \sum_a p_a j_a$ in path order. Passing to the homogeneous component of degree zero gives

$$1_X = \sum_{a,r} (p_a)_r (j_a)_{-r}$$

in the local ring $\text{End}_{\text{gr-}A_0}(X)$. Some single term $(p_a)_r (j_a)_{-r}$ is therefore invertible, exhibiting X as a graded direct summand of $M_{i_a}\langle -r \rangle$. Since M_{i_a} is already indecomposable as an ungraded module, $X \cong M_{i_a}\langle -r \rangle$, proving (1).

We prove (2) first for $n = 1$. The vanishing is trivial if M_i is projective. If X_i is nonprojective, Proposition 2.6(1) and Lemma 2.7 provide an exact sequence of representable functors

$$0 \longrightarrow \mathcal{C}(-, \tau X_i) \longrightarrow \bigoplus_{Y \rightarrow X_i} \mathcal{C}(-, Y) \longrightarrow \mathcal{C}(-, X_i)$$

whose cokernel is the simple functor $\mathcal{C}_0(-, X_i)$. Restricting these functors to the projective objects P_x annihilates the simple cokernel (since X_i is nonprojective) and yields a short exact sequence of graded A_0 -modules with degree-one maps. Under the equivalence $\text{add } \mathcal{C} \simeq \text{mod } A_0$, this short exact sequence is the almost split sequence ending at M_i ; inserting degree shifts so that its maps become homogeneous of degree zero writes it as

$$0 \longrightarrow M_{\tau i}\langle 2 \rangle \longrightarrow \bigoplus_{Y \rightarrow X_i} M_Y\langle 1 \rangle \longrightarrow M_i \longrightarrow 0.$$

Now suppose for a contradiction that $0 \rightarrow M_j\langle -t \rangle \rightarrow E \xrightarrow{v} M_i \rightarrow 0$ is a nonsplit graded extension with $t > -2$. It remains nonsplit as an ungraded sequence (since any ungraded section would have a degree-zero component that is a graded section). By the lifting property of the right almost split map, v factors through $\bigoplus_{Y \rightarrow X_i} M_Y\langle 1 \rangle \rightarrow M_i$, and taking the degree-zero component produces a graded factorization. The induced graded map on kernels $M_j\langle -t \rangle \rightarrow M_{\tau_i}\langle 2 \rangle$ has underlying internal degree $-t - 2 < 0$ and therefore vanishes by (10). Consequently, the factorization descends to a section of the almost split sequence ending at M_i , a contradiction. Thus $\text{Ext}_{A_0}^1(M_i, M_j)_t = 0$ for all $t > -2$.

For $n \geq 2$, observe that in a minimal graded projective presentation of a nonprojective module M_i , every projective summand $P\langle a \rangle$ of the projective cover has $a \geq 1$ because its nonzero map to M_i lies in positive degree; similarly, every indecomposable summand $Z\langle b \rangle$ of the first syzygy $\Omega^1 M_i$ maps into $P\langle a \rangle$ via a radical morphism of positive degree, so $b - a \geq 1$ and hence $b \geq 2$. By induction on r , every indecomposable summand of the r th syzygy $\Omega^r M_i$ has the form $Z\langle b \rangle$ with $b \geq 2r$. Dimension shifting then gives

$$\text{Ext}_{A_0}^n(M_i, M_j)_t = \text{Ext}_{A_0}^1(\Omega^{n-1} M_i, M_j)_t = 0 \quad \text{for all } t > -2n,$$

because each indecomposable summand $Z\langle b \rangle$ of $\Omega^{n-1} M_i$ has $b \geq 2(n-1)$ and thus contributes $\text{Ext}_{A_0}^1(Z, M_j)_{t+b} = 0$ since $t+b > -2$. $\square$

6.3. Lifting actions and morphisms

Before lifting modules to a positive filtered deformation B , we place B in a split relative normal form over its degree-zero semisimple subalgebra.

For an algebra Λ over a commutative ring R and a unital subalgebra $S \subseteq \Lambda$, the space of *normalized S -relative Hochschild n -cochains* ($n \geq 1$) is

$$C_S^n(\Lambda, \Lambda) = \text{Hom}_{S-S}((\Lambda/S)^{\otimes_S n}, \Lambda), \quad (11)$$

with $C_S^0(\Lambda, \Lambda) = \{a \in \Lambda : as = sa \text{ for all } s \in S\}$ in degree zero and the standard Hochschild differential (in particular, $(df)(a, b) = af(b) - f(ab) + f(a)b$ for a 1-cochain f). Viewed as an R -multilinear map on Λ^n , a normalized S -relative n -cochain is S -bilinear in the outer variables, S -balanced across each adjacent pair of inputs, and vanishes whenever any input belongs to S . We will use these cochains over fields and over $\mathbb{Z}/4$.

Lemma 6.4. *Let k be a field, let $\Lambda = \bigoplus_{d \geq 0} \Lambda_d$ be a finite-dimensional nonnegatively graded k -algebra whose degree-zero part $S = \Lambda_0$ is split semisimple, and let B be a positive filtered deformation of Λ . Then S lifts to a unital subalgebra of B , and the filtration $F^\bullet B$ splits as a filtration of S -bimodules. Under the resulting filtered S -bimodule identification $B \cong \Lambda$ (inducing the given identification $\text{gr}_F B = \Lambda$), the multiplication of B takes the form $\mu_0 + \sum_{q > 0} \mu_q$, where μ_0 is the multiplication of Λ and each $\mu_q \in C_S^2(\Lambda, \Lambda)$ is a normalized S -relative 2-cochain raising internal degree by q .*

Proof. Set $\mathfrak{n} = F^1 B$, so that $\mathfrak{n}$ is a nilpotent two-sided ideal of B with $B/\mathfrak{n} \cong S$. Because idempotents can be lifted modulo any nilpotent ideal—iterating $a \mapsto a + (1 - 2a)(a^2 - a)$ across successive square-zero quotients—and orthogonal idempotents lift successively in complementary corners, we may lift all diagonal matrix units of the matrix blocks of S to a complete set of orthogonal idempotents in B .

Within a single matrix block $M_n(k)$ of S , let $p_1, \dots, p_n \in B$ be the lifted diagonal idempotents, and choose elements $u_i \in p_i B p_1$ and $v_i \in p_1 B p_i$ lifting the elementary matrix units E_{i1} and E_{1i} (with $u_1 = v_1 = p_1$). Since $v_i u_i \equiv p_1 \pmod{\mathfrak{n}}$, the element $v_i u_i$ is invertible in $p_1 B p_1$; replacing v_i by $(v_i u_i)^{-1} v_i$ ensures that $v_i u_i = p_1$. Then $u_i v_i$ is an idempotent in $p_i B p_i$ congruent to p_i modulo $\mathfrak{n}$, so $p_i - u_i v_i$ is an idempotent in $\mathfrak{n}$ and therefore vanishes. Thus $u_i v_i = p_i$, and the products $u_i v_j$ satisfy the matrix-unit relations of $M_n(k)$. Since idempotents in distinct matrix blocks are orthogonal, these matrix units define a unital algebra embedding $S \hookrightarrow B$ lifting $B/\mathfrak{n} \cong S$.

Finally, because $S \otimes_k S^{\text{op}}$ is a finite product of matrix algebras over k , it is semisimple, so every short exact sequence of S -bimodules splits. Choosing S -bimodule splittings $F^d B / F^{d+1} B \hookrightarrow F^d B$ (using the subalgebra S itself in degree $d = 0$) identifies B with Λ as a filtered S -bimodule inducing the prescribed isomorphism $\text{gr}_F B \cong \Lambda$. Under this identification, the multiplication of B is S -bilinear, restricts to the multiplication of S on S , and raises grading degree by at least 0, with degree-zero component equal to μ_0 . $\square$

Proposition 6.5. *Let k be a field, let Λ be a finite-dimensional nonnegatively graded k -algebra with split semisimple degree-zero part $S = \Lambda_0$, and let $M_1, \dots, M_h$ be finite-dimensional graded right Λ -modules satisfying*

$$\text{Ext}_{\Lambda}^2(M_i, M_i)_q = 0, \quad \text{Ext}_{\Lambda}^1(M_i, M_j)_q = 0 \quad \text{for all } i, j \text{ and all } q > 0.$$

Let B be a positive filtered deformation of Λ .

- (1) *Each M_i admits a filtered right B -module lift M_i^B with $\text{gr } M_i^B = M_i$, unique up to a filtered B -module isomorphism inducing the identity on M_i .*
- (2) *Every homogeneous Λ -module homomorphism $M_i \rightarrow M_j$ of degree $r \geq 0$ lifts to a filtered B -module homomorphism $M_i^B \rightarrow M_j^B$ in filtration degree r with that prescribed leading term. Consequently, the associated graded Hom spaces in nonnegative degrees between the lifted modules coincide with the original graded Hom spaces.*
- (3) *Suppose further that $\text{Hom}_{\Lambda}(M_i, M_j)_r = 0$ for all $r < 0$ and $\text{Hom}_{\Lambda}(M_i, M_j)_0 = \delta_{ij}k$. Then the full subcategory E of $\text{mod } B$ on $M_1^B, \dots, M_h^B$, filtered by map degree, has associated graded category equal to the graded Hom category on $M_1, \dots, M_h$; if the latter is generated in degree one, then the map-degree filtration on E coincides with its categorical radical filtration. Moreover, if $M_i = e_x \Lambda$ for a primitive idempotent $e_x \in S$, we may take $M_i^B = e_x B$.*

Proof. Identify B as a filtered S -bimodule with Λ via Lemma 6.4, so that the multiplication $a * b$ of B has the form $\mu_0(a, b) + \sum_{q>0} \mu_q(a, b)$. Because S is semisimple, any filtered right B -module lift of M_i can likewise be identified with M_i as a filtered right S -module, so that its B -action takes the form $m \cdot a = \nu_0(m, a) + \sum_{q>0} \nu_q(m, a)$, where ν_0 is the Λ -action on M_i and each $\nu_q : M_i \otimes_S (\Lambda/S) \rightarrow M_i$ is a right S -linear map raising internal degree by q . Recall that the normalized S -relative bar resolution of a graded right Λ -module M has degree- n term

$$M \otimes_S (\Lambda/S)^{\otimes_S n} \otimes_S \Lambda,$$

which is a graded projective right Λ -module because S is semisimple; applying $\text{Hom}_{\Lambda}(-, N)$ computes $\text{Ext}_{\Lambda}^n(M, N)$ via normalized S -relative bar cochains.

Suppose by induction on $t \geq 1$ that we have chosen $\nu_1, \dots, \nu_{t-1}$ so that the associativity identity $(m \cdot a) \cdot b = m \cdot (a * b)$ holds in internal degrees less than t . A direct check using the associativity of $*$ and the lower-degree action identities shows that the degree- t failure of $(m \cdot a) \cdot b = m \cdot (a * b)$ is a normalized S -relative bar 2-cocycle of internal degree t with values in $\text{Hom}_k(M_i, M_i)$. Since $\text{Ext}_{\Lambda}^2(M_i, M_i)_t = 0$, this 2-cocycle is the bar coboundary of a normalized 1-cochain ν_t of internal degree t , which cancels the degree- t error. Because M_i is finite-dimensional, this induction terminates after finitely many steps and yields a filtered B -module action on M_i . Similarly, if two such actions agree in degrees less than t , their degree- t difference is a bar 1-cocycle of internal degree t ; by $\text{Ext}_{\Lambda}^1(M_i, M_i)_t = 0$, it is the coboundary of an S -linear endomorphism h_t of M_i of degree t , so conjugating by the invertible filtered automorphism $1 + h_t$ brings the two actions into agreement through degree t . This proves (1).

For (2), let $f_0 : M_i \rightarrow M_j$ be homogeneous of degree $r \geq 0$, and suppose we have constructed $f = f_0 + f_1 + \dots + f_{t-1}$ (with f_s of degree $r + s$) commuting with the B -actions in degrees less than $r + t$. The degree- $(r + t)$ error in $f(m \cdot a) = f(m) \cdot a$ is a bar 1-cocycle

in $C_S^1(M_i, M_j)$ of internal degree $r + t > 0$. Since $\text{Ext}_\Lambda^1(M_i, M_j)_{r+t} = 0$, this error is the coboundary of a map $f_t : M_i \rightarrow M_j$ of degree $r + t$, allowing us to continue the induction to obtain a filtered B -linear lift of f_0 . Since the leading homogeneous term of any nonzero filtered B -module homomorphism $M_i^B \rightarrow M_j^B$ is a nonzero homogeneous Λ -module homomorphism $M_i \rightarrow M_j$, this proves (2).

Under the hypotheses of (3), there are no nonzero homomorphisms of negative degree, so the associated graded category of E is the full graded Hom category on $M_1, \dots, M_h$. The positive-degree filtration ideal $F^1 E$ is nilpotent and $E/F^1 E \cong \prod_i k$, so $F^1 E$ is the categorical radical $\mathcal{J}(E)$ of E (and in particular the lifted modules M_i^B are indecomposable and pairwise nonisomorphic). If $\text{gr}_F E$ is generated in degree one, then each $F^q E/F^{q+1} E$ is spanned by products of q elements of $F^1 E/F^2 E$, so descending induction on q gives $F^q E = \mathcal{J}(E)^q$ for all $q \geq 1$. Finally, when $M_i = e_x \Lambda$, the lifted projective module $e_x B$ has associated graded module $e_x \Lambda$, so by the uniqueness in (1) we may take $M_i^B = e_x B$. $\square$

Applying Proposition 6.5 to $\Lambda = A_0$ and its degree-normalized indecomposable modules $M_1, \dots, M_h$ (whose Hom and Ext groups satisfy the required bounds by (10) and Lemma 6.3(2)) shows that for every positive filtered deformation B of A_0 , the lifted modules M_i^B form a filtered category with associated graded category $\mathcal{C}$, and its projective corner recovers $\text{End}_B(\bigoplus_x e_x B)^{\text{op}} = B$.

In Section 7 and Section 10 we also need the first-order version of this lifting calculation over $R[\varepsilon]/(\varepsilon^2)$, where the coefficient ring R is either a field or $\mathbb{Z}/4$.

Lemma 6.6. *Let R be a field or $\mathbb{Z}/4$, let Λ be a finitely generated free nonnegatively graded R -algebra with $\Lambda_0 = S = R^n$, and let $M_1, \dots, M_h$ be finitely generated graded right Λ -modules that are free over R . Suppose that each graded module $\text{Hom}_\Lambda(M_i, M_j)$ is free over R and concentrated in nonnegative degrees, and that*

$$\text{Ext}_\Lambda^2(M_i, M_i)_t = 0, \quad \text{Ext}_\Lambda^1(M_i, M_j)_t = 0 \quad \text{for all } i, j \text{ and all } t > 0.$$

Let $\mu \in C_S^2(\Lambda, \Lambda)$ be a normalized S -relative Hochschild 2-cocycle of internal degree $q > 0$, and form the graded first-order deformation $\Lambda_\mu = (\Lambda \otimes_R R[\varepsilon]/(\varepsilon^2), m + \varepsilon\mu)$ with $\deg \varepsilon = -q$. Then:

- (1) *Each M_i lifts to a graded right Λ_μ -module M_i^μ free over $R[\varepsilon]/(\varepsilon^2)$ (taking $M_i^\mu = e_x \Lambda_\mu$ when $M_i = e_x \Lambda$), uniquely up to a degree-zero Λ_μ -module isomorphism reducing to the identity modulo ε .*
- (2) *Every homogeneous Λ -linear map $f : M_i \rightarrow M_j$ lifts to a homogeneous Λ_μ -linear map $M_i^\mu \rightarrow M_j^\mu$ of the same degree, and for any homogeneous R -basis of $\text{Hom}_\Lambda(M_i, M_j)$, any chosen homogeneous lifts form an $R[\varepsilon]/(\varepsilon^2)$ -basis of $\text{Hom}_{\Lambda_\mu}(M_i^\mu, M_j^\mu)$.*
- (3) *If Λ and the M_i carry a compatible grading by an abelian group L and μ is homogeneous of L -degree χ , then assigning ε the L -degree $-\chi$ allows all module and morphism lifts to be chosen homogeneous for the L -grading as well.*

Proof. Because the vertex components of Λ and M_i are direct summands of finitely generated free R -modules, they are projective over R , so the normalized S -relative bar resolution of M_i consists of graded projective Λ -modules and computes $\text{Ext}_\Lambda^\bullet(M_i, M_j)$ over R (including when $R = \mathbb{Z}/4$). Writing a candidate lifted action on $M_i \otimes_R R[\varepsilon]/(\varepsilon^2)$ as $m \cdot a = ma + \varepsilon\nu_i(m, a)$ with $\nu_i \in C_S^1(M_i, M_i)_q$, the associativity condition modulo ε^2 is

$$\nu_i(ma, b) - \nu_i(m, ab) + \nu_i(m, a)b = m\mu(a, b).$$

Since $d\mu = 0$, the right-hand side is a relative bar 2-cocycle of internal degree $q > 0$, so the vanishing of $\text{Ext}_\Lambda^2(M_i, M_i)_q$ guarantees the existence of ν_i . The difference of any two solutions is a relative bar 1-cocycle of degree q , which by $\text{Ext}_\Lambda^1(M_i, M_i)_q = 0$ is the coboundary of a degree- q endomorphism h of M_i , so the two lifts are identified by the automorphism $1 + \varepsilon h$. This proves (1).

Similarly, a homogeneous map $f : M_i \rightarrow M_j$ of degree $r \geq 0$ lifts to a Λ_μ -linear map $f + \varepsilon g$ if and only if

$$g(ma) - g(m)a = \nu_j(f(m), a) - f(\nu_i(m, a)).$$

The right-hand side is a relative bar 1-cocycle of internal degree $r + q > 0$, so the vanishing of $\text{Ext}_\Lambda^1(M_i, M_j)_{r+q}$ provides the correction g . If $F_1, \dots, F_s$ are homogeneous lifts of a homogeneous R -basis $f_1, \dots, f_s$ of the entire module $\text{Hom}_\Lambda(M_i, M_j)$, then subtracting from any $\Phi \in \text{Hom}_{\Lambda_\mu}(M_i^\mu, M_j^\mu)$ the R -linear combination of $F_1, \dots, F_s$ matching its reduction modulo ε leaves an element of the form εH where $H \in \text{Hom}_\Lambda(M_i, M_j)$ (when Φ is homogeneous of degree r , its reduction modulo ε lies in degree r and H lies in degree $r + q$), which is therefore an R -linear combination of $\varepsilon F_1, \dots, \varepsilon F_s$. Conversely, if $\sum_l (a_l + \varepsilon b_l) F_l = 0$ with $a_l, b_l \in R$, reducing modulo ε gives $\sum_l a_l f_l = 0$, so $a_l = 0$; the remaining equation $\varepsilon \sum_l b_l f_l = 0$ in the free $R[\varepsilon]/(\varepsilon^2)$ -module $\text{Hom}_R(M_i, M_j) \otimes_R R[\varepsilon]/(\varepsilon^2)$ then forces $\sum_l b_l f_l = 0$ and hence $b_l = 0$. This proves (2). Finally, since the bar differentials preserve the L -grading, projecting any solution onto its homogeneous component of the required L -degree gives (3). $\square$

7. PARAMETER WEIGHTS AND THE CRITERION FOR RECOVERING THE ALGEBRA

We continue under the assumption $\text{char } k = 2$, with $A_0 = kP$ ($P = \text{ray } A_0$) and $\mathcal{C} = \text{gr}_{\mathcal{J}} \text{ind}(\text{mod } A) \simeq \text{ind}(\text{mod } A_0)$. In this section we compute the universal group of additive weights on the ray category P , show how ray weights extend to mesh weights on $\mathcal{C}$, and use the first-order module lifting of Section 6 to embed the positive-degree Hochschild cohomology $\text{HH}^2(A_0)_{>0}$ into the mesh cokernel $\bigoplus_{m \geq 2} \text{coker } D_m(\mathcal{C})$. Crucially, the weight lattice $L = W(P)$ separates the residual cohomology classes attached to distinct periodic components even when they share the same categorical degree.

7.1. Normalizing the scalar mesh coefficients

By Lemma 4.4, we may assume from now on that all scalar coefficients in the mesh relations of $\mathcal{C}$ are equal to 1: explicitly, rescaling each arrow a of Γ_A by $\mu_a \in k^*$ and dividing the mesh relation at each nonprojective vertex v by $\eta_v \in k^*$ normalizes the coefficients $c_{v,a}$ to 1 whenever

$$\mu_{\sigma a} \mu_a = \eta_v / c_{v,a} \quad \text{for every arrow } a : u \rightarrow v. \quad (12)$$

7.2. The integral group of ray weights

We now establish an integral counterpart of the scalar weight trivialization of Lemma 4.13. This result holds in arbitrary characteristic (compare the fundamental-group and coefficient analysis in Bretscher and Gabriel [8, Sections 2.11–2.12]) and will also be used in Part III to lift modulo-two weights on smaller ray tables.

Definition 7.1. For a finite ray category Q , its *scalar-weight group* is the abelian group

$$W(Q) = \mathbb{Z}\langle [r] : r \neq 0, 1 \rangle / \langle [rs] - [r] - [s] : rs \neq 0 \rangle,$$

where the classes of identity morphisms are set to zero. Thus homomorphisms $W(Q) \rightarrow M$ into an abelian group M correspond bijectively to additive weights on the nonzero morphisms of Q (and, for $M = k^*$ written multiplicatively, $\text{Hom}(W(Q), k^*)$ is the group of multiplicative scalar weights $a(rs) = a(r)a(s)$ for $rs \neq 0$).

Proposition 7.2. Let $A_0 = kP$ ($P = \text{ray } A_0$) for a connected basic representation-finite algebra A with n primitive idempotents, let $G = \pi_1(K)$ be the free deck group of Subsection 4.2, and let $\mathcal{L}$ be the category of lifted projectives. Let $I_{\mathcal{L}} \subset \mathbb{Z}^{(\text{ob } \mathcal{L})}$ denote the subgroup of formal integer combinations of objects of $\mathcal{L}$ whose coefficients sum to zero.

- (1) Sending a lifted ray $r \in \mathcal{L}(x, y)$ to the difference $e_y - e_x \in I_{\mathcal{L}}$ induces an isomorphism $W(P) \cong (I_{\mathcal{L}})_G$ with the group of G -coinvariants of $I_{\mathcal{L}}$. Under this identification, the coinvariant class $\theta_g = [e_{gx} - e_x]$ is independent of $x \in \text{ob } \mathcal{L}$, and the map $g \mapsto \theta_g$ embeds the abelianization G_{ab} as a direct summand of $W(P)$ with free complement $\mathbb{Z}^{n-1}$:

$$W(P) \cong \mathbb{Z}^{n-1} \oplus G_{\text{ab}}.$$

In particular, since $G = \pi_1(K)$ is finitely generated, $W(P)$ is a free abelian group of finite rank.

- (2) Every additive weight on $\mathcal{L}$ with values in an abelian group is a difference of object values.

Proof. For any finite directed thin representation-finite category T , let $U(T)$ denote the abelian group generated by symbols $[r]$ for the nonzero nonidentity morphism lines r of T , subject to the relations $[rr'] = [r] + [r']$ whenever $rr' \neq 0$. If s is a source of T and $T' = T \setminus \{s\}$, let R be the representation-finite poset incidence category obtained from T' and $\text{rad } T(s, -)$ as in Lemma 4.11. Because the tautological weight $r \mapsto [r] \in U(T')$ is additive on R , Lemma 4.9 provides elements $u_y \in U(T')$ for $y \in R$, normalized to $u_{y_D} = 0$ at a chosen base object y_D in each connected component $D \in \pi_0(R)$, such that $[b_{yz}] = u_z - u_y$ for every nonzero product $(sy)b_{yz} = sz$. Since the relations in $U(T)$ involving the new generators $[sy]$ are precisely $[sy] + [b_{yz}] = [sz]$, substituting $[sy] = e_D + u_y$ gives a canonical direct-sum decomposition

$$U(T) \cong U(T') \oplus \mathbb{Z}^{\pi_0(R)},$$

and the dual decomposition holds when adjoining a sink. Passing to the direct limit along the one-object exhaustion $\mathcal{L}_1 \subset \mathcal{L}_2 \subset \dots$ of Lemma 4.12 shows that $U(\mathcal{L})$ is a free abelian group.

Now consider the G -equivariant homomorphism $\partial : U(\mathcal{L}) \rightarrow I_{\mathcal{L}}$ sending $[r : x \rightarrow y]$ to $e_y - e_x$. Because $\mathcal{L}$ is connected, ∂ is surjective. We claim that ∂ is also injective. Since $U(\mathcal{L})$ is free abelian and k^* is infinite (so for any nonzero integer m there exists $a \in k^*$ with $a^m \neq 1$), any nonzero element $z \in \ker \partial$ would be detected by some group homomorphism $\varphi : U(\mathcal{L}) \rightarrow k^*$; however, by Lemma 4.13, every multiplicative weight $\varphi \in \text{Hom}(U(\mathcal{L}), k^*) = \text{Scal}(\mathcal{L})$ has the form $\varphi([r : x \rightarrow y]) = \zeta_y / \zeta_x$ and therefore factors through ∂ , forcing $\varphi(z) = 1$. Thus $\partial : U(\mathcal{L}) \xrightarrow{\sim} I_{\mathcal{L}}$ is a G -equivariant isomorphism, which immediately implies (2).

Because the nonzero rays of P and their nonzero products are the G -orbits of the nonzero morphism lines of $\mathcal{L}$ and their nonzero products, $W(P)$ is the group of G -coinvariants $U(\mathcal{L})_G \cong (I_{\mathcal{L}})_G$. Fix reference lifts $x_0, x_1, \dots, x_{n-1} \in \text{ob } \mathcal{L}$ of the n projective vertices of A_0 . In $(I_{\mathcal{L}})_G$, the relation $e_{gy} - e_x = (e_{gy} - e_{gx}) + (e_{gx} - e_x) \equiv (e_y - e_x) + (e_{gx} - e_x)$ shows that $\theta_g = [e_{gx} - e_x]$ is independent of x and satisfies $\theta_{gh} = \theta_g + \theta_h$. Thus $(I_{\mathcal{L}})_G$ is generated by the classes $z_v = [e_{x_v} - e_{x_0}]$ ($1 \leq v \leq n-1$) and θ_g ($g \in G$), and the homomorphism $(I_{\mathcal{L}})_G \rightarrow \mathbb{Z}^{n-1} \oplus G_{\text{ab}}$ induced by sending the object gx_v to $(e_v, [g])$ (with $e_0 = 0$) provides the inverse isomorphism, proving (1). $\square$

7.3. Integer weights on rays and meshes

We use integer weights on P and on $\mathcal{C}$ to refine the categorical grading and separate first-order deformations attached to different periodic components.

Definition 7.3. Let $A_0 = kP$ ($P = \text{ray } A_0$) and $\mathcal{C} = \text{gr}_{\mathcal{J}} \text{ind}(\text{mod } A)$.

- (1) An *integer ray weight* is a function $h : P \setminus \{0\} \rightarrow \mathbb{Z}$ that vanishes on identity rays and satisfies $h(rs) = h(r) + h(s)$ whenever $rs \neq 0$. We write $H_{\mathbb{Z}} = \text{Hom}_{\mathbb{Z}}(W(P), \mathbb{Z})$ for the group of integer ray weights, set $L = W(P)$ (called the *weight lattice* of A_0), and write $\ell(r) = [r] \in L$ for the weight class of a nonzero ray $r \in P$.

- (2) An *integer mesh weight* on $\mathcal{C}$ is a pair $w = (w, \nu)$ assigning an integer $w(a) \in \mathbb{Z}$ to each arrow a of Γ_A and an integer $\nu_v \in \mathbb{Z}$ to each nonprojective vertex v of Γ_A such that

$$w(\sigma a) + w(a) = \nu_v \quad \text{for every arrow } a : u \rightarrow v.$$

We write $H_{\mathbb{Z}}^{\text{mesh}}$ for the abelian group of integer mesh weights on $\mathcal{C}$.

Note that the categorical degree function $\deg_{\mathcal{J}}$ of Lemma 3.7 is an element of $H_{\mathbb{Z}}$. Because $L = W(P)$ is free abelian of finite rank by Proposition 7.2(1), biduality identifies L with $\text{Hom}_{\mathbb{Z}}(H_{\mathbb{Z}}, \mathbb{Z})$ via $\chi(h) = h(\chi)$ for $\chi \in L$ and $h \in H_{\mathbb{Z}}$. The algebra $A_0 = kP$ is naturally L -graded by placing each ray $r \in P$ in degree $\ell(r) \in L$; a Hochschild 2-cochain sending a pair of rays (r, s) to a multiple of u shifts L -degree by $\ell(u) - \ell(r) - \ell(s) \in L$.

Lemma 7.4. *For $A_0 = kP$ and $\mathcal{C} = \text{gr}_{\mathcal{J}} \text{ind}(\text{mod } A)$, assigning to each path in $\mathcal{C}$ the sum of the weights of its arrows under a mesh weight $w \in H_{\mathbb{Z}}^{\text{mesh}}$ induces a well-defined grading on $\mathcal{C}$ and restricts on the projective corners to an integer ray weight in $H_{\mathbb{Z}}$. This restriction fits into a short exact sequence of abelian groups*

$$0 \longrightarrow \{(\zeta_X) \in \mathbb{Z}^{\text{ob } \mathcal{C}} : \zeta_P = 0 \text{ for all projective } P\} \longrightarrow H_{\mathbb{Z}}^{\text{mesh}} \longrightarrow H_{\mathbb{Z}} \longrightarrow 0,$$

where the left-hand map sends a tuple (ζ_X) to the coboundary mesh weight $w(a) = \zeta_{\text{target}(a)} - \zeta_{\text{source}(a)}$ and $\nu_v = \zeta_v - \zeta_{\tau v}$.

Proof. For any $w \in H_{\mathbb{Z}}^{\text{mesh}}$, every term $(\sigma a)a$ of the mesh relation at a nonprojective vertex v has total weight ν_v , so the mesh ideal of $\mathcal{C}$ is homogeneous with respect to path weight. Because each nonzero homogeneous component $A_0(x, y)_d = \mathcal{C}_d(P_y, P_x)$ is one-dimensional by Lemma 3.7, all paths representing a given ray $r \in P$ have the same w -weight, and Proposition 3.8(2) shows that this weight is additive on nonzero ray products. Thus restriction defines a homomorphism $H_{\mathbb{Z}}^{\text{mesh}} \rightarrow H_{\mathbb{Z}}$.

To see that restriction is surjective, pull any $h \in H_{\mathbb{Z}}$ back to an additive weight on the category $\mathcal{L}$ of lifted projectives. By Proposition 7.2(2), there exist integers b_x for $x \in \text{ob } \mathcal{L}$ such that $h(r) = b_y - b_x$ for every nonzero morphism $r \in \mathcal{L}(x, y)$, and by G -invariance the difference $\gamma(g) = b_x - b_{gx}$ is independent of $x \in \text{ob } \mathcal{L}$ and homomorphic in $g \in G$. On the lifted projective vertices of $\tilde{\mathcal{C}}$, set $\zeta_{P_x} = -b_x$ (the sign change reflecting that $\mathcal{L}$ is the opposite category), and extend ζ to all vertices of $\tilde{\mathcal{C}}$ by choosing arbitrary integers on one representative of each G -orbit of nonprojective vertices and setting $\zeta_{g\tilde{X}} = \zeta_{\tilde{X}} + \gamma(g)$. Then the differences of ζ along lifted arrows and translation edges are G -invariant and therefore descend to an integer mesh weight on $\mathcal{C}$ whose restriction to P is $\zeta_{P_x} - \zeta_{P_y} = b_y - b_x = h(r)$.

Finally, if $w \in H_{\mathbb{Z}}^{\text{mesh}}$ restricts to zero on P , pull w back to a closed integer edge weight on the simply connected complex $\tilde{K}$ to write w as the difference of vertex integers $\zeta_{\tilde{X}}$. Because w vanishes on all morphisms between lifted projectives and $\mathcal{L}$ is connected, ζ is constant on all lifted projective vertices; subtracting this constant makes $\zeta_{P_x} = 0$ for all x . Since $\zeta_{g\tilde{X}} - \zeta_{\tilde{X}}$ is independent of $\tilde{X}$ and vanishes on lifted projectives, ζ is G -invariant and descends to a vertex assignment (ζ_X) on $\mathcal{C}$ vanishing at the projective vertices. Because every connected component of Γ_A contains at least one projective vertex, this coboundary map is injective. $\square$

For each residual component $i \in I$ and its representative path $p_i : \tau v_i \rightarrow v_i$ of length $d_i = 2s_i + 2$ from Proposition 6.1(2), define the linear functional χ_i on $H_{\mathbb{Z}}^{\text{mesh}}$ by

$$\chi_i(w) = w(p_i) - \nu_{v_i}. \quad (13)$$

Because p_i and every term of the mesh relation at v_i share the same endpoints τv_i and v_i , any vertex coboundary (ζ_X) shifts $w(p_i)$ and ν_{v_i} by the same amount $\zeta_{v_i} - \zeta_{\tau v_i}$. By Lemma 7.4, $\chi_i(w)$ therefore depends only on the restricted ray weight in $H_{\mathbb{Z}}$ and defines

an element $\chi_i \in \text{Hom}_{\mathbb{Z}}(H_{\mathbb{Z}}, \mathbb{Z}) = L$, which we call the *residual character* of the i th residual component.

More generally, each mesh weight $w \in H_{\mathbb{Z}}^{\text{mesh}}$ grades $V_{m+1}(\mathcal{C})$ by assigning weight $w(p) - \nu_v$ to a path $p : \tau v \rightarrow v$ in row v , and grades $U_m(\mathcal{C})$ by assigning weight $w(p) - w(a)$ to a path p parallel to an arrow a . The mesh derivative D_m of (9) preserves this grading, so coker $D_m(\mathcal{C})$ becomes an L -graded space in which the line $k[p_i]$ is homogeneous of L -degree χ_i (and of positive internal degree $\chi_i(\deg_{\mathcal{J}}) = d_i - 2 = 2s_i > 0$).

Proposition 7.5. *Assume $\text{char } k = 2$. In the weight lattice $L = W(P)$ of Definition 7.3, the residual characters $\{\chi_i\}_{i \in I}$ defined by (13) extend to a $\mathbb{Z}$ -basis of L .*

Proof. For each $i \in I$, identify the i th residual component with $\mathbb{Z}\Delta / \langle \tau^{s_i} \rho \rangle$ as in Proposition 6.1(1), with Δ oriented away from the root fixed by ρ . On the vertices (x, j) of $\mathbb{Z}\Delta$, set

$$F_i(x, j) = \lfloor j/s_i \rfloor,$$

and assign to every arrow and translation edge of $\mathbb{Z}\Delta$ the difference of F_i between its target and source. Because the deck generator $(x, j) \mapsto (\rho x, j - s_i)$ subtracts 1 from F_i at every vertex, these edge differences are invariant under $\langle \tau^{s_i} \rho \rangle$ and descend to a closed integer edge weight on the i th residual component. Assigning weight zero to all other periodic components of Γ_A and extending to a closed edge weight on the entire mesh complex K via Lemma 4.3 produces an integer mesh weight $w_i \in H_{\mathbb{Z}}^{\text{mesh}}$ on $\mathcal{C}$; let $h_i \in H_{\mathbb{Z}} = \text{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ be its restriction to P .

Lift the path $p_i : \tau v_i \rightarrow v_i$ of length $2s_i + 2$ to $\mathbb{Z}\Delta$ starting at a vertex (x, r) over τv_i . Its endpoint has the form $(\rho^m x, r + 1 - ms_i)$ for some $m \in \mathbb{Z}$ and lies at the same distance from the root of Δ as x . Since outward arrows in $\mathbb{Z}\Delta$ preserve the second coordinate while inward arrows increase it by 1, the path p_i of length $2s_i + 2$ traverses an equal number $s_i + 1$ of outward and inward arrows and therefore ends at second coordinate $r + s_i + 1$. Subtracting the weight of the translation edge t_{v_i} from (x, r) to $(x, r + 1)$ gives

$$h_i(\chi_i) = \chi_i(w_i) = \left\lfloor \frac{r + s_i + 1}{s_i} \right\rfloor - \left\lfloor \frac{r + 1}{s_i} \right\rfloor = 1,$$

whereas $h_i(\chi_j) = \chi_j(w_i) = 0$ for $j \neq i$ because w_i vanishes on the j th periodic component. Consequently, the homomorphism $(h_i)_{i \in I} : L \rightarrow \mathbb{Z}^I$ is a split surjection with section sending the i th standard basis vector to χ_i , so $L = \bigoplus_{i \in I} \mathbb{Z}\chi_i \oplus \ker(h_i)_{i \in I}$. Since $\ker(h_i)_{i \in I}$ is a subgroup of the finitely generated free abelian group L , it is free abelian, and any $\mathbb{Z}$ -basis of it completes $\{\chi_i\}_{i \in I}$ to a $\mathbb{Z}$ -basis of L . $\square$

7.4. Hochschild cohomology and first-order mesh deformations

We now relate the Hochschild cohomology of A_0 to the mesh derivative cokernel by lifting the degree-normalized indecomposable A_0 -modules to first-order deformations of A_0 over $k[\varepsilon]/(\varepsilon^2)$. Because the degree-zero vertex subalgebra $S = \prod_x ke_x$ is separable over k , the inclusion of the normalized S -relative cochain complex $C_S^\bullet(A_0, A_0)$ of (11) into the ordinary Hochschild cochain complex is a quasi-isomorphism [10, Theorem 1.2]. Both the internal categorical degree and the L -grading on $A_0 = kP$ induce gradings on $\text{HH}^2(A_0)$; we write $\text{HH}^2(A_0)_q$ for the component of internal degree q , $\text{HH}^2(A_0)_{>0} = \bigoplus_{q>0} \text{HH}^2(A_0)_q$, and $\text{HH}^2(A_0)_\chi$ for the component of L -weight $\chi \in L$. (In Proposition 12.5 we will show that the injection Ψ below is in fact an isomorphism and $I_0 = I$, though only injectivity is needed to prove Theorem A.)

Proposition 7.6. *Let A_0 be the standard form of Construction 2.8 and $\mathcal{C}$ its graded mesh category. For every internal degree $q > 0$, lifting degree-normalized modules to first-order deformations induces an L -graded injection*

$$\Psi : \text{HH}^2(A_0)_q \hookrightarrow \text{coker } D_{q+1}(\mathcal{C}).$$

In particular, when $\text{char } k = 2$, there is a subset $I_0 \subseteq I$ of the residual components of Proposition 6.1 such that

$$\text{HH}^2(A_0)_{>0} = \bigoplus_{i \in I_0} \text{HH}^2(A_0)_{\chi_i}, \quad \dim_k \text{HH}^2(A_0)_{\chi_i} = 1 \quad (i \in I_0).$$

Proof. Represent a cohomology class in $\text{HH}^2(A_0)_q$ by a normalized S -relative 2-cocycle $\mu \in C_S^2(A_0, A_0)_q$, and form the first-order deformation $A_{0,\mu} = (A_0 \otimes_k k[\varepsilon]/(\varepsilon^2), m + \varepsilon\mu)$ with $\deg \varepsilon = -q$. By Lemma 6.3(2) and Lemma 6.6, the degree-normalized indecomposable A_0 -modules $M_1, \dots, M_h$ lift to graded $A_{0,\mu}$ -modules M_i^μ (with $M_i^\mu = e_x A_{0,\mu}$ at the projective vertices), and each arrow a of Γ_A lifts to a degree-one morphism $\tilde{a}$ in the Hom category $\mathcal{C}_\mu$ of the lifted modules. Because the mesh relation $r_v = \sum_{a: u \rightarrow v} (\sigma a) a$ vanishes in $\mathcal{C}$, its evaluation on the lifted arrows $\tilde{a}$ in $\mathcal{C}_\mu$ vanishes modulo ε ; by the freeness of the lifted Hom spaces over $k[\varepsilon]/(\varepsilon^2)$ (Lemma 6.6(2)), this mesh error has the form εz_v for a unique element $z_v \in \mathcal{C}_{q+2}(\tau v, v)$ (since $\deg \varepsilon = -q$ and $\deg r_v = 2$). Modifying the arrow lifts by $\tilde{a} \mapsto \tilde{a} + \varepsilon f_a$ with $f_a \in \mathcal{C}_{q+1}$ changes the tuple $z = (z_v)_v$ by $D_{q+1}(f)$, while changing the module lifts M_i^μ by automorphisms reducing to the identity or replacing μ by a cohomologous cocycle $\mu + dh$ (which corresponds to applying the algebra automorphism $1 + \varepsilon h$ of $A_{0,\mu}$) leaves the class $[z] \in \text{coker } D_{q+1}(\mathcal{C})$ unchanged. Thus $[\mu] \mapsto \Psi([\mu]) = [z]$ is well-defined.

Because the first-order module-action and arrow-lifting equations in Lemma 6.6 are linear in μ and in the first-order corrections, and any product of two ε -corrections vanishes modulo ε^2 , the map Ψ is k -linear. To see that Ψ is injective, suppose $\Psi([\mu]) = 0$. Then we can adjust the arrow lifts $\tilde{a}$ by εf_a so that every mesh relation $r_v(\tilde{a})$ vanishes identically in $\mathcal{C}_\mu$. The assignment $a \mapsto \tilde{a}$ then defines a graded $k[\varepsilon]/(\varepsilon^2)$ -linear functor $\mathcal{C} \otimes_k k[\varepsilon]/(\varepsilon^2) \rightarrow \mathcal{C}_\mu$ that reduces to the identity modulo ε . By Lemma 6.6(2), the morphism spaces of $\mathcal{C}_\mu$ are free over $k[\varepsilon]/(\varepsilon^2)$ with bases lifting those of $\mathcal{C}$, so by Nakayama's lemma over the local ring $k[\varepsilon]/(\varepsilon^2)$ this functor is an isomorphism on every Hom space. Restricting to the projective vertices $e_x A_{0,\mu}$ yields a graded algebra isomorphism $A_0 \otimes_k k[\varepsilon]/(\varepsilon^2) \xrightarrow{\sim} A_{0,\mu}$ reducing to the identity modulo ε , which means that $[\mu] = 0$ in $\text{HH}^2(A_0)_q$.

Finally, choosing an L -homogeneous cocycle μ of weight $\chi \in L$ and lifting the integer ray weights to mesh weights on $\mathcal{C}$ via Lemma 7.4, Lemma 6.6(3) shows that when ε is assigned L -weight $-\chi$, the lifted modules and arrows can be chosen L -homogeneous, so the resulting mesh error z_v has weight χ relative to row v . Thus Ψ preserves L -weights. Combining the injectivity of Ψ with Proposition 6.1(2) immediately gives the stated decomposition of $\text{HH}^2(A_0)_{>0}$ in characteristic two. $\square$

7.5. Finite normalization of an associative family

We conclude Part II with a general deformation-theoretic criterion for realizing a positive filtered deformation as a specialized fibre of an associative polynomial family (compare Gerstenhaber [9]). For an algebra family over $k[t_1, \dots, t_n]$, we call its specialization at a parameter vector $\lambda \in k^n$ a *scalar fibre*, and we call it a *binary fibre* when every coordinate of λ belongs to $\{0, 1\}$.

Theorem 7.7. *Let $\Lambda = \bigoplus_{d=0}^N \Lambda_d$ be a finite-dimensional nonnegatively graded algebra over a field k whose degree-zero part $S = \Lambda_0$ is a finite product of copies of k . Let $\mu_1, \dots, \mu_n \in C_S^2(\Lambda, \Lambda)$ be normalized S -relative Hochschild 2-cocycles of positive internal degrees $q_1, \dots, q_n > 0$. Assume that their cohomology classes span $\text{HH}^2(\Lambda)_{>0}$ and that the deformed product*

$$m_t = m + \sum_{i=1}^n t_i \mu_i$$

is associative over the polynomial ring $k[t_1, \dots, t_n]$, where m is the multiplication of Λ . Then every positive filtered deformation of Λ is isomorphic to a scalar fibre (Λ, m_λ) via

a filtered isomorphism that induces the identity on $\text{gr } \Lambda = \Lambda$ and fixes the lifted vertex idempotents.

Proof. By Lemma 6.4, after lifting the vertex idempotents and splitting the filtration as S -bimodules, the given positive filtered deformation is defined on the underlying graded vector space Λ by an associative S -relative multiplication $\star$ such that $\star - m$ raises internal degree by at least 1. We construct a parameter vector $\lambda \in k^n$ and a sequence of filtered automorphisms of Λ by induction on the degree $q \geq 1$, starting with $q = 1$ and $\lambda = 0$.

Suppose that $\star$ agrees with m_λ in all internal degrees less than q , where only those coordinates λ_i with $q_i < q$ have been assigned. Let $\varphi_q \in C_S^2(\Lambda, \Lambda)_q$ denote the homogeneous component of $\star - m_\lambda$ of internal degree q . Subtracting the associativity identity for m_λ from that for $\star$ in internal degree q , all cross-terms involving both φ_q and lower-degree positive components of $\star$ or m_λ have internal degree strictly greater than q and therefore vanish, leaving the Hochschild cocycle condition $d\varphi_q = 0$. Because the classes $[\mu_i]$ span $\text{HH}^2(\Lambda)_{>0}$, there exist scalars $c_i \in k$ (for those indices i with $q_i = q$) and a normalized S -relative 1-cochain $f_q \in C_S^1(\Lambda, \Lambda)_q$ such that

$$\varphi_q = \sum_{q_i=q} c_i \mu_i + df_q.$$

Setting $\lambda_i = c_i$ for all i with $q_i = q$ keeps m_λ associative by hypothesis, and pulling the multiplication $\star$ back along the filtered S -bimodule automorphism $1 - f_q$ subtracts $(df_q)(a, b) = af_q(b) - f_q(ab) + f_q(a)b$ from its degree- q component while leaving all components of degree less than q unchanged. Thus the transformed product $\star$ agrees with m_λ in all internal degrees at most q . Since Λ is finite-dimensional, $C_S^2(\Lambda, \Lambda)_q = 0$ for all $q > N$, so this process terminates after at most N steps and yields the desired filtered isomorphism with (Λ, m_λ) . $\square$

Because the residual characters χ_i extend to a $\mathbb{Z}$ -basis of the weight lattice $L = W(P)$ by Proposition 7.5, any nonzero coordinates of the parameter vector λ can then be rescaled to 1 by a diagonal grading automorphism of Λ .

Corollary 7.8. *In the setting of Theorem 7.7, suppose that Λ also carries a compatible grading by a finite-rank free abelian group L with S in degree zero, and that each cocycle μ_i is homogeneous of L -degree $\chi_i \in L$. If $\{\chi_1, \dots, \chi_n\}$ extends to a $\mathbb{Z}$ -basis of L , then every positive filtered deformation of Λ is isomorphic to a binary fibre (Λ, m_λ) with $\lambda \in \{0, 1\}^n$. More generally, any two parameter vectors in k^n having the same zero coordinates define isomorphic fibres.*

Proof. Every group homomorphism $a : L \rightarrow k^*$ defines a grading automorphism of (Λ, m) that multiplies elements of L -weight ℓ by $a(\ell)$. Transporting the deformed product m_λ along this automorphism rescales the i th parameter according to

$$\lambda'_i = \lambda_i a(\chi_i). \quad (14)$$

Because $\{\chi_1, \dots, \chi_n\}$ extends to a $\mathbb{Z}$ -basis of the free abelian group L , we can prescribe $a(\chi_i) \in k^*$ arbitrarily for $i = 1, \dots, n$; in particular, setting $a(\chi_i) = \lambda_i^{-1}$ whenever $\lambda_i \neq 0$ and $a(\chi_i) = 1$ when $\lambda_i = 0$ transforms m_λ into a binary fibre. $\square$

Specializing Theorem 7.7 and Corollary 7.8 to $A_0 = kP$ yields our reduction criterion for Theorem A in characteristic two.

Proposition 7.9. *Let A be a basic representation-finite algebra over an algebraically closed field k of characteristic two, let $A_0 = kP$ ($P = \text{ray } A_0$), and let $I_0 \subseteq I$ index the nonzero spaces $\text{HH}^2(A_0)_{\chi_i}$ in Proposition 7.6. Suppose that we have constructed normalized S -relative 2-cochains $\mu_i \in C_S^2(A_0, A_0)$ ($i \in I_0$) satisfying:*

- (1) each μ_i is homogeneous of L -weight χ_i , and its cohomology class in $\mathrm{HH}^2(A_0)_{\chi_i}$ is nonzero;
- (2) the deformed product $m_t = m + \sum_{i \in I_0} t_i \mu_i$ is associative over $k[t_i : i \in I_0]$;
- (3) in every binary fibre (A_0, m_λ) ($\lambda \in \{0, 1\}^{I_0}$), the basis of nonzero rays $P \setminus \{0\}$ is a normed multiplicative basis, and in every scalar fibre ($\lambda \in k^{I_0}$) the map sending each $r \in P \setminus \{0\}$ to its ray in (A_0, m_λ) identifies the ray category of (A_0, m_λ) with P .

Then A is isomorphic to a binary fibre (A_0, m_λ) via a filtered isomorphism that transports $P \setminus \{0\}$ to a normed multiplicative ray basis of A and identifies $\mathrm{ray} A$ with P via initial forms.

Proof. As noted after Definition 6.2, A is a positive filtered deformation of A_0 . By Proposition 7.6, the cohomology classes $[\mu_i]$ ($i \in I_0$) in (1) form a basis of $\mathrm{HH}^2(A_0)_{>0}$. Theorem 7.7 therefore provides a filtered isomorphism between A and a scalar fibre $(A_0, m_{\lambda'})$ inducing the identity on $\mathrm{gr} A = A_0$, and Proposition 7.5 together with Corollary 7.8 rescales $(A_0, m_{\lambda'})$ via a diagonal grading automorphism on the ray basis $P \setminus \{0\}$ to a binary fibre (A_0, m_λ) . Because this diagonal rescaling preserves each homogeneous ray line in $A_0 = \mathrm{gr} A$, the composite filtered isomorphism $(A_0, m_\lambda) \xrightarrow{\sim} A$ sends the normed multiplicative basis $P \setminus \{0\}$ of (3) to a normed multiplicative basis of A consisting of one representative from each nonzero ray, and identifies $\mathrm{ray} A$ with $\mathrm{ray}(A_0, m_\lambda) = P$ via the initial-form bijection of Lemma 3.7. $\square$

Part III. Construction of multiplicative bases in characteristic two

In this final part we construct the zero-supported Hochschild two-cocycles required by Proposition 7.9. Since the multiplicative basis $P = \text{ray } A_0$ of A_0 has structure constants in $\{0, 1\}$, it defines models $A^{(2)} = \mathbb{F}_2 P$ and $A^{(4)} = (\mathbb{Z}/4)P$ with $A_0 = k \otimes_{\mathbb{F}_2} A^{(2)}$. As we show in Proposition 10.3, every cohomology class in $\text{HH}^2(A^{(2)})$ of positive internal degree can be represented via the Bockstein connecting map by $(d\delta)/2$, where δ is a one-cochain on $A^{(4)}$ lifting a derivation of $A^{(2)}$ and division by two identifies $2(\mathbb{Z}/4)$ with $\mathbb{F}_2$. For the resulting two-cocycle $(d\delta)/2$ to vanish on all nonzero basis products $rs \neq 0$ in P , the lift δ must satisfy the Leibniz equation

$$\delta(rs) = \delta(r)s + r\delta(s) \quad \text{whenever } rs \neq 0.$$

These Leibniz conditions form a system of linear congruences for the coefficients of δ , which we decouple and solve one weight character at a time. Section 8 and Section 8.4 establish local factorization restrictions on rays and classify how a primitive character shifts endpoint layers. Using these restrictions, Section 9 shows that the equations on two-sided rays reduce to the additive-weight relations of a strictly smaller representation-finite ray algebra, while the remaining one-sided equations organize into a bipartite graph whose components are controlled by excluding four-point crowns. Induction on the number of basis elements then yields integer lifts of all prescribed modulo-two derivation coefficients. Finally, Section 10 and Section 11 pass via the Bockstein map to zero-supported cocycles over k , assemble them into a global associative family of deformations, and recover the original algebra A as a binary fibre, completing the proof of Theorems A and B; Section 12 records further consequences for module categories and parameter spaces.

8. LOCAL RESTRICTIONS ON ZERO PRODUCTS AND RETURNS

Our first step in characteristic two is to constrain the products and factorizations of ray-basis elements in the standard algebra $A_0 = kP$, including return paths that leave a vertex and come back to it with a nonzero value in the local ring. These local constraints play a role analogous to the analysis of endpoint actions in [2, Sections 2.4–2.5]. Combined with the integral coefficient-lifting theorem of Section 9 and the modulo-four comparison of Section 10, they will allow us to construct cocycle representatives for every class in $\text{HH}^2(A_0)_{>0}$ that vanish on all nonzero products of basis elements.

The local structural results in this section do not themselves depend on an inductive hypothesis. In Section 9, however, lifting their associated modulo-two coefficient equations to the integers will require an induction on the size of a weighted multiplication table. We record the definition of such tables and the main inductive statements before beginning the local analysis.

Definition 8.1. A *weighted multiplication table* T consists of a finite set of objects, identity elements 1_x , and finitely many nonzero morphism elements (also called *labels*), equipped with an associative partial composition under which the product of two composable elements is either an element of T or zero, together with an additive weight $\ell: T \setminus \{0\} \rightarrow M$ valued in an abelian group M and a homomorphism $d: M \rightarrow \mathbb{Z}$ satisfying the following conditions:

- (i) $\ell(1_x) = 0$ for every object x , and $d(\ell(r)) > 0$ for every nonidentity element $r \in T \setminus \{0\}$;
- (ii) two nonzero elements of T with the same source, target, and weight are equal.

The *linearization* kT is the finite-dimensional k -linear category having the nonzero elements of T as a basis, and its scalar weight group $W(T)$ is defined by the presentation of Definition 7.1.

Proposition 8.2. *Let T be a weighted multiplication table whose linearization kT is representation-finite over the algebraically closed field k . Every additive weight $T \rightarrow \mathbb{Z}/2$ lifts to an additive weight $T \rightarrow \mathbb{Z}$.*

Proposition 8.2 will be proved in Subsection 11.2 by strong induction on the total number $n = \dim kT$ of nonzero elements of T (including identities). Assuming the statement for all tables with fewer than n nonzero elements, we first use it in characteristic two—applied strictly to the smaller centralizer table of Lemma 9.2—to construct integral solutions to the nonzero-product Leibniz equations for the standard form A_0 of any basic connected representation-finite algebra A of dimension n (Theorem 9.1). The resulting cocycles and their associative deformation families recover A and identify its ray multiplication table with that of A_0 (Lemma 11.6). Specializing to $A = kT$ for a connected table T of size n then identifies T with $\text{ray}(kT)$ and equates $W(T)$ with the free abelian weight group of its standard form (Proposition 7.2), so that every $\mathbb{Z}/2$ -valued weight on T lifts to $\mathbb{Z}$. Outside characteristic two, the corresponding recovery is already given by Corollary 5.5, and disconnected tables reduce immediately to their connected components.

From this inductive step we also obtain the main structural consequences for representation-finite algebras in arbitrary characteristic, likewise proved in Subsection 11.2.

Proposition 8.3. *Let A be a basic representation-finite algebra over k , in any characteristic, and let A_0 be the projective ray algebra of Construction 2.8. Then:*

- (1) *A admits a normed multiplicative basis consisting of one representative of each nonzero ray;*
- (2) *the initial-form bijection of Lemma 3.7 identifies the ray multiplication in $\text{ray } A$ with that in $\text{ray } A_0$;*
- (3) *the scalar weight group $W(\text{ray } A)$ is free abelian.*

Setting 8.4. Throughout Part III and Appendix C, we let $A_0 = kP$ denote the projective ray algebra of Construction 2.8, identified with kP via Proposition 4.14 and equipped with its normed homogeneous ray basis. For each vertex x , we write $R_x = k[q_x]/(q_x^{N_x})$ when $R_x \neq k$, where q_x is the chosen homogeneous generator of $\text{rad } R_x$, and we set $q_x = 0$ and $N_x = 1$ when $R_x = k$.

We use the following terminology for the rays of $A_0 = kP$. Note that an *ordinary arrow* below means an indecomposable nonidentity ray of A_0 (whose vertices correspond to the indecomposable projectives of A), whereas the arrows of the Auslander–Reiten quiver Γ_A represent irreducible morphisms among all indecomposable A -modules.

Definition 8.5. Let $A_0 = kP$ satisfy Setting 8.4.

- (1) A ray $r: x \rightarrow y$ is *generating* if it generates the full corner bimodule $A_0(x, y)$ as an (R_x, R_y) -bimodule; otherwise r is *proper*.
- (2) Given a generating ray $f: x \rightarrow y$, we say that the corner bimodule $A_0(x, y)$ (or simply f) is *cyclic at x* if $A_0(x, y) = R_x f$, and *cyclic at y* if $A_0(x, y) = f R_y$. When $A_0(x, y)$ is cyclic at x , its rays are the elements $q_x^j f$, and we call the exponent j the *endpoint layer* of $q_x^j f$; the endpoint layer at y is defined symmetrically when $A_0(x, y)$ is cyclic at y .
- (3) An *ordinary arrow* is a nonidentity ray that cannot be factored as a product of two nonidentity rays, and an *ordinary path* is a composable sequence of ordinary arrows. Within a corner or quotient category, these terms refer to factorizations inside that corner or quotient.
- (4) Let $x \neq y$, and let $f: x \rightarrow y$ and $h: y \rightarrow x$ be rays such that $fh = q_x^b \neq 0$ for some integer $b \geq 1$. We call fh a *return* at x through y , h a *reverse* of f , and b the *exponent* of the return.

The *endpoint socle* of a corner bimodule $A_0(x, y)$ is the subspace of vectors annihilated by both $\text{rad } R_x$ on the left and $\text{rad } R_y$ on the right. By Lemma 3.1, every nonzero corner bimodule $A_0(x, y)$ has a one-dimensional endpoint socle.

Definition 8.6 ([2, Section 9.1]). For a ray $r: x \rightarrow y$, let $U_r = R_x r R_y \subseteq A_0(x, y)$ denote the endpoint subbimodule it generates. For composable rays $r: x \rightarrow y$ and $s: y \rightarrow z$, their *stem product* $r * s$ is the unique ray $t: x \rightarrow z$ satisfying $U_t = U_r U_s$ when $U_r U_s \neq 0$, and $r * s = 0$ when $U_r U_s = 0$. Uniqueness is guaranteed by the linear ordering of endpoint subbimodules in Lemma 3.1.

Under the bijection between rays and the endpoint subbimodules they generate, the stem product $*$ is composition in the stem category of Kupisch [15]; see [6, Section 2.3] for the formulation used here. Because $U_r U_s = R_x r R_y s R_z$, the stem product captures the entire subspace spanned by products of representatives with arbitrary insertions from the intermediate local ring R_y . We will use it to determine which pairs of rays with $rs = 0$ can support a nonzero deformation term.

8.1. Homogeneous subcategories and a dimension criterion

To establish the local restrictions on A_0 , we show that any violation would produce a finite representation-infinite category built from rays of A_0 , contradicting the representation-finiteness of A_0 . We construct these test categories as follows. Equip A_0 with its grading by the weight lattice L of Definition 7.3 (written additively), placing each ray r in degree $\ell(r)$ and the primitive idempotents in degree zero. A *homogeneous category* is the full subcategory H of the associated graded covering category (Definition 4.5) on a finite set of distinct objects (x_i, w_i) with $w_i \in L$. Writing subscripts for L -degrees, its morphism spaces are

$$H(i, j) = A_0(x_i, x_j)_{w_j - w_i}.$$

Because the categorical degree $\deg_{\mathcal{J}}$ is one of the coordinates of the weight lattice L , each nonzero morphism space $H(i, j)$ is one-dimensional and H is a directed category. By Lemma 4.7, every such homogeneous category H is representation-finite. Passing to the graded covering allows us to separate distinct rays between the same vertices—or multiple copies of the same vertex at different weights—into distinct objects and morphisms of H . Whenever the categorical $\mathbb{Z}$ -grading $\deg_{\mathcal{J}}$ alone suffices to separate the rays under consideration, we may perform the same construction using the $\mathbb{Z}$ -graded covering category instead.

A primary source of forbidden configurations is the factorization poset induced by the action of H on a thin projective or injective H -module. Taking the quotient by the annihilator of the module in the next definition is essential, because a morphism that is nonzero in H may still act as zero on the chosen module.

Definition 8.7. Let H be a homogeneous category (with respect to either the weight lattice L or the categorical grading $\deg_{\mathcal{J}}$). Fix an object $X \in H$, set $V = H(X, -)$, and let $\text{ann } V$ be the ideal of morphisms in H that act as zero on V . The *faithful projective support* at X is the poset consisting of the objects $Y \in H$ with $V(Y) \neq 0$, ordered by $Y \leq Z$ whenever $(H / \text{ann } V)(Y, Z) \neq 0$. Equivalently, if we label the points of this poset by the nonzero rays r starting at X , then $r \leq s$ if and only if $rv = s$ for some ray v in H .

The *faithful injective support* at X is defined dually via the thin module $D_k H(-, X)$; in terms of nonzero rays ending at X , its order is given by $r \leq s$ if and only if $vr = s$ for some ray v in H .

When restricted to the objects on which V is nonzero, the quotient $H / \text{ann } V$ is isomorphic to the incidence category of the faithful projective support at X by Lemma 4.10 (positivity of the grading rules out nonidentity isomorphisms). By the same lemma, this incidence category is representation-finite, and hence so is the incidence category of every full subposet

(which is an idempotent corner of $H/\text{ann } V$). The dual statements hold for the faithful injective support.

The following standard Tits-form dimension criterion rules out the small representation-infinite quivers and posets that arise in our local arguments, sparing us from having to construct explicit one-parameter families of representations for each orientation.

Lemma 8.8. *Let $B = kQ/(\rho_1, \dots, \rho_t)$ be a finite-dimensional bound quiver algebra, where each relation ρ_i is a linear combination of radical paths sharing a common source and target. For a nonzero dimension vector $d = (d_v)$, set*

$$D = \sum_v d_v^2, \quad N = \sum_{a:v \rightarrow w} d_v d_w, \quad F = \sum_{i=1}^t d_{\text{source}(\rho_i)} d_{\text{target}(\rho_i)}.$$

If $N - F \geq D$, then B is representation-infinite.

Proof. The affine variety of B -modules of dimension vector d is cut out inside $\mathbb{A}^N$ by at most F polynomial equations, and it is nonempty because it contains the zero representation. Every irreducible component of this variety therefore has dimension at least $N - F$. On the other hand, the algebraic group $\prod_v \text{GL}_{d_v}(k)$ has dimension D , and the diagonally embedded subgroup $k^\times$ acts trivially on every representation, so every isomorphism orbit has dimension at most $D - 1$. Since $N - F > D - 1$, the representation variety cannot be covered by finitely many orbits. By the Krull–Schmidt theorem, this contradicts finite representation type. $\square$

For the relation-free quivers ($F = 0$) used repeatedly below, the critical dimension vectors achieving $N = D$ are listed in the following table:

Underlying graph	Dimension vector d	$N = D$
Crown ($2n$ vertices)	1 at every vertex	$2n$
Four-leaf star ($\tilde{D}_4$)	2 at center, 1 at each leaf	8
Tree $\tilde{D}_5$ (two trivalent vertices)	2 at trivalent vertices, 1 at leaves	12
Tree $\tilde{E}_7$ (arms 3, 3, 1)	4 at center; (3, 2, 1), (3, 2, 1), (2) on arms	48

Because N and D depend only on the underlying undirected graph when $F = 0$, these counts apply to any orientation of the listed trees. Here a *crown* means a bipartite cycle of n minimal and n maximal poset elements with no additional comparability relations. In particular, neither a four-point crown nor a four-leaf star can appear as a full subposet of any faithful projective or injective support.

We also apply Lemma 8.8 to any quotient B of A_0 (or of an idempotent corner of A_0) satisfying $(\text{rad } B)^2 = 0$. Grading B by assigning degree 1 to its arrows and degree 0 to its vertices, we pass to the full subcategory of its graded covering category on the objects $v_0 = (v, 0)$ and $v_1 = (v, 1)$ as v ranges over the vertices of B . The only nonidentity morphisms in this *two-level category* go from v_0 to w_1 , one for each arrow $v \rightarrow w$ of B . By Lemma 4.7, the two-level category is representation-finite; consequently, its underlying bipartite graph cannot contain a four-cycle (for which the all-ones dimension vector on the cycle has $N = D = 4$ and $F = 0$).

To rule out more general configurations with relations, we pull representations back along a split bimodule inclusion using the following observation.

Lemma 8.9. *Let $F: S \rightarrow B$ be a functor between finite categories, and let B_F be the category having the same object set as S and morphism spaces $B_F(x, y) = B(Fx, Fy)$. Suppose that the induced functor $S \rightarrow B_F$ is injective on morphism spaces and admits a retraction $\pi: B_F \rightarrow S$ as an S -bimodule morphism. If B is representation-finite, then so is S .*

Proof. When several objects of S map to the same object of B , the category B_F is a matrix enlargement of an idempotent corner of B , and is therefore representation-finite by Lemma 3.1(1). Applying Lemma 4.6 to the split inclusion $S \hookrightarrow B_F$ with $C = D = S$ then shows that S is representation-finite. $\square$

8.2. Unequal actions and generation at the middle endpoint

With respect to our normed homogeneous ray basis, every nonzero endpoint action on a corner bimodule can be written as a power of the radical generator at an endpoint where the bimodule is cyclic. We first analyze what happens when the radical generator at the opposite endpoint acts as a second or higher power of the cyclic generator. Together with Lemma 8.11, this bound will force any zero ray product $rs = 0$ with nonzero stem product $r * s \neq 0$ to land in the one-dimensional endpoint socle (Corollary 8.13).

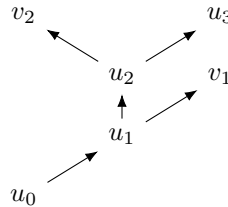
Lemma 8.10. *In Setting 8.4, suppose that $A_0(x, y) = R_x f$ has dimension l , where $f: x \rightarrow y$ is a generating ray. If $f q_y = q_x^a f \neq 0$ for some integer $a \geq 2$, then $l = a + 1$, and $f q_y$ spans the endpoint socle of $A_0(x, y)$. The dual statement holds when $A_0(x, y) = f R_y$.*

Proof. Since $q_x^a f = f q_y \neq 0$, we have $l \geq a + 1$. Suppose for a contradiction that $l \geq a + 2$, so that $q_x^{a+1} f \neq 0$. Set $d = \deg_{\mathcal{J}}(q_x)$ and $e = \deg_{\mathcal{J}}(f)$; the relation $f q_y = q_x^a f$ then gives $\deg_{\mathcal{J}}(q_y) = ad$. We first check that there is no homogeneous reverse ray $h: y \rightarrow x$ satisfying $fh = q_x^b$ with $1 \leq b \leq a + 1$. Indeed, if such an h existed, then $f(hf) = (fh)f = q_x^b f \neq 0$ because $b \leq a + 1 < l$; thus hf would be a nonzero homogeneous element of R_y of degree bd , which is possible only if bd is a multiple of $\deg_{\mathcal{J}}(q_y) = ad$, forcing $b = a$. Even for $b = a$, however, the element $h q_x f \in R_y$ has degree $(a + 1)d$, which is not divisible by ad ; hence $h q_x f = 0$, contradicting $f(h q_x f) = (fh) q_x f = q_x^{a+1} f \neq 0$.

Now consider the faithful projective support at $(x, 0)$ (with respect to the categorical grading $\deg_{\mathcal{J}}$) and restrict to the six points

$$X_i = (x, id) \quad (i \in \{0, 1, a, a + 1\}), \quad Y_j = (y, e + jd) \quad (j \in \{1, a\}),$$

which correspond to the nonzero rays q_x^i and $q_x^j f$. One point precedes another in this poset if and only if the second ray is a right multiple of the first. Thus the points $X_0 < X_1 < X_a < X_{a+1}$ form a chain, and $X_i < Y_j$ holds whenever $i \leq j$. No reverse inequality $Y_j \leq X_i$ can hold, because it would require a ray $h: y \rightarrow x$ with $q_x^j f h = q_x^i$, and hence $fh = q_x^{i-j}$ with $1 \leq i - j \leq a$, which we have just ruled out. This same exclusion prevents $X_i < Y_j$ when $i > j$. Finally, Y_1 and Y_a are incomparable: a morphism $Y_1 \rightarrow Y_a$ would have degree $(a - 1)d$, which is not a multiple of $\deg_{\mathcal{J}}(q_y) = ad$, while a morphism $Y_a \rightarrow Y_1$ would have negative degree. Relabeling X_0, X_1, X_a, X_{a+1} as u_0, u_1, u_2, u_3 and Y_1, Y_a as v_1, v_2 , we see that the induced full subposet has Hasse diagram



which is an orientation of the affine Dynkin tree $\tilde{D}_5$. Assigning dimension 2 to the trivalent vertices u_1, u_2 and dimension 1 to the four leaves gives $N = D = 12$ and $F = 0$, contradicting finite representation type by Lemma 8.8.

We conclude that $l = a + 1$. Consequently $q_x(f q_y) = q_x^{a+1} f = 0$ and $(f q_y) q_y = q_x^{2a} f = 0$ (since $2a \geq a + 1 = l$), so $f q_y$ is annihilated by both endpoint radicals and therefore spans the endpoint socle of $A_0(x, y)$. $\square$

Lemma 8.11. *In Setting 8.4, let $f: b \rightarrow c$ and $g: c \rightarrow a$ be generating rays whose corner bimodules $A_0(b, c)$ and $A_0(c, a)$ are cyclic at c but not at b and a , respectively, and let $q = q_c$ generate $\text{rad } R_c$.*

- (1) *Either $f q^2 = 0$ or $q^2 g = 0$.*
- (2) *The product subspace $f R_c g \subseteq A_0(b, a)$ has dimension at most one and is annihilated by $\text{rad } R_b$ on the left and by $\text{rad } R_a$ on the right.*

Proof. Because the local ring $R_c = A_0(c, c)$ is commutative and hence cyclic on both sides, the noncyclicity assumptions at b and a imply that $b \neq c$ and $a \neq c$ (though b and a may coincide). Moreover, a one-dimensional corner bimodule is automatically cyclic over both endpoint rings, so $\dim A_0(b, c) \geq 2$ and $\dim A_0(c, a) \geq 2$. Writing $A_0(b, c) = f R_c \cong k[q]/(q^n)$ with $n \geq 2$, take any $t \in \text{rad } R_b$ and write $t f = f \varphi(q)$ for a polynomial $\varphi(q) \in k[q]$. Nilpotence of t forces $\varphi(0) = 0$, and if the linear coefficient of φ were nonzero, then the powers of $\varphi(q)$ would generate $k[q]/(q^n)$, making $A_0(b, c) = R_b f$ cyclic at b , contrary to assumption. Thus $(\text{rad } R_b) f \subseteq f q^2 R_c$. Furthermore, for any morphism $h \in A_0(c, b)$, the composition $f h$ lies in $\text{rad } R_b$ because $b \neq c$; hence $f(h f) = (f h) f \in f q^2 R_c$, and since $\text{ann}_{R_c}(f) = (q^n) \subseteq (q^2)$, it follows that $h f \in (q^2)$. Applying the dual argument to g yields the four inclusions

$$\begin{aligned} (\text{rad } R_b) f &\subseteq f q^2 R_c, & g(\text{rad } R_a) &\subseteq R_c q^2 g, \\ A_0(c, b) f &\subseteq (q^2), & g A_0(a, c) &\subseteq (q^2). \end{aligned} \tag{15}$$

To prove (1), suppose for a contradiction that both $f q^2 \neq 0$ and $q^2 g \neq 0$. We distinguish two cases according to whether $f g$ is nonzero.

Case 1: $f g \neq 0$. Let S be the path category of the four-leaf star quiver

$$\begin{array}{ccccc} & & u & & \\ & & \downarrow^{q_1} & & \\ l & \xrightarrow{f} & c & \xrightarrow{g} & r \\ & & \downarrow^{q_2} & & \\ & & d & & \end{array}$$

in which the two paths $l \xrightarrow{f} c \xrightarrow{g} r$ and $u \xrightarrow{q_1} c \xrightarrow{q_2} d$ cross at c with no relations. We map the five vertices l, c, r, u, d of S to the objects of the $\mathbb{Z}$ -graded homogeneous category H of A_0 having underlying vertices b, c, a, c, c and grading coordinates

$$-\deg_{\mathcal{J}}(f), \quad 0, \quad \deg_{\mathcal{J}}(g), \quad -\deg_{\mathcal{J}}(q), \quad \deg_{\mathcal{J}}(q),$$

respectively, sending the arrows f, g, q_1, q_2 to the rays f, g, q, q . All four length-two paths in S then map to the nonzero rays $f g, f q, q g, q^2$ in H , so the functor $S \rightarrow H$ identifies each nonzero morphism space of S with the corresponding one-dimensional morphism space of H .

Define a linear map $\pi: H \rightarrow S$ on each morphism space to be the inverse of this identification whenever the morphism space in S is nonzero, and zero otherwise. To see that π is an S -bimodule retraction as required by Lemma 8.9, it suffices to check that π commutes with left and right multiplication by the four arrows of S . Whenever the product of two composable morphisms in H lies in a morphism space where S is nonzero, compatibility holds automatically because that homogeneous component is one-dimensional. Otherwise, we must verify that no morphism of H outside the image of S can compose with an arrow image to land inside the image of S . At the outgoing fork $c \rightarrow r, d$, a nonzero composite factoring through an ambient ray $h: r \rightarrow d$ or $h: d \rightarrow r$ would yield $gh = q$ or $qh = g$ (up to a nonzero scalar); dually, at the incoming fork $l, u \rightarrow c$, an ambient ray between l and u would yield $hf = q$ or $hq = f$. However, $gh = q$ and $hf = q$ are ruled out by (15), while $qh = g$ and $hq = f$ contradict the hypothesis that g and f are generating rays (Lemma 8.14). Since no product involving a morphism of positive degree can be

an identity, π is indeed an S -bimodule retraction. By Lemma 8.8, the four-leaf star S is representation-infinite, contradicting Lemma 8.9.

Case 2: $fg = 0$. When $fg = 0$, the last two inclusions in (15) sharpen to

$$A_0(c, b)f \subseteq (q^3), \quad gA_0(a, c) \subseteq (q^3),$$

because if hf had a nonzero q^2 -term, multiplying on the right by g would give $(hf)g = h(fg) = 0$ while $q^2g \neq 0$, and dually for gh . We now let S be the bound quiver category on seven vertices with diagram and single relation

$$\begin{array}{ccccc} u' & \xleftarrow{g_1} & u & & \\ & & \downarrow q_1 & & \\ l & \xrightarrow{f_1} & c & \xrightarrow{g_2} & r \\ & & \downarrow q_2 & & \\ l_d & \xrightarrow{f_2} & d & & \end{array} \quad f_1g_2 = 0.$$

We map l, c, r, u, d to the same graded objects as in Case 1, and we send l_d and u' to $(b, \deg_{\mathcal{J}}(q) - \deg_{\mathcal{J}}(f))$ and $(a, \deg_{\mathcal{J}}(g) - \deg_{\mathcal{J}}(q))$, respectively, mapping f_1, f_2 to f , g_1, g_2 to g , and q_1, q_2 to q . The relation $f_1g_2 = 0$ holds in H because $fg = 0$, and the three remaining length-two paths map to the nonzero rays q^2, fq, qg . Defining the projection $\pi: H \rightarrow S$ as before, we verify that no ambient morphism in H can compose with an arrow of S to produce a nonzero morphism in the image of S . The possible unwanted products and the reasons they vanish are summarized below:

target ray in S	ambient product in H	reason for exclusion
q or q^2	gh	$gA_0(a, c) \subseteq (q^3)$
qg	gv ($v \in \text{rad } R_a$)	inclusion (15) and $q^3g \notin k \cdot qg$
g	qh	g is a generating ray
q or q^2	hf	$A_0(c, b)f \subseteq (q^3)$
fq	vf ($v \in \text{rad } R_b$)	inclusion (15) and $fq^3 \notin k \cdot fq$
f	hq	f is a generating ray.

At u , the first three rows rule out factorizations between $g_1: u \rightarrow u'$ and the paths through c ; at c , they rule out factorizations between the two outgoing branches g_2 and q_2 . The last three rows provide the dual incoming exclusions at c and d , while the remaining vertices have at most one incident branch in the relevant direction. Thus π is a bimodule retraction (note that Lemma 8.9 permits distinct vertices of S to share the same underlying graded object in H). Assigning dimension 3 to c , dimension 2 to u and d , and dimension 1 to the four outer vertices u', l, r, l_d gives $N = 22$, $D = 21$, and $F = 1$ from the single relation $f_1g_2 = 0$. Since $N - F = D$, Lemma 8.8 and Lemma 8.9 again yield a contradiction, completing the proof of (1).

For (2), consider the action of $\text{rad } R_b$ on the left or $\text{rad } R_a$ on the right of fR_cg . By (15), any such product is a linear combination of terms fq^mg with $m \geq 2$. Since either $fq^2 = 0$ or $q^2g = 0$ by (1), every such term vanishes. Thus fR_cg is annihilated by both $\text{rad } R_b$ and $\text{rad } R_a$, so it lies inside the one-dimensional endpoint socle of $A_0(b, a)$. $\square$

8.3. Stem products and generating paths

We next show that the stem product of Definition 8.6 is determined purely by the ray category P via an explicit local formula, and can be nonzero even when the ordinary ray product vanishes.

Proposition 8.12 ([2, Section 9.1]). *In Setting 8.4, the stem product on $A_0 = kP$ is determined by the ray category P : for any composable rays $r: x \rightarrow y$ and $s: y \rightarrow z$,*

$$r * s = \begin{cases} rq_y^n s, & \text{where } n \geq 0 \text{ is the least integer such that } rq_y^n s \neq 0, \\ 0, & \text{if } rq_y^n s = 0 \text{ for all } n \geq 0, \end{cases}$$

with $q_y^0 = 1_y$ and $q_y = 0$ when $R_y = k$. The operation $*$ is associative and agrees with the ray product rs whenever $rs \neq 0$. Moreover, the same formula computes the product of endpoint subbimodules in any algebra having ray category P and satisfying the endpoint-chain properties of Lemma 3.1.

Proof. Let $r: x \rightarrow y$ and $s: y \rightarrow z$ be composable rays in P , choose representatives $f \in r$ and $h \in s$ in an algebra with ray category P , and let u generate $\text{rad } R_y$ (with $u = 0$ if $R_y = k$). For each $j \geq 0$, set $V_j = R_x f u^j R_y h R_z$. We have $f u^j \neq 0$ if and only if the ray product $r q_y^j$ is nonzero: indeed, if $f u^j \neq 0$, then $f u^j$ generates the subbimodule $U(f u^j)$ and therefore lies outside its radical $R(f u^j) \supseteq M(f, u^j)$, which means $r q_y^j \neq 0$ by (6); conversely, if $f u^j = 0$, then $f u^j \in M(f, u^j)$ and $r q_y^j = 0$. When $r q_y^j \neq 0$, its ray product with s vanishes if and only if $f u^j h \in V_{j+1}$, or equivalently $V_j = V_{j+1}$. Consequently, if $n \geq 0$ is the smallest integer for which $r q_y^n s \neq 0$, then $U_r U_s = V_0 = V_n$, and hence

$$U_r U_s = U_{r q_y^n s}. \quad (16)$$

If no such n exists, the chain of equalities $V_j = V_{j+1}$ and the nilpotence of u force $U_r U_s = 0$. Thus $r * s$ is given by the stated formula in P . Associativity of $*$ follows immediately from the associativity of subbimodule multiplication, and when $rs \neq 0$ the least index is $n = 0$, giving $r * s = rs$. $\square$

Corollary 8.13. *In Setting 8.4, let $r: x \rightarrow y$ and $s: y \rightarrow z$ be composable rays such that $rs = 0$ and $r * s \neq 0$. Then r and s are generating rays, $r * s = r q_y s$, and $U_{r * s}$ is the one-dimensional endpoint socle of $A_0(x, z)$.*

Proof. If the corner bimodule $A_0(x, y)$ containing r were cyclic at x , then $r q_y \in \text{rad } A_0(x, y)$ would equal $q_x^a r$ for some $a \geq 1$. For every $n \geq 1$ we would then have $r q_y^n s = q_x^{an} (rs) = 0$, giving $r * s = 0$ by Proposition 8.12, a contradiction. By the dual argument for s , neither $A_0(x, y)$ nor $A_0(y, z)$ is cyclic at its outer endpoint, so both are cyclic only at y . Write $r = f q_y^i$ and $s = q_y^j g$ where f and g are the generating rays of $A_0(x, y)$ and $A_0(y, z)$. By Lemma 8.11, $f q_y^m g = 0$ for all $m \geq 2$, and the entire subspace $f R_y g$ lies in the endpoint socle of $A_0(x, z)$. Since $rs = f q_y^{i+j} g = 0$ while $r q_y^n s = f q_y^{i+j+n} g \neq 0$ for some $n \geq 1$, the only possibility is $i = j = 0$ and $n = 1$. Thus $r = f$ and $s = g$ are generating rays, $r * s = r q_y s$, and $U_{r * s}$ is the endpoint socle of $A_0(x, z)$. $\square$

Lemma 8.14. *In Setting 8.4, let $f: x \rightarrow z$ be a generating ray whose corner bimodule $A_0(x, z)$ has dimension at least two. Then f admits no factorization of the form $f = a q_y b$ through a local radical generator q_y . In particular, every ordinary path representing f is vertex-simple (visits no vertex twice), and every nonempty proper subsegment of such a path represents a generating ray.*

Proof. Suppose for a contradiction that $f = a q_y b$ for some rays $a: x \rightarrow y$ and $b: y \rightarrow z$. If $a = 1_x$, then $f = q_x b \in (\text{rad } R_x) A_0(x, z)$ lies in the radical of $A_0(x, z)$, contradicting the assumption that f generates $A_0(x, z)$; the case $b = 1_z$ is ruled out similarly. If the corner bimodule $A_0(x, y)$ is cyclic at x , then $a q_y \in (\text{rad } R_x) A_0(x, y)$, which again places $f = a q_y b$ inside $(\text{rad } R_x) A_0(x, z)$, and symmetrically if $A_0(y, z)$ is cyclic at z . In the remaining case, both $A_0(x, y)$ and $A_0(y, z)$ are cyclic only at y . Writing $a = a' q_y^i$ and $b = q_y^j b'$ with a', b' generating, we find $f \in a' R_y b'$. By Lemma 8.11(2), $a' R_y b'$ is contained in the one-dimensional endpoint socle of $A_0(x, z)$, whereas the generating ray f of a corner

bimodule of dimension at least two cannot lie in its endpoint socle. This contradiction rules out any factorization $f = aq_yb$. Finally, any repeated vertex along an ordinary path representing f , or any proper (non-generating) subsegment of that path, would produce a factorization of f through a local radical generator q_y . $\square$

Example 3.5 illustrates the difference between the ordinary ray product and the stem product: there $[\alpha][\beta] = 0$ in the ray category, whereas $U_\alpha U_\beta = ku$ is nonzero because inserting the middle radical generator q yields $\alpha q \beta = u \neq 0$, so that $[\alpha] * [\beta] = [u]$.

8.4. Successor coefficients and primitive returns

We next study the coefficients of normalized Hochschild one-cochains δ that shift ray weights by a fixed character $\chi \in L$. The Leibniz condition $\delta(rs) = \delta(r)s + r\delta(s)$ on nonzero ray products $rs \neq 0$ is equivalent to the vanishing of the coboundary $d\delta$ on all nonzero products of basis elements. In Section 9 we will lift solutions of these nonzero-product Leibniz equations from $\mathbb{Z}/2$ to $\mathbb{Z}$. Applied to a derivation over $\mathbb{F}_2$, such an integral lift produces a two-cocycle $(d\delta)/2$ over $\mathbb{F}_2$ that vanishes on all nonzero ray products and represents the connecting homomorphism class of δ in $\mathrm{HH}^2(\mathbb{F}_2 P)_\chi$ (Proposition 10.3).

Definition 8.15. Evaluating characters in L on the categorical degree $\deg_{\mathcal{J}} \in H_{\mathbb{Z}}$, we say that a character $\chi \in L$ is *positive* if $\chi(\deg_{\mathcal{J}}) > 0$, and *primitive* if it cannot be written as $n\eta$ for an integer $n > 1$ and a character $\eta \in L$.

By Proposition 7.5 and the identity $\chi_i(\deg_{\mathcal{J}}) = 2s_i$, every residual character χ_i is both positive and primitive.

Setting 8.16. For the remainder of this section and Section 9, we fix a positive primitive character $\chi \in L$. For each vertex x , we define

$$\widehat{q}_x = \begin{cases} q_x, & \text{if } q_x \neq 0 \text{ and } \ell(q_x) = \chi, \\ 0, & \text{otherwise.} \end{cases}$$

Thus q_x continues to denote the actual homogeneous generator of $\mathrm{rad} R_x$ from Setting 8.4, whereas $\widehat{q}_x$ retains q_x only when its weight equals the fixed character χ .

Definition 8.17. Work with $A_0 = kP$ and a positive primitive character χ as in Settings 8.4 and 8.16.

- (1) A vertex x is *supporting* (for χ) if $\widehat{q}_x \neq 0$, that is, if $q_x \neq 0$ and $\ell(q_x) = \chi$. More generally, x has *primitive local weight* if $q_x \neq 0$ and $\ell(q_x)$ is primitive in L .
- (2) A nonidentity ray $r: x \rightarrow y$ is *shiftable* if there is a ray $r^+: x \rightarrow y$ of weight $\ell(r) + \chi$, which we call the *successor* of r .
- (3) The *endpoint type* of a shiftable ray $r: x \rightarrow y$ is classified by which of the endpoint actions $\widehat{q}_x r$ and $r \widehat{q}_y$ are nonzero:

Name	Symbol	Defining condition
left-sided	L	$\widehat{q}_x r = r^+ \neq 0, r \widehat{q}_y = 0$
right-sided	R	$\widehat{q}_x r = 0, r \widehat{q}_y = r^+ \neq 0$
two-sided	B	$\widehat{q}_x r = r \widehat{q}_y = r^+ \neq 0$.

We say that r is *one-sided* if its endpoint type is L or R. We also say that the source x *controls* r when $\widehat{q}_x r \neq 0$, and that the target y *controls* r when $r \widehat{q}_y \neq 0$.

To see why every shiftable ray $r: x \rightarrow y$ falls into one of the three types in Definition 8.17(3), recall that the corner bimodule $A_0(x, y)$ is cyclic over at least one of its endpoint rings R_x or R_y , so its rays are successive powers of that endpoint's radical generator applied to the generating ray of $A_0(x, y)$. The weight difference $\ell(r^+) - \ell(r) = \chi$ must therefore be a positive integer multiple of the weight of that radical generator. Because χ is primitive, the multiple is necessarily 1, which means that r^+ is obtained from r by

a single multiplication by $\widehat{q}_x$ on the left or by $\widehat{q}_y$ on the right (and r^+ is unique because each weight space in $A_0(x, y)$ is at most one-dimensional). Notice the distinction between cyclicity in Definition 8.5, which is a property of the entire corner bimodule $A_0(x, y)$ over R_x or R_y , and the endpoint type L, R, B , which records whether the action of weight χ on a specific shiftable ray r remains nonzero on the left, on the right, or on both sides.

For example, if $A_0(x, y)$ is a three-dimensional corner bimodule generated on the left at a supporting vertex x , its rays are f , $\widehat{q}_x f$, and $\widehat{q}_x^2 f$, which lie in endpoint layers 0, 1, and 2, respectively. The rays f and $\widehat{q}_x f$ are shiftable, with successors $\widehat{q}_x f$ and $\widehat{q}_x^2 f$, whereas the socle ray $\widehat{q}_x^2 f$ is unshiftable. If the right action of $\widehat{q}_y$ satisfies $f\widehat{q}_y = \widehat{q}_x f$, then both f and $\widehat{q}_x f$ have type B ; if instead $f\widehat{q}_y = 0$, then both have type L .

Any normalized one-cochain δ of weight χ vanishes on the primitive idempotents 1_x and on all unshiftable rays, and is determined on shiftable nonidentity rays r by scalar coefficients d_r via $\delta(r) = d_r r^+$. On a nonzero product $rs = t$ of nonidentity rays, the Leibniz equation $\delta(rs) = \delta(r)s + r\delta(s)$ compares $d_t t^+$ with $d_r r^+ s + d_s r s^+$. We refer to the products $r^+ s$ and rs^+ as the *successor substitutions* of rs ; each substitution either equals t^+ or vanishes.

Before tabulating these equations, we note a basic consequence of primitive local weight that will be used throughout our analysis.

Lemma 8.18. *In Setting 8.4, let $f: x \rightarrow y$ be a generating ray, and suppose that $\ell(q_x)$ is primitive. If $q_x f \neq 0$, then $A_0(x, y) = R_x f$. If in addition $A_0(x, y)$ is cyclic at y , then $\ell(q_y) = \ell(q_x)$ and $q_x f = f q_y \neq 0$. The dual assertions hold when $\ell(q_y)$ is primitive and $f q_y \neq 0$.*

Proof. If $A_0(x, y)$ is cyclic at y , then $q_x f \neq 0$ must take the form $q_x f = f q_y^a$ for some integer $a \geq 1$. Comparing weights gives $\ell(q_x) = a\ell(q_y)$, and the primitivity of $\ell(q_x)$ forces $a = 1$. Iterating the relation $q_x f = f q_y$ then shows that $R_x f = f R_y = A_0(x, y)$. If $A_0(x, y)$ is not cyclic at y , then Lemma 3.1(3) immediately implies that $A_0(x, y) = R_x f$. The second assertion follows by the same weight comparison, and the dual statements follow by passing to the opposite algebra. $\square$

Lemma 8.19. *In Settings 8.4 and 8.16, let $rs = t$ be a nonzero product of nonidentity rays. If t is unshiftable, then both successor substitutions $r^+ s$ and rs^+ vanish (whenever defined), so $\delta(rs) = \delta(r)s + r\delta(s)$ imposes no condition on the coefficients d_r, d_s . If t is shiftable, the possible endpoint types of (r, s) and the resulting Leibniz equations for d_t are given by the following table (where $*$ denotes an arbitrary shiftable or unshiftable ray):*

Type of $t = rs$	Types of (r, s)	Leibniz equation
L	$(L, *)$	$d_t = d_r$
L	(B, L)	$d_t = d_r + d_s$
R	$(*, R)$	$d_t = d_s$
R	(R, B)	$d_t = d_r + d_s$
B	(B, B) or (L, R)	$d_t = d_r + d_s$

(17)

Proof. Let $rs = t \neq 0$ with $r: x \rightarrow y$ and $s: y \rightarrow z$. If t is unshiftable, then $A_0(x, z)$ has no ray of weight $\ell(t) + \chi$, so neither $r^+ s$ nor rs^+ can be nonzero.

Assume now that t is shiftable. Suppose first that t has type L , so that $t^+ = \widehat{q}_x r s \neq 0$ while $rs\widehat{q}_z = 0$. Since $\widehat{q}_x r s \neq 0$, we have $\widehat{q}_x r \neq 0$, so r is shiftable of type L or B and $r^+ s = \widehat{q}_x r s = t^+$.

- If r has type L , then $r\widehat{q}_y = 0$. If s is shiftable, its successor s^+ equals either $\widehat{q}_y s$ or $s\widehat{q}_z$; in either case rs^+ equals $r\widehat{q}_y s = 0$ or $rs\widehat{q}_z = 0$. Thus only the first substitution $r^+ s = t^+$ is nonzero, yielding $d_t = d_r$.
- If r has type B , then $r\widehat{q}_y = \widehat{q}_x r$, so $r\widehat{q}_y s = \widehat{q}_x r s = t^+ \neq 0$. Hence $\widehat{q}_y s \neq 0$, which means s is shiftable with a nonzero left action, while $s\widehat{q}_z$ cannot equal $\widehat{q}_y s$ because $rs\widehat{q}_z = 0 \neq r\widehat{q}_y s$. Thus s has type L and $rs^+ = r\widehat{q}_y s = t^+$, giving $d_t = d_r + d_s$.

The two rows for t of type R follow symmetrically by passing to the opposite algebra.

Finally, suppose that t has type B, so that $t^+ = \widehat{q}_x r s = r s \widehat{q}_z \neq 0$. Then $\widehat{q}_x r \neq 0$ and $s \widehat{q}_z \neq 0$, so $r \in \{L, B\}$ and $s \in \{R, B\}$. The mixed pairs (L, B) and (B, R) cannot occur: for (L, B) we would have $t^+ = r s \widehat{q}_z = r \widehat{q}_y s = 0$, and for (B, R) we would have $t^+ = \widehat{q}_x r s = r \widehat{q}_y s = 0$. In the remaining cases (L, R) and (B, B), both $r^+ s = \widehat{q}_x r s$ and $r s^+ = r s \widehat{q}_z$ equal t^+ , which gives $d_t = d_r + d_s$. $\square$

We now turn to factorizations of a primitive local generator q_x through other vertices. Any factorization $q_x = fh$ through a vertex $y \neq x$ has q_x as its output, and when $\ell(q_x) = \chi$ is shiftable (so $q_x^2 \neq 0$), Lemma 8.19 yields a linear relation linking d_f , d_h , and d_{q_x} . We first describe the corner subcategories formed by the vertices through which such factorizations pass.

Definition 8.20. Using the local radical generators q_x of Setting 8.4 (independently of the fixed character χ):

- (1) A *primitive reverse* of a ray $f: x \rightarrow y$ is a ray $h: y \rightarrow x$ satisfying $fh = q_x \neq 0$ and $hf = q_y \neq 0$ (neither f nor h is required to be shiftable).
- (2) A *return partner* of x is a vertex $y \neq x$ for which there exists a generating ray $f: x \rightarrow y$ that admits a primitive reverse and satisfies $q_x f = f q_y \neq 0$.
- (3) Writing V_x for the set consisting of x together with all of its return partners, the *primitive return corner* S_x is the full corner subcategory of A_0 on V_x . We write $y \in S_x$ as shorthand for $y \in V_x$.

Lemma 8.21. *Work in Setting 8.4, with primitive reverses and return corners as in Definition 8.20.*

- (1) *Let $f: x \rightarrow y$ ($x \neq y$) be a generating ray whose corner bimodule $A_0(x, y)$ is cyclic at both endpoints with $q_x f = f q_y \neq 0$, and suppose that $\ell(q_x)$ is primitive. If $f A_0(y, x) \neq 0$, then the generating ray h of $A_0(y, x)$ satisfies*

$$fh = q_x, \quad hf = q_y.$$

- (2) *Let x be a vertex of primitive local weight. The generating rays $f: x \rightarrow y$ ($y \neq x$) that admit a primitive reverse and satisfy $q_x f = f q_y \neq 0$ are linearly ordered by factorization, and the primitive return corner S_x is a basic cyclic Nakayama algebra.*
- (3) *As x ranges over all vertices of primitive local weight with $N_x \geq 3$, the vertex sets V_x are pairwise disjoint or equal, and every factorization $q_x = uv$ into nonidentity rays $u: x \rightarrow y$ and $v: y \rightarrow x$ has its intermediate vertex y in S_x .*

Proof. (1) Because $A_0(x, y) = f R_y$ is cyclic at y , multiplying the generating ray h of $A_0(y, x)$ by powers of q_y or q_x simply multiplies fh by powers of q_x . The assumption $f A_0(y, x) \neq 0$ therefore ensures that $fh = q_x^a \neq 0$ for some least return exponent $a \geq 1$. If $a \geq 2$, then the faithful projective support at x contains the four-point subposet with lower elements q_x, f and upper elements $q_x^a, q_x f$. All four comparisons from $\{q_x, f\}$ to $\{q_x^a, q_x f\}$ hold because $q_x f = f q_y$ and $fh = q_x^a$. On the other hand, q_x and f are incomparable (since f is generating and $a \geq 2$ is minimal), and by left cancellation q_x^a and $q_x f$ are likewise incomparable, while neither upper element can precede a lower element by degree. Thus $\{q_x, f, q_x^a, q_x f\}$ forms a forbidden four-point crown, a contradiction. Hence $a = 1$ and $fh = q_x$. Since $f(hf) = (fh)f = q_x f = f q_y \neq 0$, we also have $hf = q_y$. (Note that h had to be generating from the outset, as any proper endpoint factor of h would strip a power of q_x or q_y from $fh = q_x$, factoring an identity morphism through a distinct vertex.)

(2) By Definition 8.20 and Lemma 8.18, the targets of the specified generating rays $f: x \rightarrow y$ are precisely the return partners of x , and each associated corner bimodule $A_0(x, y)$ is cyclic at both endpoints. Given two such rays $f_i: x \rightarrow y_i$ and $f_j: x \rightarrow y_j$, their primitive reverses show that both f_i and f_j precede both $q_x f_i$ and $q_x f_j$ in the faithful

projective support at x . By left cancellation of q_x , f_i and f_j are comparable if and only if $q_x f_i$ and $q_x f_j$ are comparable; moreover, no upper element $q_x f_i$ can precede a lower element f_j because a generating ray f_j cannot factor through q_x (Lemma 8.14). If f_i and f_j were incomparable, the four rays $\{f_i, f_j, q_x f_i, q_x f_j\}$ would therefore form a four-point crown. Consequently, these outgoing generating rays form a chain under factorization. Since their categorical degrees lie strictly between 0 and $\deg_{\mathcal{J}}(q_x)$ and no two distinct rays from x to the same partner have the same weight, the return partners of x may be listed in strictly increasing factorization order as $y_1, \dots, y_t$.

For $1 \leq i < j \leq t$, let $h_{ij}: y_i \rightarrow y_j$ be the ray satisfying $f_i h_{ij} = f_j$, and let $h_i: y_i \rightarrow x$ be the primitive reverse of f_i . Then

$$f_i h_{ij} = f_j, \quad h_{ij} h_j = h_i, \quad h_{ji} := h_j f_i: y_j \rightarrow y_i.$$

Cancellation and Lemma 8.14 imply that all h_{ij} and h_{ji} are generating rays and are mutual primitive reverses. The full corner S_x is therefore generated by the oriented cycle of ordinary arrows inside S_x ,

$$x \rightarrow y_1 \rightarrow \dots \rightarrow y_t \rightarrow x,$$

in which every simple path is a generating ray, one full turn around the cycle at any vertex is its local radical generator, and the only relations are monomial nilpotence relations. Thus S_x is a cyclic Nakayama algebra.

(3) Let a be any vertex of primitive local weight $\xi = \ell(q_a)$, and let $b, c \in V_a \setminus \{a\}$ be two distinct return partners of a , reached by generating rays $u: a \rightarrow b$ and $v: a \rightarrow c$ with primitive reverses $\bar{u}, \bar{v}$. By the chain property in (2), we may assume after swapping b and c if necessary that $v = ut$ for a ray $t: b \rightarrow c$. Setting $h = \bar{v}u: c \rightarrow b$, we have

$$ht = \bar{v}ut = \bar{v}v = q_c, \quad u(th) = (ut)\bar{v}u = v\bar{v}u = q_a u \neq 0.$$

Since q_a, q_b , and q_c all have weight ξ , the nonzero ray $th \in R_b$ has weight ξ and therefore equals q_b . Neither t nor h can be a proper ray, since pulling out a local generator from $th = q_b$ or $ht = q_c$ would factor 1_b or 1_c through a distinct vertex. By associativity,

$$q_b t = (th)t = t(ht) = tq_c, \quad q_c h = (ht)h = h(th) = hq_b.$$

If either $N_b \geq 3$ or $N_c \geq 3$, then $q_b^2 = tq_c h$ or $q_c^2 = hq_b t$ is nonzero, which forces $q_b t = tq_c \neq 0$ and $q_c h = hq_b \neq 0$; hence b and c are mutual return partners whenever $\max(N_b, N_c) \geq 3$. Likewise, if $N_a \geq 3$, then $u\bar{u}q_a = q_a^2 \neq 0$ forces $q_b \bar{u} = \bar{u}q_a \neq 0$, so $a \in V_b$; if both $N_a \geq 3$ and $N_b \geq 3$, applying the preceding partner comparison at a and at b shows that $V_a = V_b$.

Now suppose that $N_x \geq 3$ and $N_{x'} \geq 3$, and let $v \in V_x \cap V_{x'}$. Then v has the same primitive local weight as x and x' . If $N_v \geq 3$, the equality just proved gives $V_x = V_v = V_{x'}$. If instead $N_v = 2$, then since $N_x \geq 3$ and $N_{x'} \geq 3$, the reverse rays from v to x and to x' show as above that $x, x' \in V_v$. Applying the partner comparison at v to its two partners x and x' (which satisfy $N_x, N_{x'} \geq 3$) shows that x and x' are mutual return partners, and hence $V_x = V_{x'}$. Thus the sets V_x ($N_x \geq 3$) are pairwise disjoint or equal.

Finally, suppose that $q_x = uv$ is a factorization of q_x into nonidentity rays $u: x \rightarrow y$ and $v: y \rightarrow x$ at a vertex x of primitive local weight ξ with $N_x \geq 3$ (so $q_x^2 \neq 0$). Then $u(vu) = (uv)u = q_x u \neq 0$ (since $q_x^2 = q_x uv \neq 0$), so $vu = q_y^a \neq 0$ for some $a \geq 1$. Comparing weights gives $\xi = a\ell(q_y)$, so primitivity of ξ forces $a = 1$ and $vu = q_y$. Because $q_x^2 = uvuv \neq 0$, we have $q_x u = uq_y \neq 0$ and $vq_x = q_y v \neq 0$, and both u and v are generating rays because $uv = q_x$ cannot contain an extra local radical factor. Thus $y \in V_x$, so the factorization passes through S_x . $\square$

Lemma 8.22. *In Setting 8.4, let x be a vertex of primitive local weight with $N_x \geq 3$. Then the generating rays $f: x \rightarrow y$ satisfying $q_x f \neq 0$ and $y \notin S_x$ are linearly ordered by factorization. The dual statement holds for incoming generating rays $g: y \rightarrow x$ with $gq_x \neq 0$ and $y \notin S_x$.*

Proof. Suppose for a contradiction that $f: x \rightarrow y$ and $h: x \rightarrow z$ are two incomparable generating rays with $q_x f \neq 0$, $q_x h \neq 0$, and $y, z \notin S_x$. In the faithful projective support at x , consider the five rays

$$1_x, \quad q_x, \quad q_x^2, \quad q_x f, \quad q_x h.$$

Clearly $1_x < q_x$, and q_x precedes each of q_x^2 , $q_x f$, and $q_x h$. By left cancellation of q_x , the elements $q_x f$ and $q_x h$ are incomparable because f and h are incomparable. Neither $q_x^2 \leq q_x f$ nor $q_x f \leq q_x^2$ can hold: the first inequality would cancel to $q_x \leq f$, contradicting Lemma 8.14, while the second would cancel to $fa = q_x$ for some ray $a: y \rightarrow x$, placing y in S_x by Lemma 8.21(3). The same holds for $q_x h$, and degree considerations rule out any remaining comparisons. The five points therefore form a four-leaf star with center q_x and leaves $1_x, q_x^2, q_x f, q_x h$, which is impossible by Lemma 8.8. $\square$

8.5. First and last arrows of generating rays

To construct our integral coefficient lift in Section 9, we will define the coefficient d_f of a generating one-sided ray f in terms of the first (or last) ordinary arrow along a path representing f . For this prescription to be well defined, all ordinary paths representing f must share the same initial (or terminal) arrow. Constraints on multiple paths representing the same ray play a central role in [2, 5]; we establish the exact uniqueness property we need in the next proposition.

Proposition 8.23. *In Settings 8.4 and 8.16, all ordinary paths representing a generating left-sided ray begin with the same ordinary arrow. Dually, all ordinary paths representing a generating right-sided ray end with the same ordinary arrow.*

The proof of Proposition 8.23 is given in Appendix C.1, where we show that two distinct initial arrows for a generating left-sided ray would allow one to construct a bimodule retraction onto the incidence category of a representation-infinite tree.

The Leibniz equations of Lemma 8.19 also constrain products rs of generating rays whose output rs is a *proper* shiftable ray. A key source of such proper products is a path that leaves the primitive return corner S_x and returns to x with value $\hat{q}_x^2$ at a vertex where $\hat{q}_x^3 \neq 0$.

Definition 8.24. In Settings 8.4 and 8.16, a *nondeep external return* at x is a pair of generating rays $e: x \rightarrow y$ of type L and $l: y \rightarrow x$ of type R, with $y \notin S_x$, such that

$$el = \hat{q}_x^2 \quad \text{and} \quad \hat{q}_x^3 \neq 0.$$

The adjective *nondeep* reflects the condition $\hat{q}_x^3 \neq 0$, which means that the return value $el = \hat{q}_x^2$ is shiftable (with successor $\hat{q}_x^3$) rather than lying in the socle of R_x .

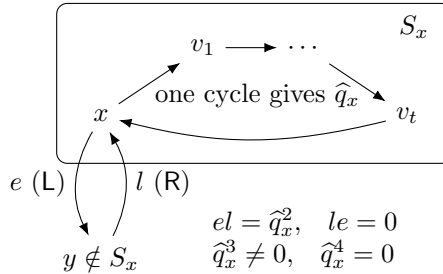


FIGURE 1. A primitive return corner S_x together with a nondeep external return (e, l) through $y \notin S_x$. The cycle arrows depict the Nakayama generator cycle of S_x , whereas e and l are generating rays that may be composite paths in the quiver of A_0 . The relations $le = 0$ and $\hat{q}_x^4 = 0$ (so $N_x = 4$) are established in Lemma C.2.

In the configuration of Figure 1, both e and l are controlled at x : we have $\widehat{q}_x e = e^+ \neq 0$ and $l \widehat{q}_x = l^+ \neq 0$, whereas $e \widehat{q}_y = 0$ and $\widehat{q}_y l = 0$. Because $\widehat{q}_x = q_x \neq 0$, Lemma C.2 in Appendix C.2 shows that $le = 0$ and $N_x = 4$.

The next proposition, proved in Appendix C.2, catalogs all proper shiftable products of generating rays and describes how nondeep external returns interact with other rays at x .

Proposition 8.25. *Work in Settings 8.4 and 8.16, with endpoint layers as in Definition 8.5, endpoint types as in Definition 8.17, and primitive return corners S_x as in Definition 8.20.*

- (1) *If a generating ray $f: x \rightarrow y$ has type L (respectively, R), then $\dim A_0(x, y) \leq 3$, with equality implying $N_x = 3$ (respectively, $N_y = 3$). Moreover, whenever a nonzero product rs and its first factor r both have type L, they have the same endpoint layer; the dual statement holds when rs and s both have type R.*
- (2) *If $r: x \rightarrow y$ of type B and $s: y \rightarrow z$ of type L are generating rays whose product rs is proper and shiftable, then rs has endpoint layer 1, the source x has local length $N_x = 3$ ($\widehat{q}_x^2 \neq 0$ and $\widehat{q}_x^3 = 0$), and r admits a primitive reverse (so $y \in S_x$). The dual statement holds for generating rays of types (R, B).*
- (3) *Let $r: x \rightarrow y$ of type L and $s: y \rightarrow z$ of type R be generating rays whose product rs is proper and shiftable. Then rs has endpoint layer 2:*
 - (a) *If $x = z$, so that $rs = \widehat{q}_x^2 \in R_x$, then (r, s) is a nondeep external return at x (Definition 8.24).*
 - (b) *If $x \neq z$, write $rs = \widehat{q}_x^2 f = f \widehat{q}_z^2$ where $f: x \rightarrow z$ is the generating ray of $A_0(x, z)$. Then at least one of the factorizations*

$$r = fa, \quad as = \widehat{q}_z^2 \quad \text{or} \quad s = bf, \quad rb = \widehat{q}_x^2$$

holds, where (a, s) (respectively, (r, b)) is a nondeep external return at z (respectively, at x) and a (respectively, b) is a generating one-sided ray.

- (4) *If x admits a nondeep external return, then there is no generating two-sided ray $f: x \rightarrow z$ with $z \notin S_x$ and $\widehat{q}_x f \neq 0$, and dually no generating two-sided ray $g: z \rightarrow x$ with $z \notin S_x$ and $g \widehat{q}_x \neq 0$.*

With these local restrictions in hand, we are ready to lift the modulo-two coefficient equations of Lemma 8.19 to the integers: the two-sided rays are governed by a smaller centralizer table to which the induction hypothesis applies, while Proposition 8.23 and Proposition 8.25 allow us to extend the resulting integral lift consistently to all one-sided rays.

9. INTEGRAL LIFTING VIA CENTRALIZERS AND FOUR-POINT CROWNS

The local restrictions established in Section 8 govern how endpoint layers and endpoint types behave under nonzero compositions of rays. Continuing in Settings 8.4 and 8.16 for a fixed positive primitive character $\chi \in L$, we now use those restrictions to lift modulo-two solutions of the nonzero-product Leibniz equations to the integers. The equations involving only two-sided rays turn out to be the additive-weight relations of a strictly smaller weighted multiplication table obtained from a centralizer quotient, to which our inductive hypothesis applies. Once the two-sided coefficients are lifted, the remaining equations for generating one-sided rays reduce to a linear system on a bipartite graph, and the absence of four-point crowns ensures that the right-hand sides are constant on each connected component.

9.1. The coefficient equations and their two-sided part

For each shiftable nonidentity ray r , let d_r denote its scalar coefficient in a normalized one-cochain $\delta(r) = d_r r^+$ of weight χ , and set $d_r = 0$ when r is an identity or an unshiftable ray. For any nonzero product $rs = t$ of nonidentity rays whose output t is shiftable, the

Leibniz identity $\delta(rs) = \delta(r)s + r\delta(s)$ takes the form

$$d_t = \epsilon_L d_r + \epsilon_R d_s, \quad \epsilon_L = \begin{cases} 1, & \text{if } r^+s = t^+, \\ 0, & \text{otherwise,} \end{cases} \quad \epsilon_R = \begin{cases} 1, & \text{if } rs^+ = t^+, \\ 0, & \text{otherwise.} \end{cases} \quad (18)$$

By Lemma 8.19, if $t = rs \neq 0$ is unshiftable, then $\epsilon_L = \epsilon_R = 0$ and the Leibniz condition on (r, s) holds trivially as $0 = 0$. No condition is imposed on pairs (r, s) with $rs = 0$.

Theorem 9.1. *Let $A_0 = kP$ and χ satisfy Settings 8.4 and 8.16. Assume that Proposition 8.2 holds for every weighted multiplication table whose linearization is representation-finite of dimension strictly less than $\dim A_0$. Then every modulo-two solution $(\bar{d}_r)$ of the nonzero-product Leibniz equations (18) lifts to an integer solution (d_r) , and in particular reduces to a solution over $\mathbb{Z}/4$.*

The proof of Theorem 9.1 proceeds in three steps: first we isolate a smaller weighted multiplication table that governs the coefficients of all two-sided rays; next we express the coefficients of generating one-sided rays in terms of their initial or terminal one-sided ordinary arrows and solve the resulting bipartite system; finally, in Subsection 9.4, we verify that our formulas automatically satisfy every remaining equation in (18).

To construct the algebra governing the two-sided rays, set $\hat{q} = \sum_x \hat{q}_x \in A_0$ and define

$$C_\chi = \text{Cent}_{A_0}(\hat{q}), \quad \mathfrak{a}_\chi = \{c \in C_\chi : \hat{q}c = 0\}, \quad B_\chi = C_\chi / \mathfrak{a}_\chi.$$

Lemma 9.2. *The quotient algebra B_χ is basic and representation-finite. The images of the supporting vertex idempotents 1_x ($\hat{q}_x \neq 0$) and the two-sided rays of A_0 form a multiplicative basis of B_χ , defining a weighted multiplication table Q_χ with the weights and positive degree inherited from P so that $B_\chi = kQ_\chi$. Moreover, if $B_\chi \neq 0$, then Q_χ has strictly fewer nonzero elements than P .*

Proof. For any ray $r: x \rightarrow y$, the commutator $\hat{q}r - r\hat{q} = \hat{q}_x r - r\hat{q}_y$ vanishes if and only if r is an identity, an unshiftable ray, or a two-sided ray (for a one-sided ray, $\hat{q}_x r - r\hat{q}_y = \pm r^+ \neq 0$, and distinct rays in $A_0(x, y)$ have distinct weights). Thus C_χ has a basis consisting of all vertex idempotents 1_x , all two-sided rays, and all unshiftable rays. Among these basis elements, multiplication by $\hat{q}$ annihilates precisely the non-supporting idempotents ($\hat{q}_x = 0$) and the unshiftable rays, so they span the ideal $\mathfrak{a}_\chi$. Consequently, the supporting idempotents and the two-sided rays project to a basis of B_χ .

Let $\pi: A_0 \rightarrow B_\chi$ be the linear projection onto this basis that sends all one-sided rays, unshiftable rays, and non-supporting idempotents to zero. We claim that π is a C_χ -bimodule morphism extending the quotient map $C_\chi \rightarrow B_\chi$. Indeed, for any composable rays r and s , if their product rs is a retained nonidentity ray in B_χ , then rs has type B; by the table (17) of Lemma 8.19, the pair (r, s) must have types (B, B) or (L, R). If at least one factor belongs to C_χ , the case (L, R) cannot occur because C_χ contains no one-sided rays. Hence no discarded ray can multiply an element of C_χ to produce a retained ray in B_χ . Conversely, if both factors are retained in B_χ but their product in A_0 is unshiftable (or zero), that product lies in $\mathfrak{a}_\chi$ and vanishes on both sides of the bimodule identity. Since identity factors manifestly respect π , it follows that π is a C_χ -bimodule retraction of the quotient $C_\chi \rightarrow B_\chi$ through $C_\chi \hookrightarrow A_0$. By Lemma 4.6, B_χ is therefore representation-finite.

Because the retained basis elements of B_χ inherit associative multiplication, identities, additive weights, endpoint-and-weight separation, and positive categorical degree from P , they form a weighted multiplication table Q_χ in the sense of Definition 8.1 with $B_\chi = kQ_\chi$. Its positive-degree ideal is nilpotent with semisimple quotient $\prod_{\hat{q}_x \neq 0} k$, so B_χ is basic. Finally, if $B_\chi \neq 0$, then $\hat{q} \neq 0$, and the highest nonzero power $\hat{q}^d$ of the nilpotent element $\hat{q}$ is unshiftable and hence lies in $\mathfrak{a}_\chi \setminus \{0\}$. Therefore

$$\dim B_\chi < \dim C_\chi \leq \dim A_0,$$

so Q_χ has strictly fewer nonzero elements than P . $\square$

Write $W_\chi = W(Q_\chi)$ for the scalar weight group of Q_χ . By definition, W_χ is the abelian group generated by symbols $[r]$ for two-sided rays r , subject to the relations $[rs] = [r] + [s]$ whenever r , s , and rs are all two-sided. Comparing with the (B, B) row of (17), we see that these relations are precisely the equations in (18) in which both inputs and the output are two-sided. Any modulo-two solution of (18) therefore restricts on two-sided rays to a homomorphism $W_\chi \rightarrow \mathbb{Z}/2$, which lifts to a homomorphism $W_\chi \rightarrow \mathbb{Z}$ by Lemma 9.2 and the inductive hypothesis of Theorem 9.1.

9.2. Coefficients of generating one-sided rays

We next determine how the coefficients of generating one-sided rays depend on the coefficients of two-sided rays and one-sided ordinary arrows. The following lemma constrains the two-sided prefixes and suffixes that can appear along an ordinary path representing a one-sided ray.

Lemma 9.3. *In Settings 8.4 and 8.16, let b be a two-sided ray. If s is a ray of type L such that bs is nonzero and shiftable, then every ordinary path representing b consists entirely of two-sided ordinary arrows. Dually, if r has type R and rb is nonzero and shiftable, then every ordinary path representing b consists entirely of two-sided ordinary arrows.*

Proof. Suppose for a contradiction that b admits an ordinary path containing at least one arrow that is not two-sided, and read the path from left to right up to the first arrow a that is not of type B. Because $\hat{q}b \neq 0$, every prefix of the path has a nonzero left action by $\hat{q}$. Along the two-sided arrows preceding a , this left action commutes across each arrow to reach the source of a ; hence $\hat{q}a \neq 0$, which means that a has type L. Consequently, the prefix u ending at a also has type L ($\hat{q}u \neq 0$ and $u\hat{q} = 0$). Since b itself is two-sided, the remaining suffix v of the path must be a nonidentity ray, and the factorization $b = uv$ has output of type B and first factor u of type L. By the table (17), the second factor v must have type R.

Now consider the product $bs = uvs$. Because bs is shiftable and b is two-sided, we have

$$\hat{q}bs = b\hat{q}s = u(v\hat{q}s) \neq 0,$$

so in particular $vs \neq 0$ and $v\hat{q}s = v^+s = vs^+ \neq 0$. However, v has type R and s has type L, so both outer actions on vs vanish: $\hat{q}(vs) = (\hat{q}v)s = 0$ and $(vs)\hat{q} = v(s\hat{q}) = 0$. Thus vs cannot have a successor of weight $\ell(vs) + \chi$, contradicting $v\hat{q}s \neq 0$. The statement for rb follows by passing to the opposite algebra. $\square$

Because every nonidentity ray of A_0 has positive categorical degree, repeatedly factoring any decomposable ray into two rays of smaller degree shows that every nonidentity ray can be represented by an ordinary path. We now introduce formal symbols X_a for each ordinary arrow a of type L and Y_b for each ordinary arrow b of type R, and work in the abelian group formed by the direct sum of W_χ with the free abelian group on these symbols.

Lemma 9.4. *Let r be a generating ray of type L, choose an ordinary path representing r , let p be its maximal two-sided prefix, and let a be the first arrow of the path that has type L. Dually, for a generating ray s of type R, choose an ordinary path representing s , let b be its last arrow of type R, and let p' be its maximal two-sided suffix. Then the formal expressions*

$$d_r = [p] + X_a, \quad d_s = Y_b + [p'] \quad (19)$$

(where an empty prefix or suffix has class $0 \in W_\chi$) do not depend on the choice of ordinary paths representing r and s , and they satisfy every Leibniz equation in (18) whose output is a generating one-sided ray. In particular, the first L-arrow a along an ordinary path for r , and the last R-arrow b along an ordinary path for s , are independent of the chosen path.

Proof. We prove the statements for a generating left-sided ray r by induction on its categorical degree $\deg_{\mathcal{J}}(r)$. Any ordinary path representing r must contain at least one

arrow that is not two-sided, since a shiftable product of two-sided rays is again two-sided. Because $\widehat{q}r \neq 0$, every prefix of the path has a nonzero left action, and commuting $\widehat{q}$ across the initial two-sided arrows shows that the first non-two-sided arrow a has $\widehat{q}a \neq 0$ and hence is of type L. Moreover, every prefix extending past a also has type L: if some longer proper prefix u had type B, then in the factorization $r = uv$ the suffix v would have type L by (17), and Lemma 9.3 applied to $uv = r$ would force every ordinary path for u to consist solely of two-sided arrows, contradicting the presence of a in u .

By Proposition 8.23, all ordinary paths representing r begin with the same initial ordinary arrow u . If u has type L, then p is empty, $a = u$ is the first arrow of every path for r , and every path yields $d_r = X_u$. If instead u has type B, write $r = uv$. The suffix rays v obtained from any two ordinary paths for r share the same endpoints and the same weight $\ell(r) - \ell(u)$, so they are the same ray v . By Lemma 8.14, v is a generating ray, and by (17) it has type L. Since $\deg_{\mathcal{J}}(v) < \deg_{\mathcal{J}}(r)$, the induction hypothesis ensures that the first L-arrow a of v and the sum $[p_v] + X_a$ are independent of the path chosen for v . Prepending u gives $[p] = [u] + [p_v]$ in W_χ , so $[p] + X_a$ is independent of the path for r . The statements for a generating right-sided ray s follow by duality.

Finally, for any nonzero product $rs = t$ whose output t is generating and one-sided (say of type L), we concatenate an ordinary path for r with an ordinary path for s to obtain an ordinary path for t . If r has type L, then r already contains the first L-arrow a of t and shares the same two-sided prefix p , so $d_t = [p] + X_a = d_r$. If r has type B and s has type L, then by Lemma 9.3 every arrow in the path of r is two-sided, so the two-sided prefix of $t = rs$ is rp , which satisfies $[rp] = [r] + [p]$ in W_χ ; hence $d_t = [r] + [p] + X_a = d_r + d_s$. Both cases match (17), and the right-sided case is dual. $\square$

Consider now a nonzero product $rs = t$ in which r has type L, s has type R, and the output t is a *generating* two-sided ray. Concatenating ordinary paths for r and s yields an ordinary path for t . Writing $r = par'$ and $s = s'bp'$ where p is the two-sided prefix of r , a is its first L-arrow, b is the last R-arrow of s , and p' is its two-sided suffix, we have $t = pfp'$, where the middle subsegment $f = ar's'b$ is a generating ray by Lemma 8.14 and is two-sided because the outer actions on t commute across p and p' . Since $[t] = [p] + [f] + [p']$ in W_χ while $d_r = [p] + X_a$ and $d_s = Y_b + [p']$, subtracting $[p] + [p']$ reduces the Leibniz equation $d_r + d_s = d_t$ to

$$X_a + Y_b = [f], \quad (20)$$

where f is a generating two-sided ray represented by an ordinary path that begins with a and ends with b .

9.3. Four-point crowns and equal right-hand sides

Because the same variable X_a or Y_b can appear in several equations of the form (20), we must check that these equations are mutually consistent over $\mathbb{Z}$. Excluding four-point crowns shows that any two equations sharing a variable have the exact same right-hand side $[f]$.

Lemma 9.5. *Any two two-sided rays that admit ordinary paths beginning with the same L-arrow a are comparable under right factorization. Dually, any two two-sided rays that admit ordinary paths ending with the same R-arrow b are comparable under left factorization. Consequently, any two equations of the form (20) (with generating two-sided outputs) that share a variable X_a or Y_b have the same output ray f .*

Proof. Let f and t be two-sided rays admitting ordinary paths that start with the same L-arrow $a: x \rightarrow y$. Write $f = av$ and $t = aw$. Because f is two-sided, we have $\widehat{q}_x f = f \widehat{q}_z = a(v \widehat{q}_z) \neq 0$, so a precedes $\widehat{q}_x f$ in the faithful projective support at x , and likewise a precedes $\widehat{q}_x t$. Since $\widehat{q}_x$ obviously precedes both $\widehat{q}_x f$ and $\widehat{q}_x t$, all four upward comparisons between the lower pair $\{\widehat{q}_x, a\}$ and the upper pair $\{\widehat{q}_x f, \widehat{q}_x t\}$ hold in the faithful projective support at x .

Notice that $\hat{q}_x$ and a are incomparable: $\hat{q}_x \leq a$ would factor the ordinary arrow a through $\hat{q}_x$, while $a \leq \hat{q}_x$ would give $ah = \hat{q}_x$ for some ray $h: y \rightarrow x$, forcing $a(ha) = \hat{q}_xa \neq 0$ and hence $ha = \hat{q}_y$ (by primitivity of χ), which would make $a\hat{q}_y = \hat{q}_xa \neq 0$ two-sided rather than left-sided. Nor can either upper element precede a (which would again factor a through $\hat{q}_x$) or $\hat{q}_x$ (by positive degree). Therefore, if $\hat{q}_xf$ and $\hat{q}_xt$ were incomparable, the four rays $\{\hat{q}_x, a, \hat{q}_xf, \hat{q}_xt\}$ would form a forbidden four-point crown. Thus $\hat{q}_xf$ and $\hat{q}_xt$ are comparable, and left cancellation of $\hat{q}_x$ shows that f and t are comparable under right factorization.

Suppose now that f and t are *generating* two-sided outputs of two equations (20) that share the variable X_a . By what we have just shown, f and t are comparable; assume without loss of generality that $t = fh$ for a ray h . If h were a nonidentity ray, then writing $f = av$ and $t = aw$ (where $v = r's'b$ and w are generating rays of type R by Lemma 8.14 and (17)) and canceling a would give $vh = w$. Because $w = vh$ is shiftable, v has type R, and $t = fh$ is two-sided (so h has type B), the dual of Lemma 9.3 would imply that every ordinary path representing h consists of two-sided arrows. Concatenating a path for v with a path for h would then produce an ordinary path for w ending in a two-sided arrow, whereas w also has an ordinary path ending in its terminal R-arrow from (20). This contradicts the dual of Proposition 8.23, which requires all ordinary paths representing the generating right-sided ray w to share the same final ordinary arrow. Thus h is an identity and $f = t$. The case of a shared variable Y_b follows dually. $\square$

We also need to incorporate the equations coming from nondeep external returns (Definition 8.24).

Lemma 9.6. *Let $el = \hat{q}_x^2$ (with $\hat{q}_x^3 \neq 0$) be a nondeep external return at x , where $e: x \rightarrow y$ and $l: y \rightarrow x$ are generating rays of types L and R, respectively. Then every ordinary path representing e begins with a single left-sided arrow a , and every ordinary path representing l ends with a single right-sided arrow b (so both the two-sided prefix of e and the two-sided suffix of l are empty).*

Proof. The first arrow $u: x \rightarrow v$ of an ordinary path for e satisfies $\hat{q}_xu \neq 0$ (since $\hat{q}_xe \neq 0$) and therefore has type L or B. Suppose for a contradiction that u has type B, and write $e = uc$. Because e and $e^+ = \hat{q}_xe$ are distinct rays in $A_0(x, y)$, we have $\dim A_0(x, y) \geq 2$, so Lemma 8.14 implies that $x \neq v$ and that both u and c are generating rays. Since u is generating and two-sided with $\hat{q}_xu \neq 0$ and χ is primitive, Lemma 8.18 shows that $A_0(x, v)$ is cyclic at both endpoints. Furthermore, $u(cl) = el = \hat{q}_x^2 \neq 0$, so $uA_0(v, x) \neq 0$. Lemma 8.21(1) therefore provides a generating primitive reverse $g: v \rightarrow x$ satisfying $ug = \hat{q}_x$ and $gu = \hat{q}_v$.

Comparing $u(cl) = \hat{q}_x^2 = u(\hat{q}_vg) \neq 0$ shows that cl and $\hat{q}_vg$ have the same endpoints and weight, so $cl = \hat{q}_vg$. Its successor $\hat{q}_v^2g$ is nonzero because $u(\hat{q}_v^2g) = \hat{q}_x^3 \neq 0$. On the other hand, applying (17) to $e = uc$ shows that c has type L, while l has type R by hypothesis. Thus $cl = \hat{q}_vg$ is a proper shiftable product of generating rays of types (L, R) whose endpoint layer is 1, contradicting Proposition 8.25(3) (which forces any such product to have endpoint layer 2).

Consequently, the initial arrow of every ordinary path for e has type L, and by Proposition 8.23 (or Lemma 9.4) this initial arrow a is independent of the path. The statement for l follows by passing to the opposite algebra. $\square$

By Lemma 9.6, if (e, l) is a nondeep external return at x with initial arrow a of e and terminal arrow b of l , then $d_e = X_a$ and $d_l = Y_b$. Since $\hat{q}_x$ is two-sided and $\hat{q}_x^3 \neq 0$, we have $[\hat{q}_x^2] = 2[\hat{q}_x]$ in W_χ , so the Leibniz equation $d_e + d_l = d_{\hat{q}_x^2}$ for $el = \hat{q}_x^2$ becomes

$$X_a + Y_b = 2[\hat{q}_x]. \quad (21)$$

Moreover, neither X_a nor Y_b can appear in any generating-output equation (20). Indeed, if a generating two-sided ray f had an ordinary path starting with a , Lemma 9.5 would make f and $\hat{q}_x^2$ comparable under right factorization; however, $\hat{q}_x^2 h = f$ would contradict the fact that f is generating (Lemma 8.14), while $fh = \hat{q}_x^2$ would give f (and hence its initial arrow a) a nonzero return to x , forcing a to be two-sided by Lemma 8.18 and Lemma 8.21(1), a contradiction. The dual argument excludes Y_b from (20). Finally, if two nondeep-return equations (21) share X_a , their base vertex x is the source of a , and if they share Y_b , their base vertex x is the target of b ; in either case their right-hand sides $2[\hat{q}_x]$ coincide.

We now form the bipartite graph whose vertices are the formal variables X_a and Y_b and whose edges correspond to the equations (20) and (21), labeled by their right-hand sides in W_χ . By Lemma 9.5 and the preceding paragraph, any two edges incident to a common vertex carry the same label $\omega \in W_\chi$, so the edge labels are constant on each connected component of the bipartite graph. On a connected component with constant edge label ω , the system of equations $X_a + Y_b = \omega$ has the general solution

$$X_a = \omega + z, \quad Y_b = -z,$$

parametrized by a single free variable z per connected component (with an isolated vertex likewise contributing one free variable). Letting T denote the quotient of the direct sum of W_χ and the free abelian group on all symbols X_a, Y_b by the relations (20) and (21), and writing m for the number of connected components of the bipartite graph, we obtain an isomorphism

$$T \cong W_\chi \oplus \mathbb{Z}^m. \quad (22)$$

In particular, even when $\omega = 2[\hat{q}_x]$ in (21), the equation $X_a + Y_b = 2[\hat{q}_x]$ imposes no modulo-two divisibility obstruction on lifting X_a and Y_b , because each variable appears with coefficient 1 and the right-hand side is constant across each connected component.

9.4. Verification of all nonzero-product equations

We now complete the proof of Theorem 9.1 by assigning coefficients in the group $T \cong W_\chi \oplus \mathbb{Z}^m$ of (22) to all rays of A_0 and verifying that they satisfy every nonzero-product Leibniz equation (18).

Proof of Theorem 9.1. We first define a universal solution with values in the group T of (22):

- For every two-sided ray b , set $d_b = [b] \in W_\chi \subseteq T$.
- For every generating one-sided ray r (of type L) or s (of type R), define d_r and d_s in T by the path formulas (19).
- For every proper shiftable one-sided ray, write it as $\hat{q}_x^j r$ or $s \hat{q}_z^j$ with r, s generating and $j \geq 1$, and set

$$d_{\hat{q}_x^j r} = j[\hat{q}_x] + d_r, \quad d_{s \hat{q}_z^j} = d_s + j[\hat{q}_z] \quad (23)$$

(note that $\hat{q}_x^{j+1} r \neq 0$ implies $\hat{q}_x^2 \neq 0$, so $\hat{q}_x$ is two-sided and $[\hat{q}_x] \in W_\chi$ is defined, and dually for $[\hat{q}_z]$).

- For all vertex idempotents and unshiftable rays, set $d_r = 0$.

Work in the integral ray algebra $\mathbb{Z}P$ (which is free over $\mathbb{Z}$ on the nonzero rays of P , with basis products having coefficients 0 or 1), let $\delta: \mathbb{Z}P \rightarrow (\mathbb{Z}P) \otimes_{\mathbb{Z}} T$ be the resulting normalized $\mathbb{Z}$ -linear one-cochain $\delta(r) = r^+ \otimes d_r$, and let $\mu = d\delta$ be its Hochschild coboundary with values in $(\mathbb{Z}P) \otimes_{\mathbb{Z}} T \cong \bigoplus_{r \in P \setminus \{0\}} T$. We verify that $\mu(r, s) = 0$ in $(\mathbb{Z}P) \otimes_{\mathbb{Z}} T$ for every pair of composable rays (r, s) with $rs \neq 0$ by checking five exhaustive cases.

Case 1: One of the factors is a local ray. When r or s is an identity, $\mu(r, s) = 0$ by normalization. Suppose that the left factor is a nonidentity local ray $c = q_x^j$ ($j \geq 1$) with $cs \neq 0$. If the output cs is unshiftable, then $\mu(c, s) = 0$ by Lemma 8.19. Assume therefore that cs is shiftable.

If x is a supporting vertex ($q_x = \widehat{q}_x$), then s cannot have type R (since $q_x s \neq 0$), so by (17) the local ray $c = \widehat{q}_x^j$ has type B and s has type B or L. When s has type B, the relation $d_{cs} = d_c + d_s$ holds in W_χ . When s has type L, write $s = \widehat{q}_x^i f$ with f generating ($A_0(x, y)$ is cyclic at x by Lemma 8.18). Note that f also has type L, since if $f \widehat{q}_y = \widehat{q}_x f$ were nonzero, then $s \widehat{q}_y = \widehat{q}_x^{i+1} f = \widehat{q}_x s \neq 0$ would make s two-sided. Because $c = \widehat{q}_x^j$ is a product of j copies of $\widehat{q}_x$ in W_χ , we have $d_c = j[\widehat{q}_x]$, and (23) gives

$$d_{cs} = (j + i)[\widehat{q}_x] + d_f = j[\widehat{q}_x] + (i[\widehat{q}_x] + d_f) = d_c + d_s,$$

matching (17).

If x is not a supporting vertex ($\ell(q_x) \neq \chi$), we claim that no nonzero product $cs = q_x^j s$ can be shiftable. Indeed, if the corner bimodule $A_0(x, y)$ containing s is cyclic at x , then adjacent rays in $A_0(x, y)$ differ in weight by $\ell(q_x) \neq \chi$, and since χ is primitive no multiple of $\ell(q_x)$ can equal χ ; hence $A_0(x, y)$ contains no shiftable rays at all. If instead $A_0(x, y)$ is cyclic only at y with generating ray f , then $A_0(x, y)$ can contain shiftable rays only when $\ell(q_y) = \chi$, in which case $\ell(q_x) \neq \ell(q_y)$ forces any nonzero left action $q_x f = f q_y^n$ to have $n \geq 2$. By Lemma 8.10 (applied in the opposite algebra), $q_x f$ then lies in the endpoint socle of $A_0(x, y)$, so $cs = q_x^j s$ is again unshiftable (and $cs^+ = 0$). Thus $\mu(c, s) = 0$ in all cases, and the dual argument gives $\mu(r, c) = 0$ for any local ray c on the right.

Moreover, if $cr = 0$ or $rc = 0$ for a local ray c , then $c * r = 0$ and $r * c = 0$ by Corollary 8.13 (since a nonidentity local ray $c = q_x^j$ is never the generating ray of a non-cyclic corner), so $\mu(c, r) = 0$ and $\mu(r, c) = 0$ even when the product vanishes. Expanding the cocycle identity $d\mu = 0$ on the triples (c, r, s) , (r, s, c) , and (r, c, s) for any local ray c therefore yields the local-linearity relations in $(\mathbb{Z}P) \otimes_{\mathbb{Z}} T$:

$$\mu(cr, s) = c\mu(r, s), \quad \mu(r, sc) = \mu(r, s)c, \quad \mu(rc, s) = \mu(r, cs). \quad (24)$$

Case 2: Generating outputs or two-sided inputs. Every nonzero product whose output is generating and one-sided satisfies $\mu(r, s) = 0$ by Lemma 9.4, and every nonzero product of types (L, R) with a generating two-sided output is one of the defining relations (20) of T . Likewise, every product of two-sided rays (whether generating or proper) with a shiftable output has a two-sided output and holds by the defining relations of W_χ .

Case 3: Generating inputs of types (B, L) or (R, B) with a proper output. Suppose that $r: x \rightarrow y$ of type B and $s: y \rightarrow z$ of type L are generating rays whose product rs is proper and shiftable. By Proposition 8.25(2), $rs = \widehat{q}_x f$ has endpoint layer 1 (where $f: x \rightarrow z$ is generating of type L) and r admits a primitive reverse $u: y \rightarrow x$ with $ru = \widehat{q}_x$ and $ur = \widehat{q}_y$. Since $r(uf) = (ru)f = \widehat{q}_x f = rs \neq 0$, the product uf is a nonzero ray in $A_0(y, z)$ with the same endpoints and weight as s , so $uf = s$ by weight uniqueness in $A_0(y, z)$. Moreover, $\widehat{q}_y s = \widehat{q}_y u f = u \widehat{q}_x f = urs \neq 0$, so u has type B. Because $s = uf$ is a generating left-sided output with u of type B and f of type L, Case 2 gives $d_s = [u] + d_f$. Since $[\widehat{q}_x] = [r] + [u]$ in W_χ , formula (23) yields

$$d_{rs} = [\widehat{q}_x] + d_f = [r] + [u] + d_f = d_r + d_s,$$

so $\mu(r, s) = 0$. The case of types (R, B) follows dually.

Case 4: Generating inputs of types (L, R) with a proper output. Let $r: x \rightarrow y$ of type L and $s: y \rightarrow z$ of type R be generating rays whose product rs is proper and shiftable. By Proposition 8.25(3), if $x = z$, then (r, s) is a nondeep external return at x , so $\mu(r, s) = 0$ is one of the defining relations (21) of T . If $x \neq z$, then Proposition 8.25(3)(b) gives either $r = fa$ and $as = \widehat{q}_z^2$ (with f, a generating and (a, s) a nondeep external return at z) or the dual factorization. In the first case, expanding $d\mu(f, a, s) = 0$ gives

$$f\mu(a, s) - \mu(fa, s) + \mu(f, as) - \mu(f, a)s = 0.$$

Here $\mu(a, s) = 0$ by the local-return case just noted, $\mu(f, as) = \mu(f, \widehat{q}_z^2) = 0$ by Case 1, and $\mu(f, a) = 0$ by Case 2 because $fa = r$ is a generating left-sided ray. Hence $\mu(r, s) = \mu(fa, s) = 0$, and the dual factorization is handled symmetrically.

Case 5: Proper inputs. Suppose finally that at least one of r or s is a proper ray and rs is shiftable.

- If the output rs has type B, then (r, s) has types (B, B) (already covered in Case 2) or (L, R). For (L, R), write $r = \widehat{q}_x^i r_0$ and $s = s_0 \widehat{q}_z^j$ with r_0, s_0 generating; the local-linearity relations (24) give $\mu(r, s) = \widehat{q}_x^i \mu(r_0, s_0) \widehat{q}_z^j = 0$ by Case 2 or Case 4 applied to (r_0, s_0) .
- If the output rs has type L and r has type B, write $r = \widehat{q}_x^i r_0$ and $s = \widehat{q}_y^j s_0$ with r_0, s_0 generating; then (24) and the two-sidedness of r_0 give $\mu(r, s) = \widehat{q}_x^i \mu(r_0 \widehat{q}_y^j, s_0) = \widehat{q}_x^{i+j} \mu(r_0, s_0) = 0$ by Case 2 or Case 3 applied to (r_0, s_0) .
- If both r and rs have type L, write $r = \widehat{q}_x^i r_0$ with r_0 generating. By Proposition 8.25(1), rs also has endpoint layer i , which means $rs = \widehat{q}_x^i(r_0 s)$ with $r_0 s$ generating (and s is generating by Lemma 8.14). Then (24) gives $\mu(r, s) = \widehat{q}_x^i \mu(r_0, s) = 0$ by Case 2.

The remaining cases with output of type R follow by duality. By Lemma 8.19, this exhausts all nonzero products with shiftable output, so our universal coefficients in T satisfy every equation in (18).

Lifting a prescribed modulo-two solution. Now let $(\bar{d}_r)$ be any solution of (18) over $\mathbb{Z}/2$. Its restriction to two-sided rays defines a homomorphism $\bar{w}: W_\chi \rightarrow \mathbb{Z}/2$, which lifts by Lemma 9.2 and our inductive hypothesis to a homomorphism $w: W_\chi \rightarrow \mathbb{Z}$.

Evaluating the modulo-two equations (17) along the chosen ordinary paths for generating one-sided rays r and s forces

$$\bar{d}_r = \bar{d}_p + \bar{d}_a, \quad \bar{d}_s = \bar{d}_b + \bar{d}_{p'},$$

in the notation of (19), and evaluating them on local products forces $\bar{d}_{\widehat{q}_x^j r} = j \bar{d}_{\widehat{q}_x} + \bar{d}_r$ and $\bar{d}_{s \widehat{q}_z^j} = \bar{d}_s + j \bar{d}_{\widehat{q}_z}$ as in (23). Moreover, splitting a generating two-sided ray f of (20) after its first L-arrow shows that $\bar{d}_a + \bar{d}_b = \bar{d}_f = \bar{w}([f])$, while for any nondeep external return $el = \widehat{q}_x^2$ Lemma 9.6 gives $\bar{d}_a + \bar{d}_b = \bar{d}_e + \bar{d}_l = \bar{d}_{\widehat{q}_x^2} = 2 \bar{d}_{\widehat{q}_x} = 0 = \bar{w}(2[\widehat{q}_x])$. Thus the modulo-two arrow coefficients $(\bar{d}_a, \bar{d}_b)$ satisfy every edge relation of the bipartite graph.

On each connected component of the bipartite graph with constant edge label $\omega \in W_\chi$, the modulo-two values therefore take the form $\bar{d}_a = \bar{w}(\omega) + \bar{z}$ and $\bar{d}_b = -\bar{z}$ for a unique element $\bar{z} \in \mathbb{Z}/2$. Choosing an integer lift $z \in \mathbb{Z}$ of $\bar{z}$, we specialize the component's formal variables to $X_a = w(\omega) + z$ and $Y_b = -z$ (and for each isolated vertex we lift its modulo-two value directly to $\mathbb{Z}$). Together with $w: W_\chi \rightarrow \mathbb{Z}$, these choices define a group homomorphism $T \rightarrow \mathbb{Z}$ whose composition with the universal coefficients in T yields an integer solution (d_r) of (18) reducing modulo two to $(\bar{d}_r)$. $\square$

10. HOCHSCHILD REPRESENTATIVES FROM THE INTEGERS MODULO FOUR

We now construct a cocycle generator for each one-dimensional weight space $\mathrm{HH}^2(A_0)_{\chi_i}$ ($i \in I_0$) that is supported strictly on zero ray products (r, s) with $rs = 0$ and takes values in their stem-product lines U_{r*s} . Because $A_0 = k \otimes_{\mathbb{F}_2} \mathbb{F}_2 P$ and Hochschild cohomology commutes with field extensions, it suffices to construct these cocycles over the prime field $\mathbb{F}_2$; their k -linear combinations will then span all of $\mathrm{HH}^2(A_0)_{>0}$.

The strategy is based on the Bockstein connecting homomorphism for the short exact coefficient sequence $0 \rightarrow \mathbb{F}_2 \xrightarrow{2} \mathcal{O} \rightarrow \mathbb{F}_2 \rightarrow 0$, where

$$\mathbb{F}_2 = \mathbb{Z}/2, \quad \mathcal{O} = \mathbb{Z}/4,$$

and the inclusion $\mathbb{F}_2 \hookrightarrow \mathcal{O}$ is given by $a \mapsto 2a$. Setting $A^{(2)} = \mathbb{F}_2 P$ and $A^{(4)} = \mathcal{O}P$, we show in Proposition 10.3 that every cohomology class in $\mathrm{HH}^2(A^{(2)})_{>0}$ lies in the image of the connecting map from $\mathrm{HH}^1(A^{(2)})_{>0}$, and is therefore represented by $(d\delta)/2$ for any $\mathcal{O}$ -linear one-cochain δ lifting a derivation $\bar{\delta}$ of $A^{(2)}$. By Theorem 9.1, the lift δ can be chosen so that $d\delta$ vanishes on all nonzero ray products, which forces $(d\delta)/2$ to be supported exclusively on pairs of rays with zero product. Throughout this section, all cochains are normalized with respect to the vertex idempotents and are homogeneous with respect to both the categorical grading and the weight lattice L . For background on the Bockstein homomorphism in Hochschild cohomology, see [10, Sections 4–5] and [16, Section 2].

10.1. The model over the two-element field and its module lifts

The all-ones mesh relations of Section 7 define a graded $\mathbb{F}_2$ -linear category $\mathcal{C}^{(2)}$ satisfying $k \otimes_{\mathbb{F}_2} \mathcal{C}^{(2)} = \mathcal{C}$. Its full subcategory on the projective objects P_x has opposite endomorphism algebra $\mathrm{End}(\bigoplus_x P_x)^{\mathrm{op}} = A^{(2)}$, since each nonzero homogeneous component of $A^{(2)}(x, y)$ is one-dimensional over $\mathbb{F}_2$ and every nonzero product of basis rays has coefficient 1. For each object X_i of $\mathcal{C}^{(2)}$, define the graded right $A^{(2)}$ -module $M_i^{(2)}$ by

$$M_i^{(2)} e_x = \mathcal{C}^{(2)}(P_x, X_i).$$

Evaluation on the projective objects identifies $\mathcal{C}^{(2)}$ with the full graded homomorphism category of the modules $M_i^{(2)}$, because it does so after the faithfully flat base change $- \otimes_{\mathbb{F}_2} k$. Since tensoring a finitely generated projective resolution with k commutes with graded Ext , Lemma 6.3 implies that

$$\mathrm{Ext}_{A^{(2)}}^n(M_i^{(2)}, M_j^{(2)})_d = 0 \quad (n \geq 1, d \geq 0). \quad (25)$$

When $X_i = P_x$ is a projective object, $M_i^{(2)} = e_x A^{(2)}$ is the corresponding indecomposable projective $A^{(2)}$ -module.

Proposition 10.1. *The graded $A^{(2)}$ -modules $M_i^{(2)}$ lift to graded $A^{(4)}$ -modules $M_i^{(4)}$ that are free over $\mathcal{O} = \mathbb{Z}/4$, and their full graded homomorphism category E_4 has the following properties:*

- (1) *Each morphism space of E_4 is a free $\mathcal{O}$ -module, and $E_4/2E_4 = \mathcal{C}^{(2)}$.*
- (2) *At the projective objects $X_i = P_x$, we have $M_i^{(4)} = e_x A^{(4)}$, so the projective corner of E_4 has opposite endomorphism algebra $\mathrm{End}(\bigoplus_x e_x A^{(4)})^{\mathrm{op}} = A^{(4)}$.*
- (3) *The category E_4 admits a quadratic scalar mesh presentation over $\mathcal{O}$ whose coefficients are units reducing modulo 2 to the all-ones mesh presentation of $\mathcal{C}^{(2)}$.*

Moreover, all of these lifts may be chosen to preserve the simultaneous integer weights in L : lifting a basis of $H_{\mathbb{Z}}$ to mesh weights via Lemma 7.4 equips the modules $M_i^{(2)}$ with compatible L -gradings, and all obstruction and lifting steps can be carried out within each weight component.

Proof. Choose arbitrary degree-preserving lifts of the action matrices of $A^{(2)}$ on $M_i^{(2)}$ to free graded $\mathcal{O}$ -modules. Their associativity defect is divisible by 2; dividing it by 2 and reducing modulo 2 produces a degree-zero cocycle representing a class in $\mathrm{Ext}_{A^{(2)}}^2(M_i^{(2)}, M_i^{(2)})_0$, which vanishes by (25). Subtracting 2 times a bounding one-cochain kills the defect modulo 4 (since $(2\mathcal{O})^2 = 0$), yielding an associative graded $A^{(4)}$ -module structure $M_i^{(4)}$. The same bar-resolution calculation using the vanishing of Ext^1 in nonnegative degrees shows that $M_i^{(4)}$ is unique up to isomorphism and that every homogeneous morphism of nonnegative degree between the modules $M_i^{(2)}$ lifts to E_4 . At the projective objects $X_i = P_x$, we take $M_i^{(4)} = e_x A^{(4)}$ to be the regular projective $A^{(4)}$ -modules, which gives (2).

To verify (1), fix any pair of objects X_i, X_j and lift an $\mathbb{F}_2$ -basis $\bar{f}_1, \dots, \bar{f}_r$ of $\mathcal{C}^{(2)}(X_i, X_j)_d$ ($d \geq 0$) to homogeneous $A^{(4)}$ -module homomorphisms $f_1, \dots, f_r \in E_4(X_i, X_j)_d$. Any $g \in E_4(X_i, X_j)_d$ that reduces to zero modulo 2 has matrix entries in $2\mathcal{O}$; dividing by 2 gives an $A^{(2)}$ -module homomorphism of degree d , which is an $\mathbb{F}_2$ -linear combination of $\bar{f}_1, \dots, \bar{f}_r$, so $f_1, \dots, f_r$ span $E_4(X_i, X_j)_d$ over $\mathcal{O}$. Likewise, if $\sum_{m=1}^r a_m f_m = 0$ with $a_m \in \mathcal{O}$, reducing modulo 2 shows that $a_m = 2b_m$ for all m , and dividing by 2 and reducing modulo 2 again shows that every b_m is even, so $a_m = 0$ in $\mathcal{O}$. The same two-step reduction shows that $E_4(X_i, X_j)_d = 0$ for $d < 0$.

For (3), Nakayama's lemma implies that the degree-one morphisms generate E_4 as an $\mathcal{O}$ -linear category. Because $(E_4)_2$ is a free $\mathcal{O}$ -module, the composition map from length-two paths onto $(E_4)_2$ splits, so its kernel is a free $\mathcal{O}$ -module direct summand. Lifting the mesh relations of $\mathcal{C}^{(2)}$ to a basis of this kernel defines a quadratic $\mathcal{O}$ -linear category $\tilde{E}_4$ equipped with a graded surjection $\tilde{E}_4 \rightarrow E_4$ that reduces modulo 2 to the identity on $\mathcal{C}^{(2)}$. Each graded piece of $\tilde{E}_4$ is finitely generated over $\mathcal{O}$, and in degrees above the top degree of $\mathcal{C}^{(2)}$ its reduction modulo 2 is zero, so $N = 2N$ and hence $N = 4N = 0$; thus $\tilde{E}_4$ is finite over $\mathcal{O}$. Since E_4 is free over $\mathcal{O}$, the surjection $\tilde{E}_4 \rightarrow E_4$ splits as a map of $\mathcal{O}$ -modules, and its kernel vanishes by Nakayama's lemma because it reduces to zero modulo 2. Finally, because each arrow $\tau X \rightarrow Y$ in a mesh of $\mathcal{C}^{(2)}$ has a unique mate $Y \rightarrow X$ and the irreducible morphism spaces are one-dimensional, every length-two path between τX and X is a mesh path, so the lifted relations are quadratic scalar mesh relations with unit coefficients in $\mathcal{O}^\times$. $\square$

10.2. Comparison with the mesh derivative

Let $Q_m = \text{coker } D_m(E_4)$ denote the cokernel of the mesh derivative of E_4 in degree m , as in Lemma B.2 of Appendix B.2. For a Hochschild two-cocycle of $A^{(2)}$ of internal degree $q > 0$, the corresponding arrow-correction degree on the mesh category is $m = q + 1$, and Lemma B.2 gives

$$2Q_{q+1} = 0. \quad (26)$$

For any cochain $\bar{\mu}$ over $\mathbb{F}_2$, we write $2\tilde{\mu}$ for twice an arbitrary coefficient lift $\tilde{\mu}$ of $\bar{\mu}$ to $\mathcal{O} = \mathbb{Z}/4$; since $2\mathcal{O} \cong \mathbb{F}_2$, the cochain $2\tilde{\mu}$ depends only on $\bar{\mu}$.

Lemma 10.2. *Let $\bar{\mu}$ be a normalized Hochschild two-cocycle of $A^{(2)}$ of internal degree $q > 0$. Then $2\tilde{\mu}$ is a coboundary in the normalized Hochschild complex of $A^{(4)}$. The same statement holds within each weight component $\chi \in L$ with $\chi(\deg_{\mathcal{J}}) = q > 0$.*

Proof. Set $R = \mathcal{O}[\varepsilon]/(\varepsilon^2)$ with $\deg(\varepsilon) = -q$, and equip $B = A^{(4)} \otimes_{\mathcal{O}} R$ with the deformed multiplication $m + 2\varepsilon\tilde{\mu}$. Its associator is $2\varepsilon d\tilde{\mu} = 0$ because $d\bar{\mu} = 0$ over $\mathbb{F}_2$, so B is an associative graded R -algebra.

Over the reduced ring $\mathbb{F}_2[\varepsilon]/(\varepsilon^2)$, we may apply Lemma 6.6 to the first-order deformation $m + \varepsilon\bar{\mu}$ of $A^{(2)}$ and the modules $M_i^{(2)}$, since the required Ext^2 and Ext^1 groups vanish by (25). Let $\bar{\nu}_i$ be the resulting first-order corrections to the module actions and let $\bar{g}_a$ be the corrections to the generating arrow morphisms. On the free R -modules $M_i^{(4)} \otimes_{\mathcal{O}} R$, we then define deformed actions $\nu_i^{(4)} + 2\varepsilon\tilde{\nu}_i$ and deformed arrow maps $a^{(4)} + 2\varepsilon\tilde{g}_a$. Because every correction term carries a factor of 2ε and $(2\varepsilon)^2 = 0$, the module-action and module-homomorphism conditions depend only on the reductions $\bar{\nu}_i$ and $\bar{g}_a$ modulo 2 and are therefore satisfied over R . At the projective objects P_x , we take the lifted modules to be the regular projective B -modules $e_x B$, whose action corrections match the deformed multiplication $m + 2\varepsilon\tilde{\mu}$ of B .

Now evaluate the quadratic scalar mesh relations of E_4 on the deformed arrow maps $a^{(4)} + 2\varepsilon\tilde{g}_a$. The constant terms vanish in E_4 , and because the unit coefficients of the mesh relations in E_4 reduce to 1 modulo 2, the remaining residual error at each mesh ending at X is $2\varepsilon\tilde{z}_X$, where $\tilde{z}_X$ is the mesh error of the modulo-two lift. Choose a homogeneous

morphism $z_X \in E_4(\tau X, X)_{q+2}$ lifting $\bar{z}_X$; since the underlying representation spaces are free over $\mathcal{O}$, evaluating $2\varepsilon z_X$ produces precisely the residual error $2\varepsilon \bar{z}_X$. By (26), the class of $2z = (2z_X)$ vanishes in $Q_{q+1} = \text{coker } D_{q+1}(E_4)$, so there exist homogeneous $A^{(4)}$ -module morphisms $f_a \in E_4(\text{source } a, \text{target } a)_{q+1}$ satisfying

$$D_{q+1}(f) = 2z.$$

Replacing the deformed arrow maps by $a^{(4)} + 2\varepsilon \tilde{g}_a - \varepsilon f_a$ preserves the B -module homomorphism property (since f_a is $A^{(4)}$ -linear and $\varepsilon^2 = 0$) and subtracts $\varepsilon D_{q+1}(f) = 2\varepsilon z$ from the mesh errors, killing them completely. The corrected arrow maps therefore satisfy all relations of E_4 and define an R -linear functor from $E_4 \otimes_{\mathcal{O}} R$ to the category of lifted B -modules.

Restricting this functor to the projective objects P_x and passing to opposite endomorphism algebras via Proposition 10.1(2) yields an R -algebra homomorphism

$$A^{(4)} \otimes_{\mathcal{O}} R \longrightarrow B$$

that reduces to the identity modulo ε and fixes the vertex idempotents. On the underlying free R -module it has the form $\text{id} + \varepsilon h$ for a normalized one-cochain h of $A^{(4)}$, and multiplicativity $(\text{id} + \varepsilon h)(ab) = (\text{id} + \varepsilon h)(a) \cdot_B (\text{id} + \varepsilon h)(b)$ expands to

$$h(ab) = h(a)b + ah(b) + 2\tilde{\mu}(a, b),$$

so $2\tilde{\mu} = -dh$ is a coboundary. Giving ε weight $-\chi$ keeps the entire construction homogeneous of weight zero, so h may be taken in weight χ . $\square$

Proposition 10.3. *The Bockstein connecting homomorphism associated to the short exact coefficient sequence $0 \rightarrow \mathbb{F}_2 \xrightarrow{2} \mathcal{O} \rightarrow \mathbb{F}_2 \rightarrow 0$ is surjective in every positive internal degree $q > 0$:*

$$\beta: \text{HH}^1(A^{(2)})_q \rightarrow \text{HH}^2(A^{(2)})_q.$$

Explicitly, for any derivation $\bar{\delta}$ of $A^{(2)}$ and any normalized $\mathcal{O}$ -linear one-cochain δ of $A^{(4)}$ lifting $\bar{\delta}$, the class $\beta([\bar{\delta}])$ is represented by $(d\delta)/2$, where division by 2 denotes the inverse of the additive isomorphism $\mathbb{F}_2 \xrightarrow{\sim} 2\mathcal{O}$. The same statement holds within each weight component $\chi \in L$ with $\chi(\deg_{\mathcal{I}}) > 0$.

Proof. The short exact sequence of normalized Hochschild cochain complexes

$$0 \longrightarrow C^*(A^{(2)}) \xrightarrow{2} C^*(A^{(4)}) \longrightarrow C^*(A^{(2)}) \longrightarrow 0$$

induces a long exact sequence in cohomology. By Lemma 10.2, multiplication by 2 from $\text{HH}^2(A^{(2)})_q$ to $\text{HH}^2(A^{(4)})_q$ is zero for every $q > 0$, so the connecting homomorphism $\beta: \text{HH}^1(A^{(2)})_q \rightarrow \text{HH}^2(A^{(2)})_q$ is surjective. By the standard definition of the connecting homomorphism, $\beta([\bar{\delta}])$ is represented by $(d\delta)/2$ for any cochain lift $\delta \in C^1(A^{(4)})$, and all maps respect the weight grading by L . $\square$

10.3. A coefficient criterion for supported representatives

Before combining Proposition 10.3 with Theorem 9.1, we isolate the elementary homological principle that produces cocycle representatives supported on a prescribed subset of basis elements. It holds for any prime p ; we will apply it with $p = 2$.

Lemma 10.4. *Let p be a prime, let C^* be a cochain complex of free $\mathbb{Z}/p^2$ -modules, and set $\bar{C}^* = C^*/pC^*$. Let $U \subseteq C^2$ be a submodule spanned by a subset of a fixed $\mathbb{Z}/p^2$ -basis of C^2 , and assume that:*

- (1) *the map $H^2(\bar{C}^*) \xrightarrow{p} H^2(C^*)$ induced by the inclusion $\mathbb{Z}/p \hookrightarrow \mathbb{Z}/p^2$ is zero;*
- (2) *every one-cocycle $\bar{\delta} \in Z^1(\bar{C}^*)$ lifts to a one-cochain $\delta \in C^1$ satisfying $d\delta \in U$.*

Then every cohomology class in $H^2(\bar{C}^*)$ is represented by a two-cocycle lying in the subspace $U/pU \subseteq \bar{C}^2$.

Proof. By hypothesis (1) and the long exact cohomology sequence for $0 \rightarrow \overline{C}^* \xrightarrow{p} C^* \rightarrow \overline{C}^* \rightarrow 0$, the Bockstein connecting map $H^1(\overline{C}^*) \rightarrow H^2(\overline{C}^*)$ is surjective. Given any class in $H^2(\overline{C}^*)$, choose a preimage represented by $\bar{\delta} \in Z^1(\overline{C}^*)$ and lift $\bar{\delta}$ to $\delta \in C^1$ with $d\delta \in U$ via (2). Since $d\bar{\delta} = 0$ in $\overline{C}^2$, the coboundary $d\delta$ lies in $U \cap pC^2 = pU$ (as U is spanned by a subset of a basis of C^2). Dividing $d\delta$ by p therefore yields a representative of the given cohomology class in U/pU . $\square$

10.4. Moving the representative onto zero products

Combining Proposition 10.3 with the integral lifting of the nonzero-product Leibniz equations in Theorem 9.1 now yields a cocycle generator for each residual line $\mathrm{HH}^2(A_0)_{\chi_i}$ ($i \in I_0$) that vanishes on all nonzero ray products and takes values in the stem-product lines on zero ray products.

Corollary 10.5. *Assume that Proposition 8.2 holds for all weighted multiplication tables with fewer than $\dim A_0$ nonzero elements, as in Theorem 9.1. Then for each $i \in I_0$, the one-dimensional weight space $\mathrm{HH}^2(A_0)_{\chi_i}$ has a nonzero cocycle representative supported exclusively on composable pairs of rays (r, s) satisfying $rs = 0$ and $r * s \neq 0$, where $*$ is the stem product of Definition 8.6, and its value on each such pair (r, s) lies in the one-dimensional socle line U_{r*s} . Moreover, this cocycle may be chosen over $\mathbb{F}_2$ and extended by scalars to k .*

Proof. Fix $i \in I_0$ and work first over $\mathbb{F}_2$ with $A^{(2)} = \mathbb{F}_2 P$ and $A^{(4)} = \mathcal{O}P$. By Proposition 7.5, the residual character $\chi_i \in L$ is part of a $\mathbb{Z}$ -basis of L and satisfies $\chi_i(\deg_{\mathcal{J}}) = 2s_i > 0$, so χ_i is positive and primitive as required by Setting 8.16. Let C^* be the weight- χ_i subcomplex of the normalized Hochschild complex of $A^{(4)}$, so that $\overline{C}^*$ is the weight- χ_i subcomplex for $A^{(2)}$, and let $U \subseteq C^2$ be the free $\mathcal{O}$ -submodule of cochains that vanish on all pairs (r, s) with $rs \neq 0$ or $r * s = 0$ and take values in U_{r*s} when $rs = 0$ and $r * s \neq 0$.

Condition (1) of Lemma 10.4 holds by Lemma 10.2. For condition (2), any derivation $\bar{\delta} \in Z^1(\overline{C}^*)$ of weight χ_i acts on shiftable rays by $\bar{\delta}(r) = \bar{d}_r r^+$ and vanishes on identities and unshiftable rays, where the coefficients $(\bar{d}_r)$ satisfy the nonzero-product Leibniz equations (18) modulo 2. By Theorem 9.1, $(\bar{d}_r)$ lifts to a solution (d_r) over $\mathcal{O} = \mathbb{Z}/4$, so the lifted one-cochain $\delta(r) = d_r r^+ \in C^1$ satisfies $(d\delta)(r, s) = 0$ whenever $rs \neq 0$. On any pair (r, s) with $rs = 0$, we have $\delta(rs) = 0$, and any outer successor substitution $r^+ = \hat{q}_x r$ or $s^+ = s \hat{q}_z$ contributes $\hat{q}_x(rs) = 0$ or $(rs)\hat{q}_z = 0$ to $(d\delta)(r, s)$. Thus the only possible nonzero contributions to $(d\delta)(r, s) = d_r r^+ s + d_s r s^+$ come from middle actions $r^+ = r \hat{q}_y$ or $s^+ = \hat{q}_y s$, both of which evaluate to $r \hat{q}_y s$. By Corollary 8.13 and Proposition 8.12, $r \hat{q}_y s$ vanishes if $r * s = 0$ and equals $r * s \in U_{r*s}$ if $r * s \neq 0$. Hence $d\delta \in U$, verifying condition (2) of Lemma 10.4.

Lemma 10.4 therefore produces a nonzero cocycle generator $(d\delta)/2 \in U/2U$ of $\mathrm{HH}^2(A^{(2)})_{\chi_i}$, and tensoring with k over $\mathbb{F}_2$ gives the desired generator of $\mathrm{HH}^2(A_0)_{\chi_i}$. $\square$

11. ASSOCIATIVE FAMILIES AND COMPLETION OF THE PROOF

Under the inductive hypothesis of Corollary 10.5, the cocycles supported on zero ray products span the positive-degree Hochschild cohomology $\mathrm{HH}^2(A_0)_{>0}$. In this section we organize all such cocycles into a single polynomial family of associative multiplications, verify that every specialization with parameters in $\{0, 1\}$ has a normed multiplicative basis, and apply Proposition 7.9 to recover the original algebra A and close the induction of Proposition 8.2.

11.1. Indicator cocycles on zero products

The stem product $r * s$ specifies the only possible nonzero value for a deformation term on a pair of rays (r, s) with $rs = 0$. Imposing the Hochschild cocycle condition on all

such pairs simultaneously turns out to require only pairwise equalities between their scalar coefficients, partitioning the active zero pairs into equivalence classes whose indicator sums define independent deformation parameters.

We first record this phenomenon as a general observation about two compatible associative multiplication tables on the same basis.

Lemma 11.1. *Let S be a finite set with an absorbing element 0, and let m and m_* be two associative multiplications on S that share a common finite set of orthogonal idempotents summing to the identity of both linearizations. Suppose that $m_*(a, b) = m(a, b)$ whenever $m(a, b) \neq 0$. Set*

$$Z = \{(a, b) \in (S \setminus \{0\})^2 : m(a, b) = 0 \text{ and } m_*(a, b) \neq 0\},$$

and for each $z = (a, b) \in Z$ let ϵ_z be the bilinear cochain on the free abelian group $\mathbb{Z}[S \setminus \{0\}]$ that sends (a, b) to $m_*(a, b)$ and vanishes on all other basis pairs. Assume that all nested insertions vanish:

$$\epsilon_z(\epsilon_w(a, b), c) = 0 = \epsilon_z(a, \epsilon_w(b, c)) \quad \text{for all } z, w \in Z \text{ and } a, b, c \in S \setminus \{0\}.$$

Then there is an equivalence relation $\sim$ on Z such that, over any commutative ring R , the Hochschild two-cocycles of $(R[S \setminus \{0\}], m)$ of the form $\sum_{z \in Z} a_z \epsilon_z$ ($a_z \in R$) are precisely the R -linear combinations of the class sums

$$\mu_E = \sum_{z \in E} \epsilon_z \quad (E \in Z/\sim).$$

Moreover, the deformed multiplication $m + \sum_{E \in Z/\sim} T_E \mu_E$ is associative over the polynomial ring $R[T_E : E \in Z/\sim]$, and at every specialization with $T_E \in \{0, 1\}$ the set $S \setminus \{0\}$ is a multiplicative basis.

Proof. Set $\epsilon = \sum_{z \in Z} \epsilon_z$, so that $m_* = m + \epsilon$. Because both m and m_* are associative and all quadratic terms $\epsilon(\epsilon(a, b), c)$ and $\epsilon(a, \epsilon(b, c))$ vanish by hypothesis, the Hochschild coboundary $d\epsilon$ with respect to m vanishes identically over $\mathbb{Z}$.

Now consider a general R -linear combination $\mu = \sum_{z \in Z} a_z \epsilon_z$ and evaluate its coboundary on a triple of nonzero basis elements $a, b, c \in S \setminus \{0\}$ (writing xy for $m(x, y)$):

$$(d\mu)(a, b, c) = a\mu(b, c) - \mu(ab, c) + \mu(a, bc) - \mu(a, b)c.$$

If $abc \neq 0$, then $ab \neq 0$, $bc \neq 0$, $(ab)c \neq 0$, and $a(bc) \neq 0$, so all four terms vanish because μ is supported on Z . When $abc = 0$, the surviving terms depend on whether ab and bc are zero:

ab	bc	$(d\mu)(a, b, c)$
0	0	$a\mu(b, c) - \mu(a, b)c$
0	$\neq 0$	$\mu(a, bc) - \mu(a, b)c$
$\neq 0$	0	$a\mu(b, c) - \mu(ab, c)$
$\neq 0$	$\neq 0$	$\mu(a, bc) - \mu(ab, c).$

In every row, $(d\mu)(a, b, c)$ is the difference of at most one positive term and at most one negative term, each of which is a single coefficient a_z times a nonzero element of S (since $a\epsilon_{(b, c)}(b, c)$ lies in S). Specializing to $a_z = 1$ for all $z \in Z$, the identity $d\epsilon = 0$ over $\mathbb{Z}$ shows that on each triple (a, b, c) either both terms vanish or both terms are a_z and $a_{z'}$ times the exact same basis element of $S \setminus \{0\}$. Consequently, $(d\mu)(a, b, c) = 0$ imposes either no condition or a simple equality $a_z = a_{z'}$. Generating an equivalence relation $\sim$ on Z by these equalities proves that the cocycles of the form $\sum_{z \in Z} a_z \epsilon_z$ have R -basis $\{\mu_E : E \in Z/\sim\}$.

Finally, the associator of $m + \sum_E T_E \mu_E$ has zero linear part in the variables T_E because each μ_E is a cocycle, and zero quadratic part because all nested insertions vanish. When every T_E is specialized to 0 or 1, the deformed product of $a, b \in S \setminus \{0\}$ equals $m(a, b)$ if $(a, b) \notin Z$, and equals either 0 or $m_*(a, b)$ if $(a, b) \in Z$; in every case it belongs to S . $\square$

We now specialize Lemma 11.1 to the ray category P of $A_0 = kP$. Let Z be the set of composable pairs $z = (r, s)$ of nonidentity rays of P satisfying $rs = 0$ and $r * s \neq 0$, and for each $z = (r, s) \in Z$ let ϵ_z denote the bilinear cochain that sends (r, s) to $r * s$ and vanishes on all other pairs of rays (including identities).

Proposition 11.2. *Let Z and ϵ_z be as above for $A_0 = kP$ in Setting 8.4. Over any commutative coefficient ring R :*

- (1) *Every nontrivial cocycle condition on a linear combination $\sum_{z \in Z} a_z \epsilon_z$ ($a_z \in R$) takes the form $a_z = a_{z'}$.*
- (2) *If $\sim$ denotes the equivalence relation on Z generated by these equalities, then the Hochschild two-cocycles of the form $\sum_{z \in Z} a_z \epsilon_z$ have R -basis*

$$\mu_E = \sum_{z \in E} \epsilon_z \quad (E \in Z/\sim). \quad (27)$$

- (3) *For any two cochains μ, ν in the span of $\{\epsilon_z : z \in Z\}$ and any rays a, b, c ,*

$$\mu(\nu(a, b), c) = 0, \quad \mu(a, \nu(b, c)) = 0. \quad (28)$$

Proof. We first verify (3). By Corollary 8.13, any nonzero value $u = r * s$ of $\nu(a, b)$ spans the endpoint socle of its corner bimodule and is therefore annihilated by both endpoint radicals. For any composable ray c , if $uc \neq 0$, then $(u, c) \notin Z$ and hence $\mu(u, c) = 0$; if instead $uc = 0$, then $uq_y^n c = 0$ for all $n \geq 1$ because $uq_y = 0$, so $u * c = 0$ by Proposition 8.12 and again $\mu(u, c) = 0$. The identity $\mu(a, \nu(b, c)) = 0$ follows dually.

Now view the ordinary ray product $m(r, s) = rs$ and the stem product $m_*(r, s) = r * s$ as binary operations on the finite set S of rays of P (including 0). By Proposition 8.12, m_* is associative and agrees with m whenever $m(r, s) \neq 0$, so $m_* = m + \sum_{z \in Z} \epsilon_z$. Since all nested insertions vanish by (3), Lemma 11.1 immediately yields (1) and (2). $\square$

We call the cocycles μ_E of (27) the *indicator cocycles* on A_0 . Using them to build an affine family of multiplications verifies the associativity and basis hypotheses of Proposition 7.9, after which Corollary 10.5 will allow us to select a subfamily spanning $\text{HH}^2(A_0)_{>0}$.

Proposition 11.3. *Let $A_0 = kP$ satisfy Setting 8.4, let E range over the equivalence classes in $Z/\sim$ defining the indicator cocycles μ_E in (27), and let $(T_E)_{E \in Z/\sim}$ be independent polynomial variables. On the free $k[T_E]$ -module having the nonzero rays of P as basis, define the multiplication*

$$r \cdot s = \begin{cases} rs, & \text{if } rs \neq 0 \text{ in } P, \\ T_E(r * s), & \text{if } (r, s) \in E \in Z/\sim, \\ 0, & \text{otherwise,} \end{cases} \quad (29)$$

where $r * s$ is the stem product of Definition 8.6. Then:

- (1) *The multiplication (29) is associative over $k[T_E]$.*
- (2) *Every scalar fiber (obtained by specializing the variables T_E to elements of k) is a positive filtered deformation of A_0 .*
- (3) *Every scalar fiber has ray category P , with the nonzero rays of P as representatives, and every binary fiber (with $T_E \in \{0, 1\}$ for all E) has a normed multiplicative basis consisting of the nonzero rays of P .*

Proof. Assertion (1) follows directly from Lemma 11.1 and Proposition 11.2. For (2), the vertex idempotents are unchanged, and for every pair $(r, s) \in E$ with middle vertex y , Corollary 8.13 shows that $r * s = rq_y s$, so the indicator term $\epsilon_{(r, s)}$ has weight

$$c_E = \ell(r * s) - \ell(r) - \ell(s) = \ell(q_y),$$

which has positive categorical degree $\deg_{\mathcal{J}}(q_y) > 0$ (and is constant across each equivalence class E because the defining equations $a_z = a_{z'}$ arise from homogeneous triples). Thus every scalar fiber is a positive filtered deformation of A_0 .

For (3), Corollary 8.13 shows that $(r, s) \in Z$ requires both r and s to be generating rays between distinct vertices. Consequently, all products involving a local ray q_x^j or a proper ray remain unchanged in every scalar fiber. In particular, the local rings R_x and their left and right actions on every corner bimodule coincide with those of A_0 , so the basis elements represent the exact same $(R_x^\times \times R_y^\times)$ -orbits and generate the same endpoint subbimodules U_r . Furthermore, for any composable rays $r: x \rightarrow y$ and $s: y \rightarrow z$ and any $n \in \text{rad } R_y$, the product rn is a linear combination of proper rays, so $(rn) \cdot s = (rn)s$; hence the middle-radical insertion subspace $M(r, s) = R_x r(\text{rad } R_y) s R_z$ of Definition 3.4 is identical in A_0 and in the fiber algebra. Because each deformation term $T_E(r * s) = T_E(rq_y s)$ lies inside $M(r, s)$, the deformed product $r \cdot s$ belongs to $M(r, s)$ if and only if $rs = 0$ in A_0 (and when $rs \neq 0$, $r \cdot s = rs$). Thus every scalar fiber has ray category P .

Finally, at any binary specialization $T_E \in \{0, 1\}$, all products of basis elements are either basis elements or zero. Because the endpoint subbimodule chains are unchanged, every corner of a two-sided ideal is an endpoint subbimodule and is therefore spanned by a subset of the basis rays. Intersecting with powers of the radical shows that the basis is normed. $\square$

Example 11.4. In the nine-dimensional algebra B of Example 3.5, replacing the single zero product $\alpha\beta = 0$ by $\alpha \cdot_t \beta = tu$ (where $u = \alpha q \beta = [\alpha] * [\beta]$) while keeping all other basis products fixed yields the one-parameter family

$$B_t = kQ/(q^2, \alpha\beta - t\alpha q\beta).$$

The same nine paths form a basis of B_t , and at $t = 0$ and $t = 1$ they form a normed multiplicative basis; assigning degree 1 to q, α, β makes B_t a positive filtered deformation of B . Notice that the linear change of basis $\alpha' = \alpha - t\alpha q$ satisfies $\alpha'\beta = 0$ and $\alpha'q = \alpha q$, giving an isomorphism $B_t \cong B$ for every $t \in k$. Equivalently, the indicator cocycle $\epsilon_{(\alpha, \beta)}$ is the coboundary $d\delta$ of the cochain $\delta(\alpha) = \alpha q$. Thus an individual indicator cocycle μ_E can be a coboundary; to obtain cocycles that represent nonzero classes in $\text{HH}^2(A_0)_{>0}$, we must combine Proposition 11.3 with Corollary 10.5.

Remark 11.5. In [2, Sections 9.2–9.7], Bongartz, Gabriel, Roiter, and Scherback construct associative algebras whose basis multiplications agree with the ray product except on specified local configurations where they equal the stem product, using these "mixtures" to classify representation-finite algebras in characteristic two. From the deformation-theoretic perspective, the indicator cocycles μ_E in (27) provide canonical cocycle coordinates for these mixtures: if $V = \bigoplus_{E \in Z/\sim} k\mu_E$ denotes their span, then under the inductive hypothesis of Corollary 10.5, that corollary together with Proposition 7.6 yields the short exact sequence

$$0 \longrightarrow V \cap B^2(A_0) \longrightarrow V \longrightarrow \text{HH}^2(A_0)_{>0} \longrightarrow 0,$$

where $B^2(A_0)$ is the space of normalized Hochschild two-coboundaries.

11.2. Existence of multiplicative bases in every characteristic

We now assemble the preceding results to prove the inductive step for Proposition 8.2 and deduce our main theorems. We first show that if Proposition 8.2 holds for all weighted multiplication tables of dimension strictly less than $\dim A$, then every basic connected representation-finite algebra A of dimension $\dim A$ in characteristic two satisfies all three conclusions of Proposition 8.3.

Lemma 11.6. *Assume that $\text{char } k = 2$ and let A be a basic connected representation-finite algebra over k . If Proposition 8.2 holds for all weighted multiplication tables with fewer than $\dim A$ nonzero elements, then all three conclusions of Proposition 8.3 hold for A .*

Proof. Since $\dim A_0 = \dim A$, the smaller-table hypothesis applies to the centralizer tables Q_χ in Theorem 9.1. For each $i \in I_0$, Corollary 10.5 therefore provides a cocycle generator of the one-dimensional weight space $\mathrm{HH}^2(A_0)_{\chi_i}$ supported on active zero pairs Z . By Proposition 11.2(2), this cocycle is a k -linear combination of indicator cocycles μ_E of weight χ_i . Consequently, for each $i \in I_0$ we can choose a single equivalence class $E_i \in Z/\sim$ of weight χ_i whose indicator cocycle $\mu_i := \mu_{E_i}$ represents a nonzero cohomology class in $\mathrm{HH}^2(A_0)_{\chi_i}$.

Because the characters χ_i ($i \in I_0$) are pairwise distinct, the classes E_i are distinct. Specializing the family of Proposition 11.3 by setting $T_{E_i} = t_i$ for $i \in I_0$ and $T_E = 0$ for all other classes $E \notin \{E_i : i \in I_0\}$ produces the polynomial family of multiplications

$$m_t = m + \sum_{i \in I_0} t_i \mu_i \quad (30)$$

over $k[t_i : i \in I_0]$. By Proposition 11.3, m_t is associative, every scalar fiber has ray category P , and every binary fiber ($t_i \in \{0, 1\}$) has the nonzero rays of P as a normed multiplicative basis. Together with our choice of μ_i as a basis of $\mathrm{HH}^2(A_0)_{>0} = \bigoplus_{i \in I_0} \mathrm{HH}^2(A_0)_{\chi_i}$, all hypotheses of Proposition 7.9 are satisfied. Hence there exists a binary specialization A_t ($t_i \in \{0, 1\}$) and a filtered algebra isomorphism $A_t \xrightarrow{\sim} A$ whose associated graded isomorphism $\mathrm{gr} A_t \xrightarrow{\sim} \mathrm{gr} A$ rescales each homogeneous ray line of A_0 . Transporting the normed multiplicative ray basis of A_t across this isomorphism gives a normed multiplicative ray basis of A , proving conclusion (1) of Proposition 8.3.

Furthermore, because $\mathrm{ray} A_t = P = \mathrm{ray} A_0$ by Proposition 11.3(3) and the filtered isomorphism $A_t \xrightarrow{\sim} A$ induces a diagonal rescaling on $\mathrm{gr} A_t = A_0 = \mathrm{gr} A$, the initial-form bijection of Lemma 3.7 identifies the ray multiplication in $\mathrm{ray} A$ with that in $\mathrm{ray} A_0$, proving conclusion (2). Finally, $W(\mathrm{ray} A) \cong W(P)$ is free abelian by Proposition 7.2, proving conclusion (3). $\square$

Proof of Proposition 8.2. We proceed by strong induction on the number $n = \dim kT$ of nonzero elements of T . The base case $n = 0$ is vacuous, and if T is disconnected, it decomposes as a disjoint union of smaller weighted multiplication tables whose linearizations are representation-finite direct factors of kT , so the claim follows componentwise.

Assume now that T is connected with n nonzero elements, and set $B = kT$. Because nonidentity elements of T have strictly positive degree $d(\ell(r)) > 0$, they span a nilpotent ideal of B whose quotient is $\prod_x k \cdot 1_x$; thus B is a basic connected representation-finite algebra of dimension n . We claim that the basis elements of T represent the intrinsic rays of B and identify $T \cong \mathrm{ray} B$ as multiplication tables. By Lemma 3.1, each local ring $R_x = B(x, x)$ is a truncated polynomial ring and each corner bimodule $B(x, y)$ is cyclic over R_x or R_y . Since the product of two elements of T is either an element of T or zero and nonidentity elements have positive degree, the radical $\mathrm{rad} R_x$ and all endpoint subbimodules of $B(x, y)$ are spanned by subsets of T . Choosing elements $q_x \in T$ and $f \in T$ that project to generators of $\mathrm{rad} R_x / (\mathrm{rad} R_x)^2$ and of the top of $B(x, y)$, we see that the powers q_x^j and the products $q_x^i f$ (or $f q_y^i$) are nonzero elements of T of strictly increasing degree, and therefore form a basis for the successive endpoint layers of R_x and $B(x, y)$. Thus every nonzero ray of B contains a unique element of T . Moreover, whenever two elements $r, s \in T$ have a nonzero product $rs \in T$, its degree $d(\ell(rs)) = d(\ell(r)) + d(\ell(s))$ is strictly smaller than the degree of any middle-radical insertion $r q_y^m s$ ($m \geq 1$); hence $rs \notin M(r, s)$, which means that the ray product in $\mathrm{ray} B$ coincides with the product in T . Thus $T \cong \mathrm{ray} B$.

If $\mathrm{char} k \neq 2$, Corollary 5.5 identifies $\mathrm{ray} B$ with $\mathrm{ray} B_0$, and if $\mathrm{char} k = 2$, the strong induction hypothesis and Lemma 11.6 applied to B likewise identify $\mathrm{ray} B$ with $\mathrm{ray} B_0$. By Proposition 7.2, the scalar weight group $W(\mathrm{ray} B_0)$ is free abelian. Since $W(T) \cong$

$W(\text{ray } B) \cong W(\text{ray } B_0)$ is free abelian, every homomorphism $W(T) \rightarrow \mathbb{Z}/2$ lifts to a homomorphism $W(T) \rightarrow \mathbb{Z}$, completing the induction. $\square$

Proof of Proposition 8.3. Decomposing A into connected components, we may assume that A is connected. When $\text{char } k = 2$, Proposition 8.2 provides the smaller-table hypothesis of Lemma 11.6, which gives (1), (2), and (3). When $\text{char } k \neq 2$, Corollary 5.5 gives (1) and (2), and Proposition 7.2 gives (3). $\square$

Together with the matrix-unit extension for nonbasic algebras in Lemma 2.1, Proposition 8.3 establishes Theorem A from the introduction.

Theorem 11.7 ([2, Sections 1.2 and 1.12]). *Let A be a finite-dimensional representation-finite algebra over an algebraically closed field k .*

- (1) *The algebra A admits a normed multiplicative basis (Definition 1.1).*
- (2) *If A is basic, this basis can be chosen to consist of one representative from each nonzero ray of A , with the primitive idempotents representing the identity rays.*
- (3) *If A is basic and $\text{char } k \neq 2$, these representatives may be chosen so that they induce an algebra isomorphism $k \text{ ray } A \xrightarrow{\sim} A$.*

Proof. When A is basic, Proposition 8.3(1) gives (1) and (2), and Corollary 5.5 gives (3). When A is nonbasic, Lemma 2.1 extends the normed multiplicative basis from its basic corner to A . $\square$

Our next theorem establishes Theorem B from the introduction: it identifies the standard form of Bretscher and Gabriel [8, Section 3] with the linearization $k \text{ ray } A$ of the ray category (as in [2, Section 9]) and records the freeness of the scalar weight group.

Theorem 11.8. *Let A be a basic representation-finite algebra over an algebraically closed field k , and form $k(\Gamma_A)$ using the mesh coefficients from the almost split sequences of $\text{mod } A$. In every characteristic:*

- (1) *Taking initial forms identifies $\text{ray } A$ with $\text{ray } A_0$. In particular, $\text{ray } A$ is an associative ray category, $A_0 \cong k \text{ ray } A$ is representation-finite, and*

$$\text{add } k(\Gamma_A) \simeq \text{mod } k \text{ ray } A.$$

- (2) *The algebra A is standard if and only if $A \cong k \text{ ray } A$.*
- (3) *For every finite ray category Q whose linearization kQ is representation-finite, the scalar weight group $W(Q)$ is free abelian and*

$$k(\Gamma_{kQ}) \simeq \text{ind}(\text{mod } kQ).$$

Proof. Parts (1) and (2) follow immediately from Proposition 8.3(2), Proposition 4.14, and Corollary 4.15.

For (3), we first check that $B = kQ$ is basic. By Definition 3.9, the identity elements $e_x = 1_x$ form a complete set of orthogonal primitive idempotents in B . If two distinct vertices $x \neq y$ gave isomorphic projective modules $e_x B \cong e_y B$, there would exist elements $f \in e_x B e_y$ and $g \in e_y B e_x$ satisfying $fg = e_x$ and $gf = e_y$. Conjugation by f and g would identify the local rings R_x and R_y and their radicals, ensuring that $M(f, g) \subseteq \text{rad } R_x$ and $M(g, f) \subseteq \text{rad } R_y$. The ray products in $\text{ray } B \cong Q$ would then satisfy $[f][g] = [e_x]$ and $[g][f] = [e_y]$, making x and y isomorphic in Q and contradicting the skeletality of Q (Definition 3.9). Thus kQ is basic. Applying (1) and (2) together with Proposition 8.3(3) to each connected component of kQ (and noting that $\text{ray}(kQ) \cong Q$ by Definition 3.9, so $(kQ)_0 \cong kQ$) shows that $W(Q)$ is free abelian and $k(\Gamma_{kQ}) \simeq \text{ind}(\text{mod } kQ)$. $\square$

We emphasize that in characteristic two, the existence of a normed multiplicative ray basis in Theorem 11.7(2) does *not* imply that A is isomorphic to $k \text{ ray } A$. Outside characteristic two, the products of the chosen ray representatives match composition in

ray A . In characteristic two, however, the deformation family (30) allows a pair of rays with zero ray product $rs = 0$ to have the nonzero product $r \cdot s = r * s = rq_y s$ in A . Because $rq_y s$ lies in the middle-radical insertion subspace $M(r, s)$ of Definition 3.4, the ray product in ray A remains zero even though the product of the basis representatives in A is nonzero. Thus the deformation changes the algebra structure from standard to nonstandard while preserving the ray category ray A .

12. FURTHER CONSEQUENCES FOR DEFORMATIONS AND MESH PARAMETERS

The module-lifting techniques developed in Part II yield further structural information about the module categories of positive filtered deformations of A_0 . In this final section we show that the lifted modules of Proposition 6.5 exhaust all indecomposable modules over any positive deformation of A_0 , establish a polynomial normal form for filtered mesh categories, and prove that the comparison map Ψ of Proposition 7.6 is an isomorphism in all positive degrees. None of the results in this section is required for the proofs of Theorems A and B.

12.1. All modules over a positive deformation

When we apply Proposition 6.5 to the full collection of normalized graded indecomposable A_0 -modules, the resulting lifted modules exhaust all indecomposable modules over the deformed algebra B . Compare the correspondence between modules over a nonstandard algebra and its standard form in [3, Theorem 1.9 and Remark 1.10].

Corollary 12.1. *Let A_0 be the projective ray algebra of Construction 2.8. Every positive filtered deformation B of A_0 is representation-finite, the lifted modules M_i^B of Proposition 6.5 form a complete set of pairwise nonisomorphic indecomposable B -modules, and $\text{gr}_{\mathcal{J}} \text{ind}(\text{mod } B) \cong \mathcal{C}$.*

Proof. Set $G = \bigoplus_i M_i^B$. At the projective objects $X_i = P_x$, we may take $M_i^B = e_x B$ to be the indecomposable projective right B -modules. Likewise, equipping the k -duals of the indecomposable projective left B -modules with their dual filtrations produces filtered injective right B -modules whose associated graded A_0 -modules are degree shifts of the graded indecomposable injectives of A_0 ; after shifting their filtrations to match our degree normalization, the uniqueness statement in Proposition 6.5 identifies them with the corresponding lifted modules M_i^B . Consequently, G contains all indecomposable projective and injective right B -modules and is therefore a generator-cogenerator for $\text{mod } B$.

Let $\Lambda_B = \text{End}_B(G)^{\text{op}}$ be the opposite endomorphism algebra of G (so that $\text{Hom}_B(G, -)$ takes values in right Λ_B -modules). By Proposition 6.5(3), the associated graded algebra $\text{gr } \Lambda_B$ is the Auslander algebra of A_0 , which has global dimension at most 2. We show by a standard filtered lifting argument that $\text{gl. dim } \Lambda_B \leq 2$ as well. Because $F^1 \Lambda_B$ is a nilpotent ideal with semisimple quotient $\Lambda_B / F^1 \Lambda_B = \prod_i k$, the simple right Λ_B -modules coincide with the degree-zero simple modules of $\text{gr } \Lambda_B$. Lifting the primitive idempotents e_i and equipping each indecomposable projective $e_i \Lambda_B$ with its induced filtration, we can lift any graded projective cover $\bar{P} \rightarrow \text{gr } X$ of the associated graded module of a finitely generated filtered Λ_B -module X to a filtered morphism $P \rightarrow X$ (where P is a direct sum of filtration-shifted copies of $e_i \Lambda_B$) by lifting the images of the homogeneous generators in each idempotent component.

Because the filtration on X is finite, successive approximation in increasing filtration degree shows that $P \rightarrow X$ is a strict filtered surjection ($F^d P \twoheadrightarrow F^d X$ for all d). Equipping $K = \ker(P \rightarrow X)$ with the induced filtration $F^d K = K \cap F^d P$, strictness implies that any element of $F^d P$ whose image lies in $F^{d+1} X$ can be modified by an element of $F^{d+1} P$ to lie in $F^d K$, so the induced sequence of associated graded modules

$$0 \longrightarrow \text{gr } K \longrightarrow \text{gr } P \longrightarrow \text{gr } X \longrightarrow 0$$

is exact. Applying this filtered projective cover construction twice to any simple Λ_B -module S produces a filtered second syzygy K_2 whose associated graded module $\text{gr } K_2$ is a graded second syzygy over $\text{gr } \Lambda_B$, and hence is a graded projective $\text{gr } \Lambda_B$ -module because $\text{gl. dim } \text{gr } \Lambda_B \leq 2$. Since $\text{gr } \Lambda_B$ is positively graded, $\text{gr } K_2$ is a direct sum of shifted copies of $e_i \text{gr } \Lambda_B$; lifting its homogeneous generators yields a filtered morphism from a direct sum of shifted copies of $e_i \Lambda_B$ to K_2 that is an isomorphism on associated graded modules, and therefore an isomorphism of Λ_B -modules. Thus $\text{proj. dim}_{\Lambda_B} S \leq 2$ for every simple Λ_B -module S , so $\text{gl. dim } \Lambda_B \leq 2$.

Finally, by Auslander's theorem, for any finitely generated right B -module X , choose the beginning of an injective copresentation $0 \rightarrow X \rightarrow I^0 \rightarrow I^1$ with $I^0, I^1 \in \text{add } G$. Applying the left-exact functor $\text{Hom}_B(G, -)$ exhibits $\text{Hom}_B(G, X)$ as the kernel of a morphism between projective Λ_B -modules, hence as a second syzygy over Λ_B , which is projective because $\text{gl. dim } \Lambda_B \leq 2$. Since G is a generator for $\text{mod } B$, the functor $\text{Hom}_B(G, -)$ restricts to an equivalence between $\text{add } G$ and $\text{proj } \Lambda_B$, forcing $X \in \text{add } G$. Because the modules M_i^B have local endomorphism rings and pairwise nonisomorphic associated graded modules, they form a complete set of pairwise nonisomorphic indecomposable B -modules, and Proposition 6.5(3) gives $\text{gr}_{\mathcal{J}} \text{ind}(\text{mod } B) \cong \mathcal{C}$. $\square$

12.2. Normal forms for filtered mesh categories

Extending Proposition 5.4 from the case where the mesh derivative D_{q-1} is surjective to the general case, we can eliminate all higher-order terms in a filtered deformation of a quadratic category except for a chosen set of representatives spanning $\text{coker } D_{q-1}$. In characteristic two, Proposition 6.1 provides canonical path representatives $(p_i)_{i \in I}$ for the mesh category $\mathcal{C}$.

Proposition 12.2. *Let $C = kQ/(r_v : v \in V)$ be a finite-dimensional graded category (or algebra) over an arbitrary field k , where Q is a finite quiver and the r_v are homogeneous quadratic relations, and let D_{q-1} be the linearized substitution map of Proposition 5.4. Fix a finite family $(p_i)_{i \in I}$ of homogeneous path tuples of degrees $d_i \geq 3$ such that, for each $q \geq 3$, the classes of the tuples p_i with $d_i = q$ span $\text{coker } D_{q-1}$. Choose N such that $C_d = 0$ for all $d \geq N$, and let $\mathfrak{m}$ denote the arrow ideal of kQ . Then every filtered category E equipped with a finite filtration and an identification $\text{gr } E = C$ admits a presentation*

$$E \cong kQ / \left(\mathfrak{m}^{N+1}, r_v - \sum_{i \in I} \lambda_i (p_i)_v : v \in V \right) \quad (31)$$

for some scalars $\lambda_i \in k$, via a filtered isomorphism that induces the identity on $\text{gr } E = C$.

Proof. We adjust the lifted arrows inductively in increasing degree $q \geq 3$. Suppose that arrow lifts have been chosen so that all terms of degree strictly between 2 and q in r_v belong to the span of the retained path tuples $(p_i)_v$ with $d_i < q$. Let $z_v \in C_q$ be the leading degree- q component of the remaining error $r_v - \sum_{d_i < q} \lambda_i (p_i)_v$. By assumption, the class of $(z_v)_{v \in V}$ in $\text{coker } D_{q-1}$ can be written as $\sum_{d_i=q} \lambda_i [p_i] + D_{q-1}(g)$ for some scalars $\lambda_i \in k$ ($d_i = q$) and a degree- $(q-1)$ arrow tuple g . Modifying the arrow lifts in degree $q-1$ by $-g$ cancels $D_{q-1}(g)$ in degree q while leaving all terms of degree less than q unchanged (since substituting a degree- $(q-1)$ correction into a quadratic relation shifts degree by $q-2$, and substituting it into a previously retained term of degree $d_i \geq 3$ shifts degree to at least $d_i + q - 2 > q$). Since the filtration of E is finite, this inductive adjustment terminates at $q = N$.

The resulting arrow lifts satisfy the relations of (31) in E and therefore induce a filtered surjection from the right-hand side of (31) onto E . Passing to associated graded categories gives a composition of graded surjections $C \twoheadrightarrow \text{gr}(\text{RHS}) \twoheadrightarrow \text{gr } E = C$ that is the identity on C , so both maps are isomorphisms of finite-dimensional categories. $\square$

When $C = \mathcal{C}$ in characteristic two, Proposition 6.1 shows that we may take $(p_i)_{i \in I}$ to be the single-path residual representatives supported on the residual periodic components of $\mathcal{C}$. Similar higher-degree perturbations of mesh relations define the deformed preprojective algebras of [4], and filtered reparametrizations of arrows appear in Kaplan's comparison of additive and multiplicative preprojective algebras [14, Lemma 3.8].

12.3. Dimensions of perturbed mesh categories

Viewing finite k -linear categories via their category algebras as in the introduction, the category algebra of $\mathcal{C} \simeq \text{ind}(\text{mod } A_0)$ has global dimension at most 2: projective resolutions of its simple modules of length at most 2 are obtained by applying representable functors to the almost split sequences and to the projective-injective sequences of Proposition 2.6(2). The next lemma shows that perturbing the homogeneous relations of any such category by higher-degree terms preserves both the dimensions of all morphism spaces and the projective resolutions of the simple modules.

Lemma 12.3. *Let k be an arbitrary field, and let $C = kQ/(r_1, \dots, r_t)$ be a finite-dimensional graded category presented by homogeneous relations r_l of degree at least 2. Assume that for each vertex i , the sequence of right C -modules obtained by factoring each relation r_l starting at i through its initial arrow is a projective resolution of the simple right module S_i of length at most 2.*

Choose N such that $C_d = 0$ for all $d \geq N$, and for each $l = 1, \dots, t$ let $f_l \in kQ$ be a linear combination of paths having the same source and target as r_l and degree strictly greater than $\deg(r_l)$. Set

$$B = kQ/(\mathfrak{m}^{N+1}, r_1 - f_1, \dots, r_t - f_t).$$

Then $\text{gr}_{\mathfrak{m}} B \cong C$, every homogeneous path basis of C lifts to a basis of B , and factoring the perturbed relations $r_l - f_l$ through their initial arrows yields projective resolutions of length at most 2 for the simple right B -modules.

Proof. Taking leading terms gives a graded surjection $C \rightarrow \text{gr}_{\mathfrak{m}} B$. Since $C_d = 0$ for $d \geq N$, it follows that $\mathfrak{m}^N B = \mathfrak{m}^{N+1} B$, and because $\mathfrak{m}^{N+1} = 0$ in B , we already have $\mathfrak{m}^N B = 0$. In particular, if any path of length $N + 1$ is factored as ap with a its initial arrow and p of length N , the suffix p already vanishes in B .

With our left-to-right path convention, the indecomposable projective right B -module at vertex i is the covariant representable module $P_i = B(i, -)$. Factoring each relation $r_l - f_l$ starting at i through its initial arrow yields an exact sequence of right B -modules

$$0 \longrightarrow K_i \longrightarrow \bigoplus_{r_l: i \rightarrow j} P_j \longrightarrow \bigoplus_{a: i \rightarrow j} P_j \longrightarrow P_i \longrightarrow S_i \longrightarrow 0, \quad (32)$$

where K_i denotes the kernel at the relation term. (Exactness at the arrow term $\bigoplus_{a: i \rightarrow j} P_j$ holds because any relation in B starting at i is a sum of expressions $v(r_l - f_l)w$ and paths of length at least $N + 1$: when v is nonempty, stripping the initial arrow of v leaves $v'(r_l - f_l)w = 0$ in B ; when v is empty, stripping the initial arrow of $r_l - f_l$ lands in the image of $\bigoplus_{r_l: i \rightarrow j} P_j$; and stripping the initial arrow of a path of length $N + 1$ leaves a path of length N , which is zero in B .)

Let H and H_B be the matrices whose (i, j) -entries are $\dim C(i, j)$ and $\dim B(i, j)$, let M and R be the adjacency matrices counting arrows $i \rightarrow j$ and relations $r_l: i \rightarrow j$, set $Z = I - M + R$, and let K be the matrix with (i, j) -entry $\dim(K_i e_j)$. Taking Euler characteristics in the assumed projective resolutions over C and in (32) over B gives the matrix identities

$$ZH = I, \quad ZH_B = I + K.$$

Hence Z is invertible with $Z^{-1} = H$, and multiplying the second identity on the left by H gives $H_B = H + HK$. Because H and K have nonnegative entries, we have $H_B \geq H$

entrywise, while the graded surjection $C \twoheadrightarrow \text{gr}_{\mathfrak{m}} B$ gives $H_B \leq H$ entrywise. Therefore $H_B = H$, $K = 0$, and the surjection $C \twoheadrightarrow \text{gr}_{\mathfrak{m}} B$ is an isomorphism. $\square$

Note that for the category algebra of $C = \mathcal{C}$ (where $C(i, j)$ sits in the (i, j) -corner), the indecomposable projective right modules are the covariant functors $\mathcal{C}(X, -)$ rather than the contravariant functors $\mathcal{C}(-, X)$ used in Section 2. Consequently, the projective resolution of the simple right $\mathcal{C}$ -module at a noninjective object X is induced by the almost split sequence $0 \rightarrow X \rightarrow E_{\tau^{-1}X} \rightarrow \tau^{-1}X \rightarrow 0$ starting at X (the leftmost map $\mathcal{C}(\tau^{-1}X, -) \rightarrow \mathcal{C}(E_{\tau^{-1}X}, -)$ is injective because $E_{\tau^{-1}X} \rightarrow \tau^{-1}X$ is an epimorphism in $\text{mod } A_0$), while at an injective object X the epimorphism of Proposition 2.6(2) gives a projective resolution of length at most 1. Thus Lemma 12.3 applies directly to $C = \mathcal{C}$.

Corollary 12.4. *If the coefficients of the higher-degree perturbation terms f_l in Lemma 12.3 are replaced by independent polynomial variables over k , then any homogeneous path basis of C lifts to a free basis of the perturbed category over the polynomial ring, and remains a basis after arbitrary base change to any commutative k -algebra.*

Proof. Using the homogeneous relations r_l of C , every path in kQ can be rewritten modulo $r_l - f_l$ as a k -linear combination of the chosen basis paths plus paths of strictly higher degree; iterating this reduction terminates at degree $N + 1$ and shows that the basis paths span the perturbed category over the polynomial ring without dividing by any polynomials. Extending scalars to the rational function field and applying Lemma 12.3 shows that the basis paths are linearly independent over the function field, and hence over the polynomial ring. Since the perturbed category is free with this basis over the polynomial ring, the same holds after any commutative base change. $\square$

12.4. The full Hochschild comparison

With Corollary 12.4 in hand, we can now prove that the injective comparison map Ψ of Proposition 7.6 is also surjective, and hence an isomorphism in every positive internal degree.

Proposition 12.5. *For every internal degree $q > 0$, the comparison homomorphism*

$$\Psi: \text{HH}^2(A_0)_q \xrightarrow{\sim} \text{coker } D_{q+1}(\mathcal{C})$$

of Proposition 7.6 is an isomorphism preserving weights in L . In particular, when $\text{char } k = 2$, we have $I_0 = I$.

Proof. Injectivity and weight compatibility were established in Proposition 7.6. To prove surjectivity, fix a tuple of homogeneous morphisms $z_v \in \mathcal{C}_{q+2}(\tau v, v)$ representing a class $[z] \in \text{coker } D_{q+1}(\mathcal{C})$, and form the perturbed category $\mathcal{C}_\varepsilon$ over $k[\varepsilon]/(\varepsilon^2)$ defined by the relations $r_v - \varepsilon z_v$ (and $\mathfrak{m}^{N+1} = 0$). By Corollary 12.4, $\mathcal{C}_\varepsilon$ is a free deformation of $\mathcal{C}$ over $k[\varepsilon]/(\varepsilon^2)$. Its projective corner is therefore a first-order deformation A_ε of A_0 , and evaluating each object X_i of $\mathcal{C}_\varepsilon$ on the projective objects yields a free lift $M_i^\varepsilon = \mathcal{C}_\varepsilon(P_-, X_i)$ of the normalized A_0 -module M_i . Giving ε internal degree $-q$ (and weight $-\chi$ when z has weight χ) makes A_ε and M_i^ε homogeneous of degree zero, so by the uniqueness in Lemma 6.6 these modules are the unique graded lifts used to define Ψ . Moreover, the evaluation map from $\mathcal{C}_\varepsilon(X_i, X_j)$ to $\text{Hom}_{A_\varepsilon}(M_i^\varepsilon, M_j^\varepsilon)$ is a map of free $k[\varepsilon]/(\varepsilon^2)$ -modules of the same rank that reduces modulo ε to the isomorphism $\mathcal{C}(X_i, X_j) \xrightarrow{\sim} \text{Hom}_{A_0}(M_i, M_j)$, so it is an isomorphism. By construction of $\mathcal{C}_\varepsilon$, the lifted arrows satisfy $r_v = \varepsilon z_v$, which means that Ψ sends the cohomology class of the corner deformation A_ε to $[z]$. Thus Ψ is surjective. In characteristic two, each residual line $k[p_i]$ ($i \in I$) of Proposition 6.1 therefore lies in the image of Ψ in weight χ_i , forcing $\text{HH}^2(A_0)_{\chi_i} \neq 0$ for all $i \in I$ and hence $I_0 = I$. $\square$

12.5. Rescaling the mesh parameters

Using the all-ones mesh normalization of Section 7 and the residual paths $(p_i)_{i \in I}$ of Proposition 6.1, let $C(\lambda)$ denote the perturbed mesh category of (31) associated to a parameter vector $\lambda = (\lambda_i)_{i \in I} \in k^I$. By Lemma 12.3, $C(\lambda)$ has the same dimension as $\mathcal{C}$ for every $\lambda \in k^I$. We next show that up to isomorphism, $C(\lambda)$ depends only on the support $\{i \in I : \lambda_i \neq 0\}$ of the parameter vector λ .

Lemma 12.6. *Assume that $\text{char } k = 2$, and let $C(\lambda)$ and $C(\lambda')$ be two perturbed mesh categories of the form (31) constructed from the all-ones mesh relations of $\mathcal{C}$ and the residual path representatives $(p_i)_{i \in I}$ of Proposition 6.1. If $\lambda_i = 0 \iff \lambda'_i = 0$ for every $i \in I$, then there is an object-preserving isomorphism $C(\lambda) \cong C(\lambda')$.*

Proof. For each $i \in I$, choose a nonzero scalar $a_i \in k^\times$. On the i -th residual periodic component of Γ_A , set the arrow scale $\mu_a = a_i$ for every internal arrow a and the mesh scale $\eta_v = a_i^2$ for every internal mesh v ; on all other periodic components, set $\mu_a = 1$ and $\eta_v = 1$. Because all mesh coefficients $c_{v,a}$ equal 1, these choices satisfy the compatibility equations $\mu_a \mu_{\sigma(a)} = \eta_v$ of (12) on all periodic components. Since the cycles of the mate permutation σ consist precisely of the arrows whose endpoints are periodic (as σ^2 acts on endpoints as $\tau^{\pm 1}$), all remaining arrows belong to open σ -chains; choosing $\eta_v = 1$ on all remaining meshes and picking an arbitrary nonzero scale on the initial arrow of each open σ -chain determines $\mu_a \in k^\times$ globally so that $\mu_a \mu_{\sigma(a)} = \eta_v$ holds everywhere.

The resulting diagonal rescaling of the arrows of Q multiplies each quadratic mesh relation r_v by η_v and multiplies the residual path p_i (which is a path of length $d_i \geq 3$ internal to the i -th residual component) by $a_i^{d_i}$ while scaling its mesh r_{v_i} by a_i^2 . Dividing the v -th relation by η_v therefore transforms the parameter λ_i according to

$$\lambda_i \mapsto \lambda_i a_i^{d_i-2}.$$

Because k is algebraically closed and $d_i - 2 \geq 1$, whenever λ_i and λ'_i are both nonzero we may choose $a_i \in k^\times$ with $a_i^{d_i-2} = \lambda'_i / \lambda_i$, yielding an object-preserving isomorphism $C(\lambda) \xrightarrow{\sim} C(\lambda')$. $\square$

12.6. Polynomial comparison of the two parameter systems

Finally, we relate the algebra deformation parameters $t = (t_i)_{i \in I}$ of our family (30) to the mesh perturbation parameters $\lambda = (\lambda_i)_{i \in I}$ of the lifted module category. Because the residual characters χ_i form part of a $\mathbb{Z}$ -basis of the weight lattice L , weight homogeneity forces the relationship between the two parameter systems to be purely diagonal and linear.

Lemma 12.7. *Let L be an abelian group, and equip the polynomial ring $k[t_1, \dots, t_s]$ with an L -grading by setting $\deg(t_i) = \chi_i \in L$. Assume that the weights $\chi_1, \dots, \chi_s$ are pairwise distinct and satisfy the nonresonance condition that no χ_j can be written as $\sum_{i=1}^s n_i \chi_i$ with integers $n_i \geq 0$ and $\sum_{i=1}^s n_i \geq 2$. Then every homogeneous polynomial $F_j(t) \in k[t_1, \dots, t_s]$ of weight χ_j with zero constant term has the linear form $F_j(t) = a_j t_j$ for some scalar $a_j \in k$.*

Proof. The nonresonance hypothesis rules out every monomial of degree at least 2 from having weight χ_j , and distinctness of the χ_i leaves t_j as the only degree-one monomial of weight χ_j . $\square$

Notice that any subset of a $\mathbb{Z}$ -basis of a free abelian group L automatically satisfies the hypotheses of Lemma 12.7.

Proposition 12.8. *Let $m_t = m + \sum_{i \in I} t_i \mu_i$ be an associative polynomial family of multiplications on A_0 over $K = k[t_i : i \in I]$ satisfying the hypotheses of Proposition 7.9 (such as the family (30), where $I_0 = I$ by Proposition 12.5). Then the normalized graded A_0 -modules M_j , their morphisms, and their mesh presentation lift over the polynomial ring*

K so that the perturbed mesh relations take the normal form (31) with linear parameter polynomials

$$F_i(t) = a_i t_i \quad (a_i \in k^\times, i \in I). \quad (33)$$

Moreover, after specializing t to any point of k^I , the specialized modules form a complete set of pairwise nonisomorphic indecomposable modules over the fiber algebra A_t .

Proof. Set $K = k[t_i : i \in I]$, and lift a $\mathbb{Z}$ -basis of $H_{\mathbb{Z}}$ to mesh weights via Lemma 7.4 so that the normalized A_0 -modules M_j carry simultaneous L -gradings compatible with the L -grading of A_0 . Assign each variable t_i internal degree $-2s_i = -\chi_i(\deg_{\mathcal{J}})$ and weight $-\chi_i \in L$, so that the deformed multiplication $m_t = m + \sum_{i \in I} t_i \mu_i$ on $A_K = A_0 \otimes_k K$ is homogeneous of degree 0 and weight 0. In each internal degree and weight, choose once and for all a k -linear right inverse to the bar differentials on their images in the module bar complexes of the M_j , and choose a k -linear splitting of the mesh derivative $D_{q-1}(\mathcal{C})$ with respect to the complement spanned by the residual paths $(p_i)_{i \in I}$. Tensoring these splittings with K makes them available over K without introducing denominators.

Keeping the underlying K -modules $M_j \otimes_k K$ and their vertex idempotent actions fixed, we lift the module actions of A_K inductively in increasing internal degree $q > 0$: when the action has been constructed up to degree $q - 1$, its degree- q associativity defect is a two-cocycle of internal degree $q > 0$ in the module bar complex tensored with K , which is killed in a single step by applying our chosen right inverse (since $\text{Ext}_{A_0}^2(M_j, M_j)_{>0} = 0$ by Lemma 6.3). Because the underlying vector spaces M_j are concentrated in a finite range of internal degrees, this inductive correction terminates after finitely many steps, yielding polynomial action matrices over K . By the same token, using $\text{Ext}_{A_0}^1(M_j, M_l)_{>0} = 0$ and our chosen right inverses, every homogeneous A_0 -module homomorphism $f \in \text{Hom}_{A_0}(M_j, M_l)_r$ ($r \geq 0$) lifts in finitely many degree-raising steps to a polynomial A_K -module homomorphism $\tilde{f}$, with $\tilde{f} = f$ at $t = 0$.

Choosing a homogeneous k -basis $f_1, \dots, f_h$ of each $\text{Hom}_{A_0}(M_j, M_l)$, we see by standard degree induction on the leading term that their lifts $\tilde{f}_1, \dots, \tilde{f}_h$ form a free K -basis of the lifted homomorphism module $\mathcal{M}(j, l) := \text{Hom}_{A_K}(M_j \otimes_k K, M_l \otimes_k K)$, and that the same remains true after specializing t to any point of k^I . In particular, at every scalar specialization, the specialized modules exhaust all indecomposable modules of the fiber algebra by Corollary 12.1.

Now apply the degree-by-degree arrow adjustment of Proposition 12.2 to the lifted arrow morphisms in $\mathcal{M}$, using our K -linear splittings of $D_{q-1}(\mathcal{C}) \otimes_k K$ to eliminate all higher-degree mesh errors except for K -multiples $F_i(t)p_i$ of the residual paths p_i . Because the procedure is triangular and terminates above the top degree of $\mathcal{C}$, the coefficients $F_i(t)$ are polynomials in K satisfying $F_i(0) = 0$, and the resulting presentation is isomorphic to $\mathcal{M}$ over K .

Finally, because every step preserves the L -grading, each polynomial $F_j(t)$ is homogeneous of weight $-\chi_j$ (canceling the weight difference χ_j between p_j and its mesh relation r_{v_j}). Reversing the signs of the weights and applying Proposition 7.5 and Lemma 12.7 shows that $F_j(t) = a_j t_j$ for some scalar $a_j \in k$. Specializing $t_j = \varepsilon$ with $\varepsilon^2 = 0$ and $t_i = 0$ for $i \neq j$ reduces the construction to the first-order lift of Lemma 6.6 for $m + \varepsilon \mu_j$, where the residual mesh error is $\varepsilon a_j [p_j] = \varepsilon \Psi([\mu_j])$. Since $[\mu_j] \neq 0$ in $\text{HH}^2(A_0)_{\chi_j}$ and Ψ is injective by Proposition 7.6, we have $a_j \neq 0$. $\square$

APPENDIX A. PERIODIC MESHES AND CORRECTION VANISHING OUTSIDE
CHARACTERISTIC TWO

In this appendix we prove the two results on periodic mesh categories used in Part I: the Dynkin-tree covering description of periodic components (Proposition 4.1, used in Section 4) and the vanishing of the mesh correction cokernels $\text{coker } D_m(\mathcal{C})$ outside characteristic two (Proposition 5.3, which completes the proof of Theorem A when $\text{char } k \neq 2$). The characteristic-two and modulo-four calculations in Appendix B reuse the tree models and path identities established here, but are not needed outside characteristic two.

A.1. Finite trees and periodic mesh categories

We first prove Proposition 4.1, stated in Subsection 4.1, by lifting sectional paths to construct a repetition quiver cover. This proof also fixes the explicit Dynkin tree models used in our subsequent cokernel calculations. Both the repetition-quiver covering and its construction via sectional paths are due to Riedtmann [18, Sections 1.5–1.6 and 2.1].

Proof of Proposition 4.1. We divide the proof into four steps.

Step 1: Sectional paths and a finite rooted tree. A path in a periodic translation quiver Q is called *sectional* if it contains no consecutive pair of arrows of the form $(\sigma a)a$. Because every term in a mesh relation is of the form $(\sigma a)a$, a sectional path can never appear as a monomial in any contextual mesh relation $ur_v w$. Consequently, the indicator functional of a sectional path annihilates the mesh ideal, so every sectional path represents a nonzero morphism in the mesh category $C(Q, c)$. Since $C(Q, c)$ is finite-dimensional, there is an integer N such that all paths of length at least N vanish in $C(Q, c)$; hence every sectional path in Q has length strictly less than N .

Fix a base vertex x_0 in the connected periodic quiver Q , and let Δ be the directed graph whose vertices are all sectional paths in Q starting at x_0 (including the trivial path at x_0), with a directed edge $p \rightarrow q$ whenever q is obtained by appending a single arrow to p . By definition, Δ is a finite tree rooted at the trivial path, oriented away from the root. Its repetition quiver $\mathbb{Z}\Delta$ has vertices (p, j) for $p \in \Delta_0$ and $j \in \mathbb{Z}$, translation $\tau(p, j) = (p, j - 1)$, and for each directed edge $p \rightarrow q$ in Δ (where $q = pa$ for an arrow a of Q) a pair of arrows

$$(p, j) \longrightarrow (q, j), \quad (q, j) \longrightarrow (p, j + 1).$$

We define a translation-quiver morphism $\pi: \mathbb{Z}\Delta \rightarrow Q$ on vertices by

$$\pi(p, j) = \tau^{-j}(\text{target } p),$$

and on the two repetition arrows associated to $q = pa$ by sending $(p, j) \rightarrow (q, j)$ to $\tau^{-j}a$ and $(q, j) \rightarrow (p, j + 1)$ to $\tau^{-j}(\sigma^{-1}a)$.

At the root $p = x_0$, the children q correspond to all outgoing arrows from x_0 . At a nonroot path p ending with an arrow a , the children q correspond to all outgoing arrows from target p except $\sigma^{-1}a$, while the returning arrow $(p, j) \rightarrow (p', j + 1)$ to the parent p' of p maps to $\tau^{-j}(\sigma^{-1}a)$. Thus π restricts to a bijection on the outgoing arrows at every vertex of $\mathbb{Z}\Delta$, and compatibility with the mate permutation σ implies that π is also bijective on incoming arrows. In particular, the image of π is a union of connected components, hence equals all of Q .

Because π is a local isomorphism on incoming and outgoing stars, paths in Q lift uniquely to $\mathbb{Z}\Delta$ once a starting vertex in $\mathbb{Z}\Delta$ is fixed. Pulling back the mesh coefficients of Q along π equips $\mathbb{Z}\Delta$ with a mesh category whose contextual mesh relations from a fixed starting vertex (p, j) project bijectively onto those of Q starting at $\pi(p, j)$. Taking the direct sum over all endpoints in $\mathbb{Z}\Delta$ therefore identifies the space of paths in $\mathbb{Z}\Delta$ starting at (p, j) , modulo lifted mesh relations, with the space of paths in $C(Q, c)$ starting at $\pi(p, j)$. In particular, all paths of length at least N vanish in the lifted mesh category on $\mathbb{Z}\Delta$.

Step 2: The tree Δ is a Dynkin diagram. We next rescale the arrows of $\mathbb{Z}\Delta$ so that the lifted mesh relations take the standard ± 1 signs of the signed preprojective algebra $\Pi(\Delta)$. Along each tree edge $u \rightarrow v$ of Δ , denote the repetition arrows by $a_j: (u, j) \rightarrow (v, j)$ and $b_j: (v, j) \rightarrow (u, j+1)$. Prescribing mesh coefficients $+1$ at (v, j) and -1 at (u, j) imposes equations of the form $\mu_{a,j}\mu_{b,j-1} = \alpha_{v,j}$ and $\mu_{a,j-1}\mu_{b,j-1} = \beta_{u,j}$ on the arrow scales μ . Taking their ratio determines $\mu_{a,j}$ recursively for all $j \in \mathbb{Z}$ from $\mu_{a,0} = 1$, and then determines every $\mu_{b,j}$ (since the index j runs over $\mathbb{Z}$, there is no periodicity obstruction, and distinct edges of Δ involve disjoint sets of arrows).

Recall that the *signed preprojective algebra* $\Pi(\Delta)$ is the quotient of the path algebra of the doubled quiver $\overline{\Delta}$ (obtained by adding a reverse arrow $a^*: v \rightarrow u$ for each tree edge $a: u \rightarrow v$) by the quadratic relations

$$\sum_{a:x \rightarrow y} aa^* - \sum_{a:y \rightarrow x} a^*a = 0 \quad (x \in \Delta_0).$$

Forgetting the integer coordinate j in $\mathbb{Z}\Delta$ identifies the rescaled mesh relations on $\mathbb{Z}\Delta$ with the defining relations of $\Pi(\Delta)$, and every path in $\overline{\Delta}$ starting at u lifts uniquely to a path in $\mathbb{Z}\Delta$ starting at $(u, 0)$. Consequently, all paths of length at least N vanish in $\Pi(\Delta)$, so $\Pi(\Delta)$ is finite-dimensional.

For each vertex $i \in \Delta_0$, set $P_i = e_i \Pi(\Delta)$ and $n_i = \dim_k P_i$, and let S_i be the simple right $\Pi(\Delta)$ -module at i . Factoring paths and relations starting at i through their initial arrows produces the exact sequence of right $\Pi(\Delta)$ -modules

$$P_i \longrightarrow \bigoplus_{j \sim i} P_j \longrightarrow P_i \longrightarrow S_i \longrightarrow 0, \quad (34)$$

where $j \sim i$ ranges over the neighbors of i in Δ . (Exactness at $\bigoplus_{j \sim i} P_j$ holds because any relation in the free path algebra starting at i is a sum of contextual relations $vr_x w$: if v is nonempty, stripping its first arrow leaves a relation in $\Pi(\Delta)$ that vanishes in P_j , while if v is empty, stripping the first arrow of $r_i w$ gives an element in the image of the leftmost map $P_i \rightarrow \bigoplus_{j \sim i} P_j$.) Letting K_i denote the kernel of the leftmost map in (34), taking dimensions yields

$$2n_i - \sum_{j \sim i} n_j = 1 + \dim_k K_i > 0 \quad (i \in \Delta_0).$$

For the symmetric matrix $M = 2I - \text{Adj}(\Delta)$ and any real vector $z = (z_i)$, expanding the quadratic form gives

$$z^\top M z = \sum_{\{i,j\} \in \Delta_1} n_i n_j \left(\frac{z_i}{n_i} - \frac{z_j}{n_j} \right)^2 + \sum_{i \in \Delta_0} \frac{(Mn)_i}{n_i} z_i^2.$$

Since $n_i \geq 1$ and $(Mn)_i = 1 + \dim_k K_i > 0$ for all i , the matrix M is positive definite. This immediately rules out any vertex of degree at least 4 (which contains a $\tilde{D}_4$ star, on which $z = (2; 1, 1, 1, 1)$ satisfies $z^\top M z = 0$) as well as two trivalent vertices (which contain a $\tilde{D}_m$ subtree, on which z equal to 2 along the connecting path and 1 at the four leaves satisfies $z^\top M z = 0$). Thus Δ is either a linear path A_n or a star with three arms of lengths $p \leq q \leq r$. In the star case, the Cartan matrix of an arm of length m has inverse entry $m/(m+1)$ at its inner endpoint, so taking the Schur complement at the central vertex gives

$$2 - \frac{p}{p+1} - \frac{q}{q+1} - \frac{r}{r+1} > 0 \iff \frac{1}{p+1} + \frac{1}{q+1} + \frac{1}{r+1} > 1,$$

which forces $p = 1$ and either $q = 1$ (D_n) or $q = 2$ with $r \in \{2, 3, 4\}$ (E_6, E_7, E_8).

Step 3: The fibers of π are deck-group orbits. Write $R = \mathbb{Z}\Delta$ and let $o = (x_0, 0)$ be the root at coordinate 0. For any vertex $z \in R$ with $\pi(z) = \pi(o)$, we can construct

a translation-quiver endomorphism $H_z: R \rightarrow R$ over π by setting $H_z(o) = z$, lifting the rooted tree edges of $\Delta \times \{0\}$ uniquely from z , and extending to all of R via τ and the mate permutation σ . Because Δ is a tree, this extension is well defined and locally bijective at every vertex, hence surjective onto R . Moreover, any translation-quiver morphism $R \rightarrow R$ over π is uniquely determined by the image of a single vertex via unique lifting of undirected walks. Choosing $z' \in R$ with $H_z(z') = o$, we have $\pi(z') = \pi(o)$, and the compositions $H_z \circ H_{z'}$ and $H_{z'} \circ H_z$ fix o and z' respectively, so H_z is an automorphism of R over π .

Consequently, the deck group Θ of translation-quiver automorphisms of R over π acts freely on R and transitively on the fiber $\pi^{-1}(\pi(o))$. For any two vertices $u, v \in R$ with $\pi(u) = \pi(v)$, projecting an undirected walk from o to u down to Q and lifting its reverse from v produces a vertex $z \in \pi^{-1}(\pi(o))$ such that $H_z(u) = v$. Thus Θ acts transitively on every fiber of π , giving $R/\Theta \cong Q$.

Step 4: The deck group Θ is cyclic. Every translation-commuting automorphism of $\mathbb{Z}\Delta$ has the form

$$(v, j) \mapsto (\rho(v), j + n_v),$$

where ρ is an undirected graph automorphism of Δ and the integer shifts $n_v \in \mathbb{Z}$ satisfy $n_v = n_u$ along a directed edge $u \rightarrow v$ if ρ preserves its orientation, and $n_v = n_u + 1$ if ρ reverses its orientation. Thus the automorphism is completely determined by ρ and its shift at a single vertex, and changing the orientation of Δ simply shifts the repetition coordinates of the vertices of $\mathbb{Z}\Delta$.

When Δ is of type D_n , E_6 , E_7 , E_8 , or A_{2n-1} , every graph automorphism of Δ fixes a canonical root vertex (the trivalent vertex or the central vertex of A_{2n-1}). Orienting Δ away from this root makes every $\rho \in \text{Aut}(\Delta)$ orientation-preserving, so n_v is constant across $v \in \Delta_0$ and the full group of translation-commuting automorphisms of $\mathbb{Z}\Delta$ is $\mathbb{Z} \times \text{Aut}(\Delta)$ (with $\mathbb{Z}$ generated by τ). Since Θ acts freely on vertices and any element in the kernel of the projection $\Theta \rightarrow \mathbb{Z}$ fixes the root $(x_0, 0)$, the projection $\Theta \rightarrow \mathbb{Z}$ is injective; since $Q = \mathbb{Z}\Delta/\Theta$ is finite, its image is a nontrivial subgroup $s\mathbb{Z}$ ($s \geq 1$). Hence $\Theta = \langle \tau^s \rho \rangle$ for some $s \geq 1$ and $\rho \in \text{Aut}(\Delta)$, proving the first case of Proposition 4.1(2).

When $\Delta = A_{2n}$, label the two halves of the tree by L_i and R_i ($1 \leq i \leq n$) from the center outward, orient the central edge as $L_1 \rightarrow R_1$, and orient both halves outward ($L_i \rightarrow L_{i+1}$ and $R_i \rightarrow R_{i+1}$). The translation-quiver automorphism

$$\eta(L_i, j) = (R_i, j), \quad \eta(R_i, j) = (L_i, j + 1)$$

satisfies $\eta^2 = \tau^{-1}$, and the edge-shift equations show that every translation-commuting automorphism of $\mathbb{Z}A_{2n}$ is a power of η . Hence $\Theta = \langle \eta^r \rangle$ for some $r \geq 1$ (where even r gives a pure translation quotient and odd r twists translation by the reflection of A_{2n}), completing the proof of Proposition 4.1. $\square$

For our later calculations on the odd quotient $\mathbb{Z}A_{2n}/\langle \eta^r \rangle$ (r odd), it is convenient to identify $L_{i,j}$ with $(i, 2j)$ and $R_{i,j}$ with $(i, 2j + 1)$. The quotient quiver then has vertices (i, j) with $1 \leq i \leq n$ and $j \in \mathbb{Z}/r\mathbb{Z}$, translation $\tau(i, j) = (i, j - 2)$, and arrows

$$\varepsilon_j: (1, j) \rightarrow (1, j + 1), \quad a_{i,j}: (i, j) \rightarrow (i + 1, j), \quad b_{i,j}: (i + 1, j) \rightarrow (i, j + 2).$$

A.2. Mesh corrections outside characteristic two

We now prove the vanishing assertion $\text{coker } D_m(\mathcal{C}) = 0$ ($m \geq 2$) of Proposition 5.3 when $\text{char } k \neq 2$. By Proposition 5.1, $\text{coker } D_m(\mathcal{C})$ decomposes as the direct sum of the cokernels of the periodic components of Γ_A (and a periodic component without arrows consists of isolated vertices with $\mathcal{C}_d = 0$ for $d \geq 1$). By Proposition 4.1, each nontrivial periodic component is modeled on a Dynkin repetition quotient $\mathbb{Z}\Delta/\Theta$. We first normalize the mesh coefficients on these periodic models (diagonal rescalings of arrows and mesh relations conjugate D_m by invertible diagonal matrices, preserving both $\text{coker } D_m$ and the brick

hypothesis). We then show that $\text{coker } D_{d-1}$ is a quotient of a *twisted trace space* of the signed preprojective algebra $\Pi(\Delta)$, and compute these twisted trace spaces for each Dynkin type. Note that our normalization, the twisted-trace surjection, and the nonvanishing results in Lemmas A.5 and A.7 hold in arbitrary characteristic; we will point out explicitly where the invertibility of 2 is used.

A.2.1. Scalar coefficients in the periodic models. Consider first a rooted periodic model $Q(\Delta, s, \rho) = \mathbb{Z}\Delta / \langle \tau^s \rho \rangle$, where Δ is oriented away from a root fixed by ρ . The orbit graph $\Delta / \langle \rho \rangle$ is again a finite rooted tree. For each ρ -orbit v of vertices in Δ , let $m_v = s|v|$ denote the period of the corresponding τ -cycle in $Q(\Delta, s, \rho)$. Along any parent-child edge orbit $u \rightarrow v$, the ρ -stabilizer of a child vertex fixes its unique parent in the tree, so $|u|$ divides $|v|$ and hence $m_u \mid m_v$.

Lemma A.1. *By rescaling arrows and mesh relations in $C(Q, c)$, any system of nonzero mesh coefficients c can be normalized as follows:*

- (1) *On $Q = \mathbb{Z}\Delta / \langle \tau^s \rho \rangle$, where Δ is a finite tree oriented away from a root fixed by ρ , the coefficients c can be rescaled to the standard ± 1 coefficients of the signed preprojective algebra $\Pi(\Delta)$.*
- (2) *On the odd quotient $\mathbb{Z}A_{2n} / \langle \eta^r \rangle$ (r odd), written in the coordinates at the end of Appendix A.1, the coefficients c can be rescaled so that every length-two mesh path has coefficient $+1$.*

Proof. For (1), label the arrows in Q lying over a parent-child edge orbit $u \rightarrow v$ by $j \in \mathbb{Z}/m_v\mathbb{Z}$ as

$$a_j: (u, j \bmod m_u) \rightarrow (v, j), \quad b_j: (v, j) \rightarrow (u, j+1 \bmod m_u).$$

Let $\mu_{a,j}, \mu_{b,j} \in k^\times$ be arrow scales and let $\eta_{v,j} \in k^\times$ scale the mesh relation at (v, j) . Transforming the given mesh coefficients into the target ± 1 coefficients requires solving equations of the form

$$\mu_{a,j} \mu_{b,j-1} = t_{v,j} \eta_{v,j}, \quad \mu_{a,j-1} \mu_{b,j-1} = t_{u,j} \eta_{u,j \bmod m_u},$$

for prescribed nonzero scalars $t_{v,j}, t_{u,j} \in k^\times$ (when $|\text{orbit } v| > |\text{orbit } u|$, several child indices j share the same parent mesh $\eta_{u,j \bmod m_u}$). Dividing the two equations gives a first-order recurrence for $\mu_{a,j}$ around the cycle $j \in \mathbb{Z}/m_v\mathbb{Z}$, which has a solution if and only if the product of the ratios around the cycle equals 1:

$$\prod_{j \in \mathbb{Z}/m_v\mathbb{Z}} \eta_{v,j} = c_{uv} \left(\prod_{j \in \mathbb{Z}/m_u\mathbb{Z}} \eta_{u,j} \right)^{m_v/m_u},$$

where $c_{uv} = \prod_j (t_{u,j}/t_{v,j}) \in k^\times$. Because the orbit graph $\Delta / \langle \rho \rangle$ is a rooted tree, we may set $\eta_{\text{root},j} = 1$ at the root and inductively choose $\eta_{v,j} \in k^\times$ down the tree so that the cycle-product condition holds on every edge orbit, after which the recurrences determine all arrow scales $\mu_{a,j}, \mu_{b,j} \in k^\times$.

For (2), on $\mathbb{Z}A_{2n} / \langle \eta^r \rangle$ (r odd), we set the scales of the central arrows ε_j equal to 1 and choose the level-1 mesh scales $\eta_{1,j}$ so that each $\varepsilon_{j-2} \varepsilon_{j-1}$ has coefficient $+1$. Along each arm edge between level i and level $i+1$, the ratio recurrence shifts the index $j \in \mathbb{Z}/r\mathbb{Z}$ by 2; because r is odd, shifting by 2 generates the entire cycle $\mathbb{Z}/r\mathbb{Z}$, so the compatibility condition simply prescribes the product $\prod_{j \in \mathbb{Z}/r\mathbb{Z}} \eta_{i+1,j}$ in terms of $\prod_{j \in \mathbb{Z}/r\mathbb{Z}} \eta_{i,j}$ and can be solved inductively for $i = 1, \dots, n-1$. $\square$

A.2.2. Rotation and twisted traces. Let $\Pi = \Pi(\Delta)$ be the signed preprojective algebra of a rooted tree Δ . Given an orientation-preserving automorphism $\theta \in \text{Aut}(\Delta)$, we define the θ -twisted trace space of Π to be the graded vector space quotient

$$T_\theta(\Pi) = \left(\bigoplus_{x \in \Delta_0} \Pi(x, \theta x) \right) / \text{span}\{uv - v\theta(u) : u \in \Pi(x, y), v \in \Pi(y, \theta x)\}. \quad (35)$$

In $T_\theta(\Pi)$, the relation $[uv] = [v\theta(u)]$ allows one to cut a path uv from x to θx and move its prefix u to the end after applying θ ; in particular, $[w] = [\theta(w)]$ for every $w \in \Pi(x, \theta x)$. When $\theta = \text{id}$, $T_{\text{id}}(\Pi) = \Pi/[\Pi, \Pi]$ is the ordinary cyclic trace space of Π .

Lemma A.2. *Let $Q = Q(\Delta, s, \rho) = \mathbb{Z}\Delta/\langle \tau^s \rho \rangle$ be a rooted periodic mesh model equipped with the signed preprojective mesh coefficients of $\Pi = \Pi(\Delta)$, and let D_{d-1} be the mesh derivative (9) in path degree $d \geq 3$.*

- (1) *The cokernel $\text{coker } D_{d-1}$ vanishes unless d is even and s divides $d/2 - 1$.*
- (2) *When d is even and $s \mid (d/2 - 1)$, set $l = (1 - d/2)/s$ and $\theta = \rho^l$. Then there is a canonical surjection*

$$T_\theta(\Pi)_d \twoheadrightarrow \text{coker } D_{d-1}.$$

Proof. A path of length d in Q from $\tau(x, j) = (x, j - 1)$ to (x, j) lifts to a path in $\mathbb{Z}\Delta$ from $(x, j - 1)$ to $(\rho^l x, j - sl)$ for some integer $l \in \mathbb{Z}$. Because x and $\rho^l x$ lie at the same distance from the root of Δ , the lifted path takes equal numbers of steps away from the root (which preserve the second coordinate) and toward the root (which increment the second coordinate by 1). Hence d must be even and $(j - 1) + d/2 = j - sl$, which gives $d/2 - 1 = -sl$; this proves (1) and shows that the lifted path projects in $\Pi(\Delta)$ to a path of length d from x to $\theta x = \rho^l x$.

In the signed mesh presentation on Q , modifying an arrow a by a parallel path w of length $d - 1$ adds $c[wa] + c'[(\tau a)w]$ to $\text{im } D_{d-1}$, where c and c' are the two mesh coefficients on the tree edge underlying a . Because c and c' have opposite signs $\{+1, -1\}$, this identifies $[wa] = [(\tau a)w]$ in $\text{coker } D_{d-1}$. Iterating this single-arrow rotation d times shows that $[w] = [\tau w]$, so the class in $\text{coker } D_{d-1}$ of a lifted path depends only on its underlying path in $\Pi(\Delta)$ from x to θx , independently of the repetition coordinate j (and invariant under ρ by the deck relation). Moreover, cutting a path uv from $(x, j - 1)$ to $(\theta x, j - sl)$ and rotating the prefix u to the end via $[uv] = [v(\tau^{-1}u)]$ in Q corresponds upstairs in $\Pi(\Delta)$ to $[uv] = [v\theta(u)]$. Since contextual mesh relations in $\Pi(\Delta)$ lift to contextual mesh relations in Q , the projection $T_\theta(\Pi)_d \rightarrow \text{coker } D_{d-1}$ is well defined and surjective. $\square$

To compute $T_\theta(\Pi)$ for star-shaped trees Δ , we first express closed paths at the central vertex o in terms of the length-two return loops along its arms, following [21, Sections 11.3 and 13.1] and incorporating the arm-permuting twist θ .

Lemma A.3. *Let Δ be a star tree rooted at its central vertex o , with arms of lengths $L_1, \dots, L_t$, and for each $i = 1, \dots, t$ let $x_i \in \Pi(o, o)_2$ denote the length-two closed path that steps out along the i -th arm and immediately returns to o . Then the corner algebra $\Pi(o, o)$ is generated by $x_1, \dots, x_t$ and is a quotient of*

$$k\langle x_1, \dots, x_t \rangle / \left(\sum_{i=1}^t x_i, x_1^{L_1+1}, \dots, x_t^{L_t+1} \right). \quad (36)$$

Proof. Label the vertices along the i -th arm as $v_0 = o, v_1, \dots, v_{L_i}$. At any internal vertex v_{j-1} ($j \geq 2$), the mesh relation replaces the outward backtrack $v_{j-1} \rightarrow v_j \rightarrow v_{j-1}$ (up to sign) by the inward backtrack $v_{j-1} \rightarrow v_{j-2} \rightarrow v_{j-1}$, strictly decreasing the sum of the distances from o along the path. Iterating this replacement reduces every closed path at o to a polynomial in $x_1, \dots, x_t$. The mesh relation at o gives $\sum_{i=1}^t x_i = 0$, while pushing $x_i^{L_i+1}$ out along the i -th arm reaches the leaf relation at v_{L_i} and vanishes. $\square$

Lemma A.4. *Let Δ be a star tree with center o , and let $\theta \in \text{Aut}(\Delta)$ be an automorphism fixing o and permuting the arms. In every positive degree, $T_\theta(\Pi)_+$ is spanned by the classes of positive-degree words in the arm loops $x_1, \dots, x_t$ at o . Moreover, if the i -th arm is fixed pointwise by θ , then*

$$[x_i^q] = 0 \quad \text{in } T_\theta(\Pi) \quad \text{for all } q \geq 1. \quad (37)$$

Proof. If a path w from v to θv passes through the center o , writing $w = uv$ with $u: v \rightarrow o$ and $v: o \rightarrow \theta v$ gives $[uv] = [v\theta(u)]$ in $T_\theta(\Pi)$, which is a closed path at $\theta o = o$ and hence a linear combination of words in $x_1, \dots, x_t$ by Lemma A.3.

If instead w never visits o , then w is supported entirely on a single arm of Δ . Because w goes from v to θv , that arm must be fixed pointwise by θ , so w is a closed path and twisted rotation on that arm is ordinary cyclic rotation. Rotating w so that it begins at a vertex v_j of minimum depth along w , we see that its two incident steps at v_j form an inward-pointing turn $v_{j+1} \rightarrow v_j \rightarrow v_{j+1}$; applying the mesh relation at v_{j+1} replaces this turn by $v_{j+1} \rightarrow v_{j+2} \rightarrow v_{j+1}$ (or by 0 at the leaf), strictly increasing the sum of the vertex depths along the cyclic word. Since the depth sum of a word of fixed length is bounded above, repeating this step eventually pushes the word off the leaf and shows that $[w] = 0$. Applying the exact same outward-pushing argument to the loop power x_i^q on any θ -fixed arm yields (37). $\square$

A.2.3. *Types A and D.* Assume now that $\text{char } k \neq 2$. We apply Lemma A.2 and Lemma A.4 to rule out all nonzero classes in $\text{coker } D_{d-1}$ ($d \geq 3$) for the rooted models of types A and D:

- *Type A with trivial twist* ($\theta = \text{id}$). This covers all pure translation quotients of type A, including $\mathbb{Z}A_{2n}/\langle \eta^r \rangle$ for even r (since $\eta^r = \tau^{-r/2}$). Rooting the path A_n at one of its endpoints makes it a star with a single arm, so $\Pi(o, o)$ is generated by x_1 subject to $x_1 = 0$ by Lemma A.3; hence $T_{\text{id}}(\Pi)_+ = 0$.
- *Type A_{2n-1} with reflection* ($\theta = \rho \neq \text{id}$). Rooting A_{2n-1} at its central vertex gives two arms of length $n-1$ with loops $x_1 = x$ and $x_2 = -x$ exchanged by $\theta(x) = -x$. For every $q \geq 1$, twisted rotation of one factor x gives $[x^q] = [x^{q-1}\theta(x)] = -[x^q]$, so $2[x^q] = 0$ and hence $[x^q] = 0$ because $\text{char } k \neq 2$.
- *Type D_n with $\theta = \text{id}$ or the short-arm transposition.* Let x and y be the loops along the two short arms of length 1, and let $z = -x - y$ be the loop along the long arm of length $n-3$. Since $x^2 = y^2 = 0$, the corner $\Pi(o, o)$ is spanned by alternating words in x and y .
 - When $\theta = \text{id}$, any alternating word of odd length $2q+1 \geq 3$ has the same letters at its two ends and therefore rotates to a word containing $x^2 = 0$ or $y^2 = 0$, while $[x] = [y] = 0$ by (37). In even word length $2q$, cyclic rotation identifies $[(xy)^q] = [(yx)^q]$, while (37) applied to the fixed long arm gives $0 = [z^{2q}] = [(x+y)^{2q}] = [(xy)^q] + [(yx)^q] = 2[(xy)^q]$. Since $\text{char } k \neq 2$, we obtain $[(xy)^q] = 0$.
 - When θ interchanges x and y , any alternating word of even length $2q$ twisted-rotates to a word containing $x^2 = 0$ or $y^2 = 0$ (for instance, $[(xy)^q] = [(yx)^{q-1}y\theta(x)] = [(yx)^{q-1}y^2] = 0$). In odd word length $2q+1$, the two alternating words $w = x(yx)^q$ and $w' = y(xy)^q$ satisfy $[w] = [\theta(w)] = [w']$, while the fixed long arm gives $0 = [(-z)^{2q+1}] = [(x+y)^{2q+1}] = [w] + [w'] = 2[w]$, so again $[w] = 0$.
- *Type D_4 with a three-cycle θ .* When $\Delta = D_4$, all three arms have length 1, so $x^2 = y^2 = (x+y)^2 = 0$, which gives $yx = -xy$ and kills all words of word degree 3 (path degree 6). In word degree 2 (path degree 4), if $\theta(x) = y$, then $[xy] = [y\theta(x)] = [y^2] = 0$, and if $\theta^2(y) = x$, then $[yx] = [x\theta^2(y)] = [x^2] = 0$. Thus $T_\theta(\Pi)_d = 0$ for all $d \geq 3$.

A.2.4. *The odd quotients of even A.* Consider next the odd quotient $Q = \mathbb{Z}A_{2n}/\langle \eta^r \rangle$ (r odd), normalized by Lemma A.1(2) so that every mesh path of length 2 has coefficient +1 in the coordinates (i, j) ($1 \leq i \leq n, j \in \mathbb{Z}/r\mathbb{Z}$) of Appendix A.1. Any path from $\tau(i, j) = (i, j - 2)$ to (i, j) takes equal numbers of outward steps a (which change j by 0) and inward steps b (which change j by +2); since each ε step changes j by +1, the total change in the second coordinate $j \in \mathbb{Z}/r\mathbb{Z}$ equals the path length d . Thus a path of length d from $\tau(i, j)$ to (i, j) can exist only when $d \equiv 2 \pmod{r}$.

Because all mesh coefficients equal +1, the column of D_{d-1} corresponding to any arrow u imposes the relation $[wu] + [(\tau u)w] = 0$ in coker D_{d-1} , so rotating a path by one arrow negates its class in coker D_{d-1} while preserving its cyclic sequence of levels $i \in \{1, \dots, n\}$. Given any path of length d from τv to v that reaches a maximum level $i + 1 > 1$, the step into level $i + 1$ and the step out of it must be $a_i b_i$ (as there are no level-preserving arrows above level 1). Rotating the path if necessary so that $a_i b_i$ does not straddle the cut, we may apply the mesh relation at level i to replace $a_i b_i$ by $-b_{i-1} a_{i-1}$ when $i > 1$ (or by $-\varepsilon \varepsilon$ when $i = 1$), strictly decreasing the sum of the levels along the path. Repeating this reduction expresses every class in coker D_{d-1} as a linear combination of the classes $w_j = [\varepsilon_{j-2} \varepsilon_{j-1} \cdots \varepsilon_{j+d-3}]$ of pure level-1 paths of length d ending at $(1, j)$ ($j \in \mathbb{Z}/r\mathbb{Z}$).

Since $d \equiv 2 \pmod{r}$, the path of length $d - 1$ formed by $d - 1$ consecutive ε -arrows starting at $(1, j - 1)$ has the same source and target as $\varepsilon_{j-1}: (1, j - 1) \rightarrow (1, j)$. Evaluating D_{d-1} on this arrow correction in the two root meshes containing $\varepsilon_{j-2} \varepsilon_{j-1}$ and $\varepsilon_{j-1} \varepsilon_j$ yields

$$w_{j-1} + w_j = 0 \quad (j \in \mathbb{Z}/r\mathbb{Z}).$$

Because r is odd, going all the way around the r -cycle gives $w_j = (-1)^r w_j = -w_j$, so $2w_j = 0$ and hence $w_j = 0$ whenever $\text{char } k \neq 2$ (and for $n = 1$ the root mesh $\varepsilon_{j-2} \varepsilon_{j-1} = 0$ already kills all paths of length $d \geq 2$).

A.2.5. *Type E_6 and the small-period exclusion.* For $\Delta = E_6$, the three arms have lengths 1, 2, 2. Let x be the loop along the short arm and y, z the loops along the two arms of length 2, so that $x + y + z = 0$ and $x^2 = y^3 = z^3 = 0$. Assuming $\text{char } k \neq 2$, set $v = y - z = x + 2y$, so that $2y = v - x$ and $2z = -v - x$. Adding and subtracting $8y^3 = 0$ and $8z^3 = 0$ transforms the presentation of $\Pi(o, o)$ into

$$x^2 = 0, \quad v^3 = -xv, \quad v^2x + vxv + xv^2 = 0. \quad (38)$$

From (38) one checks directly that $\Pi(o, o)$ is spanned in each word degree $q = d/2 \geq 1$ by the following words:

Word degree q	Spanning words
1	x, v
2	xv, vx, v^2
3	xvx, xv^2, vxv
4	$\psi_4 = xv xv, \psi'_4 = vx v^2$
5	$\psi_5 = xv xv^2$
≥ 6	0.

Indeed, multiplying (38) by x gives $xv^2x = -\psi_4$, $xvxv = \psi_4$, and $xv^3 = v^3x = 0$; hence $\psi_4x = x(xvxv) = 0$, $\psi_4v = \psi_5$, $\psi'_4x = -v\psi_4 = -\psi_5$, $\psi'_4v = vxv^3 = 0$, $\psi_5v = xv xv^3 = 0$, and $\psi_5x = xv x(v^2x) = -xvx(vxv + xv^2) = -\psi_4xv = 0$.

When $\theta = \rho$ is the nontrivial reflection of E_6 , θ fixes x and negates v . In $T_\rho(\Pi)$, every word containing an odd number of v 's satisfies $[w] = [\theta(w)] = -[w]$ and therefore vanishes since $\text{char } k \neq 2$. Among words with an even number of v 's, $[x] = 0$ by (37), $[v^2] = [v\theta(v)] = -[v^2] = 0$, the last relation of (38) gives $0 = [v^2x] + [vxv] + [xv^2] = [xv^2] - [xv^2] + [xv^2] = [xv^2]$ in word degree 3, and $[\psi_4] = [(xv)\theta(xv)] = -[\psi_4] = 0$ in word degree 4. Thus $T_\rho(\Pi)_+ = 0$.

When $\theta = \text{id}$ (ordinary trace), all pure arm powers vanish by (37). Expanding $0 = 4[y^2] = [(v-x)^2]$ and $0 = 4[z^2] = [(v+x)^2]$ gives $[xv] = [vx] = [v^2] = 0$ in word degree 2. In word degree 3, $[xvx] = [x^2v] = 0$ and $[v xv] = [xv^2]$, while the last relation of (38) gives $3[xv^2] = 0$. In word degree 4, $[\psi_4] = -[xv^2x] = -[x^2v^2] = 0$ and $[\psi'_4] = [xv^3] = 0$, and in word degree 5, $[\psi_5] = -[\psi'_4x] = -[x\psi'_4] = -[\psi_5] = 0$. Consequently, the only possible nonzero class in $T_{\text{id}}(\Pi)_+$ occurs in word degree 3 (path degree $d = 6$) when $\text{char } k = 3$, where it is spanned by $[xv^2]$.

To show that even this degree-6 class in characteristic 3 cannot occur in a periodic component containing a brick, we use the following characteristic-free lemma, which will also be needed for E_7 and E_8 .

Lemma A.5. *Over every field k , every vertex of $\Pi(D_4)$ admits a nonzero closed path of path degree 4. Consequently, the pure translation mesh quotients $\mathbb{Z}D_4/\langle\tau^s\rangle$ of periods $s = 1$ and $s = 2$ contain no bricks.*

Proof. Orient the three arms of D_4 outward from the center 0 to the leaves 1, 2, 3 with arrows $a_i: 0 \rightarrow i$ and $b_i: i \rightarrow 0$, and set $x_i = a_i b_i$. The defining relations of $\Pi(D_4)$ are $b_i a_i = 0$ ($i = 1, 2, 3$) and $x_1 + x_2 + x_3 = 0$. At the center 0, the paths of length 4 are the six products $x_i x_j$ ($i \neq j$), subject only to the relations $x_i(x_1 + x_2 + x_3) = 0$ and $(x_1 + x_2 + x_3)x_j = 0$; in the order (12, 13, 21, 23, 31, 32), their classes in $\Pi(D_4)_4$ are $(z, -z, -z, z, z, -z)$ with $z = x_1 x_2 \neq 0$. Likewise, at each leaf i , the two length-4 closed paths $b_i a_j b_j a_i$ ($j \neq i$) are subject only to the single relation $\sum_{j \neq i} b_i a_j b_j a_i = 0$, so each is nonzero in $\Pi(D_4)_4$.

In $\mathbb{Z}D_4$, each of these length-4 closed paths increases the repetition coordinate by 2, and therefore projects in $\mathbb{Z}D_4/\langle\tau^s\rangle$ for $s \in \{1, 2\}$ to a nonzero radical endomorphism at every vertex, ruling out the existence of any brick. $\square$

Returning to E_6 in path degree $d = 6$, Lemma A.2(1) requires $s \mid (6/2 - 1) = 2$, so $s \in \{1, 2\}$. Deleting the outer vertices of the two long arms of E_6 at every repetition coordinate leaves a translation-invariant subquiver isomorphic to a D_4 quotient of period s , whose mesh category is the object quotient of the E_6 mesh category by (2). For a pure translation quotient ($s = 1$ or 2), Lemma A.5 shows that this D_4 quotient has no bricks; for the reflected quotient $\mathbb{Z}E_6/\langle\tau^s \rho\rangle$, $s = 2$ gives an odd twist $l = -1$ where $T_\rho(\Pi)_6 = 0$, while $s = 1$ gives a D_4 quotient with $(\tau\rho)^2 = \tau^2$, on which the degree-4 closed paths of Lemma A.5 again close up to form nonzero radical endomorphisms at every vertex. Thus no periodic E_6 component containing a brick has a nonzero correction class.

A.2.6. The remaining exceptional traces. We treat $\Delta = E_7$ and $\Delta = E_8$ simultaneously. Let x, y, z denote the length-two loops at the center o along the long, medium, and short arms, respectively (note that x now denotes the long-arm loop, whereas in E_6 it denoted the short-arm loop). By Lemma A.3, their relations in $\Pi(o, o)$ are

$$x + y + z = 0, \quad x^m = y^3 = z^2 = 0, \quad \text{where } m = \begin{cases} 4 & \text{for } E_7, \\ 5 & \text{for } E_8. \end{cases}$$

Eliminating $z = -x - y$, we see from Lemmas A.3 and A.4 that the positive-degree part of the ordinary trace space $T_{\text{id}}(\Pi)_+$ is a quotient of the positive-degree part of T_m , where for any commutative coefficient ring R we define the graded R -algebra R_m and its R -module trace quotient T_m by

$$\begin{aligned} R_m &= R\langle x, y \rangle / (x^m, y^3, (x+y)^2), \\ T_m &= R_m / ([R_m, R_m] + \text{span}_R\{x^q, y^q, (x+y)^q : q \geq 1\}), \end{aligned} \tag{39}$$

with word degree in x, y equal to half the path degree in $\Pi(\Delta)$. We first compute a spanning set and torsion relations for $(T_5)_+$ over $\mathbb{Z}$; this integral calculation reproduces

the torsion groups of [21, Theorem 13.1.1] and will be used both here (for $\text{char } k \neq 2$) and in Appendix B (for $\text{char } k = 2$ and $\mathcal{O} = \mathbb{Z}/4$).

Lemma A.6. *Let R_m and T_m be defined as in (39) over an arbitrary commutative ring, and write $[w]$ for the class of a word w in T_m .*

- (1) *Over any commutative ring, set $a = yx^3$, $b = yx^4$, and $t_j = [yx^j]$. The positive-degree part $(T_5)_+$ is spanned by the elements*

$$t_1, t_2, t_3, t_4, \quad [a^2], \quad [ab^r] \quad (r \geq 1),$$

which satisfy the torsion relations

$$\begin{aligned} 2t_1 = 2t_3 = 0, \quad 3t_2 = 0, \quad 5t_4 = 0, \\ 2[a^2] = 0, \quad 3[ab] = 0, \quad 2[ab^r] = 0 \quad (r \geq 2). \end{aligned} \tag{40}$$

Moreover, for every nonempty word W in the blocks a and b ,

$$[a^2W] = 0, \quad [abW] = -[baW], \quad [b^q] = 0 \quad (q \geq 2), \tag{41}$$

and T_4 is the quotient of T_5 obtained by setting $x^4 = 0$.

- (2) *If 2 is invertible in the coefficient ring, then $(T_5)_+$ is spanned by*

$$[yx^2], \quad [yx^4], \quad [(yx^3)(yx^4)],$$

which are annihilated by 3, 5, and 3, respectively, and $(T_4)_+$ is spanned by $[yx^2]$, annihilated by 3.

Proof. (1) In R_5 over $\mathbb{Z}$, expanding $(x + y)^2 = 0$ and $y^3 = 0$ yields the four identities

$$\begin{aligned} y^2 &= -yx - xy - x^2, \\ yxy &= xyx + x^3, \\ yx^2y &= -yx^3 - xyx^2 - x^2yx - x^3y - 2x^4, \\ 0 &= F := yx^3yx + yx^4y + xyx^3y - x^2yx^3 - x^3yx^2. \end{aligned} \tag{42}$$

(Indeed, substituting the first identity into $y^2 \cdot y = 0$ gives the second; expanding $y^2(xy) = (y^2x)y$ gives the third; and equating $(yxy)x^2y = yx^3y + x^5y$ with $yx(yx^2y) = -yx^5 + x^2yx^3 + x^3yx^2 + x^6 - yx^3yx - yx^4y$ using $x^5 = 0$ gives the fourth.)

Any cyclic word in x and y can be described by the lengths of the x -gaps between consecutive occurrences of y . Whenever a gap has length 0, 1, or 2, applying the first, second, or third identity in (42) rewrites the word in terms of words with strictly fewer occurrences of y . Because $x^5 = 0$, iterating this reduction expresses every class in $(T_5)_+$ as a linear combination of words with at most one y (namely $t_j = [yx^j]$ for $1 \leq j \leq 4$, as $[x^q] = 0$) and cyclic words in the two blocks $a = yx^3$ and $b = yx^4$.

Set $c = yx^2$. Note first that any word containing $c = yx^2$ flanked on both sides by a or b (so that the adjacent x -gaps have lengths $p, q \geq 3$) has trace zero: rotating it to the form $yx^2yx^qUx^p$ and expanding yx^2y via the third identity in (42) produces five terms, three of which contain $x^5 = 0$ and the other two of which begin with x^2 or x^3 and therefore contain $x^{p+2} = 0$ or $x^{p+3} = 0$ after cyclic rotation.

Using the relation $F = 0$ from (42), we now establish (41):

- For any nonempty word W in $\{a, b\}$, taking the trace of $Fx^2W = 0$ gives

$$[a^2W] + [bcW] + [xacW] - [x^3bW] = 0.$$

Here $[bcW] = 0$ and $[xacW] = 0$ by the flanked- c observation (or because rotating the initial x to the end of W produces $x^5 = 0$), and rotating x^3 to the end of $[x^3bW]$ likewise produces $x^6 = 0$ or $x^7 = 0$; hence $[a^2W] = 0$. When W is empty, $0 = [Fx^2] = 2[a^2] + [bc]$, and multiplying the third identity of (42) by x^4 gives $[bc] = [yx^2yx^4] = 0$, so $2[a^2] = 0$.

- Taking the trace of $Fx^3W = 0$ gives

$$[abW] + [baW] + [xaaW] = 0.$$

When W is nonempty, rotating the initial x of $[xaaW]$ to the end either creates $x^5 = 0$ (if W ends in b) or gives $[a^2W'b] = 0$ (if $W = W'a$); thus $[abW] = -[baW]$. When W is empty, all three terms equal $[ab]$, giving $3[ab] = 0$.

- Taking the trace of $Fx^4b^r = 0$ ($r \geq 0$) gives $[b^{r+2}] + [xab^{r+1}] = 0$; rotating the initial x of the second term to the end creates $x^5 = 0$, so $[b^q] = 0$ for all $q \geq 2$.

From (41), in any word in $\{a, b\}$ of block length at least 3 containing at least two a 's, anticommuting a and b brings two a 's adjacent and kills the trace via $[a^2W] = 0$; in a word with a single a and $r \geq 2$ copies of b , anticommuting a all the way through b^r and cyclically rotating back gives $[ab^r] = (-1)^r[b^ra] = (-1)^r[ab^r]$ and also $[ab \cdot b^{r-1}] = -[bab^{r-1}] = -[ab^r]$, so $2[ab^r] = 0$.

Finally, for $t_j = [yx^j]$ ($1 \leq j \leq 4$), taking the trace of the first identity in (42) gives $2t_1 = 0$; comparing the trace of the second identity with that of y^2x gives $3t_2 = 0$; comparing the trace of the third identity with that of y^2x^2 gives $2t_3 = 0$; and taking the trace of the third identity multiplied by x gives $t_4 = -4t_4$, or $5t_4 = 0$. This proves (1), and (2) follows immediately on inverting 2. $\square$

To exclude the remaining periods for E_7 and E_8 in characteristics 3 and 5, we establish a lower bound on the dimensions of path spaces in $\Pi(\Delta)$. For any finite graph Δ with adjacency matrix M , define the Chebyshev-type integer matrices U_d ($d \geq 0$) by the recurrence

$$U_0 = I, \quad U_1 = M, \quad U_{d+1} = MU_d - U_{d-1}.$$

If $U_0, U_1, \dots, U_N$ have nonnegative entries, then over every field k we have the entrywise lower bound

$$(\dim_k e_i \Pi(\Delta)_N e_j)_{i,j} \geq U_N. \quad (43)$$

Indeed, let $H(t) = \sum_{d \geq 0} (\dim_k e_i \Pi(\Delta)_d e_j) t^d$ be the matrix Hilbert series of $\Pi(\Delta)$. Recording the graded dimensions of the kernels K_i of the leftmost maps in the graded exact sequences (34) in a power series $K(t)$ with nonnegative matrix coefficients gives

$$(I - tM + t^2I)H(t) = I + K(t) \implies H(t) = \left(\sum_{d \geq 0} U_d t^d \right) (I + K(t)),$$

whose degree- N coefficient is $U_N + \sum_{r=1}^N U_{N-r} K_r \geq U_N$ whenever $U_0, \dots, U_N \geq 0$.

Using (43), we verify that every vertex of $\Pi(E_7)$ has a nonzero closed path in the top socle degree $d = 16$ (compare [14, Section 6.2]).

Lemma A.7. *Over every field k , every vertex of $\Pi(E_7)$ admits a nonzero closed path of path degree 16.*

Proof. Ordering the vertices of E_7 according to its bipartition (with the three vertices at odd distance from the center followed by the four vertices at even distance), the off-diagonal block B of the adjacency matrix M and its products $S = BB^\top$ and $T = B^\top B$ are

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad S = BB^\top = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \quad T = B^\top B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Setting $f(X) = X^3 - 6X^2 + 9X - 3I$ and $w = (-1, 0, 1, -1)^\top$, direct matrix multiplication gives $f(S) = 0$, $f(T) = -ww^\top$, and $Tf(T) = 0$. Writing $U_{2j} = \text{diag}(F_j(S), F_j(T))$, the recurrence $U_{d+1} = MU_d - U_{d-1}$ yields

$$F_0 = I, \quad F_1(X) = X - I, \quad F_2(X) = X^2 - 3X + I,$$

$$\begin{aligned} F_3(S) &= F_2(S) + I, & F_3(T) &= F_2(T) + I - ww^\top, \\ F_4(S) &= 2(S - I), & F_4(T) &= 2(T - I) + ww^\top. \end{aligned}$$

Explicit evaluation shows that all entries of F_1, F_2, F_3, F_4 on S and T are nonnegative:

$$\begin{aligned} F_2(S) &= \begin{pmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}, & F_2(T) &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \\ F_3(T) &= \begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, & F_4(T) &= \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 2 & 0 \\ 1 & 2 & 3 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}. \end{aligned}$$

Likewise, the upper-right blocks of the odd matrices U_1, U_3, U_5, U_7 are B , $(S - 2I)B$, $(S^2 - 4S + 3I)B$, and $(S - I)B$, which are nonnegative because

$$(S - 2I)B = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad S^2 - 4S + 3I = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Thus $U_0, \dots, U_8$ are entrywise nonnegative. Since $U_9 = MU_8 - U_7 = U_7$, the reversibility of the recurrence $U_{d-1} = MU_d - U_{d+1}$ implies by induction that $U_{8+j} = U_{8-j}$ for $0 \leq j \leq 8$. In particular, $U_0, \dots, U_{16}$ are all entrywise nonnegative and $U_{16} = U_0 = I$, so the claim follows from (43). $\square$

We can now complete the proof of Proposition 5.3 for E_7 and E_8 outside characteristic two. By Lemma A.6(2), the only possible nonzero classes in $(T_4)_+$ and $(T_5)_+$ when 2 is invertible are $[yx^2]$ (word degree 3, path degree $d = 6$, in characteristic 3), $[yx^4]$ (word degree 5, path degree $d = 10$, for E_8 in characteristic 5), and $[(yx^3)(yx^4)]$ (word degree 9, path degree $d = 18$, for E_8 in characteristic 3). By Lemma A.2(1), path degrees $d = 6, 10, 18$ require the period s to divide $d/2 - 1 = 2, 4, 8$, respectively.

However, periods $s = 1$ and $s = 2$ are ruled out by deleting the outer vertices to pass to $\mathbb{Z}D_4/\langle\tau^s\rangle$ and applying Lemma A.5. For periods $s = 4$ and $s = 8$, the degree-16 closed paths at every vertex of $\Pi(E_7)$ from Lemma A.7 advance the repetition coordinate by 8, and therefore project to nonzero radical endomorphisms at every vertex of $\mathbb{Z}E_7/\langle\tau^s\rangle$ (which is also an object quotient of $\mathbb{Z}E_8/\langle\tau^s\rangle$ obtained by deleting the terminal vertex of the long arm). Thus no periodic component of type E_7 or E_8 containing a brick can have period dividing 8. Since E_7 and E_8 have no nontrivial graph automorphisms, this completes the proof of Proposition 5.3.

APPENDIX B. RESIDUAL AND INTEGRAL MESH CORRECTION SPACES

In this appendix we complete the periodic trace calculations required for the characteristic-two and integral branches of the main argument. Appendix B.1 establishes Proposition 6.1 by bounding the positive-degree cokernels $\text{coker } D_m(\mathcal{C})$ ($m \geq 2$) over a field of characteristic two, and Appendix B.2 lifts these calculations to $\mathcal{O} = \mathbb{Z}/4\mathbb{Z}$ to prove Lemma B.2 (used in Section 10). Throughout, we retain the periodic tree models and trace reductions of Appendix A.

B.1. Bounds for the characteristic-two correction spaces

Throughout this subsection we assume $\text{char } k = 2$ and prove Proposition 6.1. We show that each periodic component of Γ_A contributes at most one correction line across all degrees $m \geq 2$, spanned by the class of a single row path. Defining I to be the set of periodic components for which this cokernel line is actually nonzero then yields Proposition 6.1 directly, without requiring a classification of which periodic components occur in Γ_A or a separate nonvanishing proof for each model.

As in Appendix A, all vertex deletions below retain τ -invariant subsets of vertices; by (2), the resulting object quotients are the mesh categories of the retained subquivers, allowing us to restrict nonzero paths from smaller models to rule out small periods under the brick hypothesis.

B.1.1. Trees of type A. In the preprojective category $\Pi(A_M)$, number the vertices $1, \dots, M$ from left to right and denote the rightward and leftward steps by s_+ and s_- . The interior mesh relations commute s_+s_- with s_-s_+ , while the endpoint backtracks at 1 and M vanish. Sorting the steps of any path starting at vertex i shows that the paths

$$s_+^a s_-^b \quad (0 \leq a \leq M-i, 0 \leq b \leq i-1) \quad (44)$$

span the paths from i , because any step pushing a past $M-i$ or b past $i-1$ can be commuted to an endpoint backtrack and vanishes. To see that the paths (44) are linearly independent, consider the representation of $\Pi(A_M)$ with basis vectors $v_{a,b}$ ($0 \leq a \leq M-i$, $0 \leq b \leq i-1$) placed at vertex $i+a-b$, on which s_+ and s_- increment a and b respectively (and act by zero at the boundary $a = M-i$ or $b = i-1$). Since these increments commute and satisfy the endpoint relations, the paths (44) map $v_{0,0}$ to the distinct basis vectors $v_{a,b}$. In particular, all paths of degree at least M in $\Pi(A_M)$ vanish, and for each vertex i there is a unique nonzero path of top degree $M-1$, which terminates at the reflected vertex $M+1-i$.

We now examine the three periodic models of type A:

- *Pure translation quotients* $\mathbb{Z}A_M/\langle\tau^s\rangle$. The positive ordinary trace of $\Pi(A_M)$ vanishes even in characteristic two: any closed path at i sorts to $(s_+s_-)^q$, and cyclic rotation combined with the interior commutation relation transports $(s_+s_-)^q$ step by step to an endpoint, where it vanishes (without dividing by q). Lemma A.2(1) therefore gives $\text{coker } D_{d-1} = 0$ for all $d \geq 3$ and every period $s \geq 1$.
- *Reflected odd trees* $\mathbb{Z}A_{2m+1}/\langle\tau^s\rangle$. Here the brick hypothesis forces $s > m$: if $s \leq m$, retaining the central vertex and the first s vertices of each arm yields the quotient $\mathbb{Z}A_{2s+1}/\langle\tau^s\rangle$, in which the top-degree paths of length $2s$ in $\Pi(A_{2s+1})$ connect each vertex to its ρ -reflection while advancing the repetition coordinate by s , and hence project to nonzero radical endomorphisms at every vertex. With $s > m$ established, any row path from τv to v in $\mathbb{Z}A_{2m+1}/\langle\tau^s\rangle$ has even path degree $d = 2q \leq 2m$ satisfying $s \mid (q-1)$ by Lemma A.2. Since $0 \leq q-1 < m < s$ for $q \geq 1$, this divisibility forces $q = 1$ ($d = 2$), so no correction of path degree $d \geq 3$ exists.
- *Folded even trees* $\mathbb{Z}A_{2n}/\langle\eta^r\rangle$ (r odd). In the coordinates $(i, j) \in \{1, \dots, n\} \times (\mathbb{Z}/r\mathbb{Z})$ of Appendix A.1, the arrows $\varepsilon_j: (1, j) \rightarrow (1, j+1)$, $a_{i,j}: (i, j) \rightarrow (i+1, j)$, and

$b_{i,j}: (i+1, j) \rightarrow (i, j+2)$ have the property that any path starting and ending at the same level i contains an equal number of a -steps and b -steps, so its total shift in the second coordinate j equals its path degree d . If $r \leq 2n-1$, retaining the first $(r+1)/2$ levels yields $\mathbb{Z}A_{r+1}/\langle \eta^r \rangle$; the top-degree paths of length r in $\Pi(A_{r+1})$ connect each vertex to its reflection at the same level i and shift j by $r \equiv 0 \pmod{r}$, giving nonzero radical endomorphisms at every vertex and contradicting the brick hypothesis. Thus $r > 2n-1$. Since any row path from $\tau(i, j) = (i, j-2)$ to (i, j) starts and ends at level i , its path degree d satisfies $d \equiv 2 \pmod{r}$; as $d \leq 2n-1 < r$, we must have $d = 2$, ruling out any correction of degree $d \geq 3$.

B.1.2. The two short arms of type D. For $\Delta = D_n$, set $N = n-2$. By Lemma A.3, the central corner $e_{x_0}\Pi(D_n)e_{x_0}$ is a quotient of

$$R_N = k\langle x, y \rangle / (x^2, y^2, (x+y)^N),$$

where x and y represent the length-two closed paths along the two short arms. In word degree $j < N$, R_N is spanned by the two alternating words w_j and w'_j of length j starting with x and y , respectively; in word degree N , the relation $(x+y)^N = 0$ becomes $w_N + w'_N = 0$. Multiplying $w_N + w'_N = 0$ on the left or right by x and y kills both alternating words of degree $N+1$, so $(R_N)_j = 0$ for all $j > N$.

By Lemma A.4, the positive twisted trace of $\Pi(D_n)$ is a quotient of the corresponding twisted trace of R_N :

- For the *ordinary trace* ($\theta = \text{id}$), the degree-one classes $[x]$ and $[y]$ vanish by (37). Any alternating word of odd length $j \geq 3$ begins and ends with the same letter, so cyclic rotation produces x^2 or y^2 and the class vanishes. For even j , cyclic rotation identifies $[w_j] = [w'_j]$. Hence the positive ordinary trace is at most one-dimensional in each path degree $d = 2j$ with j even and $2 \leq j \leq N$, and vanishes in all other positive degrees.
- For the *short-arm interchange* ($\theta = \iota$ swapping x and y), twisted cyclic rotation $[uv] = [\iota(u)]$ turns an alternating word of even length j into a word containing x^2 or y^2 , while for odd j it identifies $[w_j] = [w'_j]$. Hence the positive ι -twisted trace is at most one-dimensional in each path degree $d = 2j$ with j odd and $1 \leq j \leq N$, and vanishes in all other positive degrees.

In either case, Lemma A.2 shows that a nonzero cokernel in path degree $d = 2j \geq 3$ can occur only when

$$2 \leq j \leq N \quad \text{and} \quad s \mid (j-1). \quad (45)$$

(These degree and parity restrictions match Riedtmann's characteristic-two analysis of type D in [19, Section 2, p. 487, and Sections 3.3, 3.9].)

For D_4 ($N = 2$), we also check a 3-cycle ρ permuting the three arms $x, y, x+y$. Here R_2 is spanned by $1, x, y, xy = yx$ with $(R_2)_j = 0$ for $j \geq 3$. In word degree 1, twisted cyclic rotation under ρ identifies $[x] = [y] = [x+y]$, whence $[x] = 0$; in word degree 2, $[xy] = [y\rho(x)] = [y^2] = 0$ (and for ρ^2 , $[yx] = [x\rho^2(y)] = [x^2] = 0$). The only possible ordinary trace class has path degree $d = 4$ ($j = 2$), for which $s \mid (j-1)$ forces $s = 1$; since $3 \nmid s$, the induced twist $\theta = \rho^{\pm 1}$ is nontrivial, so no correction class arises from a 3-cycle on D_4 .

To show that (45) admits at most one value of j when $\theta \in \{\text{id}, \iota\}$, we bound N in terms of the period s using the top-degree socle paths of $\Pi(D_n)$ (cf. [14, Section 6.2]).

Lemma B.1. *Let $\Pi(D_n)$ be the preprojective algebra of type D_n over an arbitrary field, set $N = n-2$, and let $\nu \in \text{Aut}(D_n)$ be the identity if N is even and the short-arm interchange if N is odd. For every vertex x of D_n , there is a nonzero path of degree $2N$ in $\Pi(D_n)$ from x to νx .*

Proof. We apply the integer recurrence criterion (43). First consider the line A_M on vertices $1, \dots, M$ with adjacency matrix $\mathcal{A}$, and define $V_{-1} = 0$, $V_0 = I$, and $V_{d+1} = \mathcal{A}V_d - V_{d-1}$. On the infinite integer line $\mathbb{Z}$, the function

$$c_d(r) = \begin{cases} 1, & \text{if } |r| \leq d \text{ and } r \equiv d \pmod{2}, \\ 0, & \text{otherwise} \end{cases} \quad (c_{-1} = 0)$$

satisfies $c_{d+1}(r) = c_d(r-1) + c_d(r+1) - c_{d-1}(r)$. By the method of images, the reflected sum

$$(V_d)_{ij} = \sum_{h \in \mathbb{Z}} (c_d(i-j+2h(M+1)) - c_d(i+j+2h(M+1))) \quad (1 \leq i, j \leq M)$$

satisfies the same recurrence and initial conditions, and vanishes at the artificial boundaries $i = 0$ (since $c_d(r) = c_d(-r)$) and $i = M+1$ (by shifting $h \mapsto h-1$). For $0 \leq d \leq M-1$, only the $h = 0$ term can contribute to the first sum and only the $h = 0, -1$ terms can contribute to the second, subtracting the ranges $d \geq i+j$ and $d \geq 2M+2-i-j$. Thus, for $0 \leq d \leq M-1$, $(V_d)_{ij} = 1$ if and only if

$$|i-j| \leq d \leq \min(i+j-2, 2M-i-j) \quad \text{and} \quad d \equiv i-j \pmod{2},$$

and $(V_d)_{ij} = 0$ otherwise. In particular, $V_0, \dots, V_{M-1}$ are entrywise nonnegative, and V_{M-1} is the reflection matrix $i \mapsto M+1-i$.

Now set $M = 2N+1$ and shift the vertex labels of A_{2N+1} to $-N, \dots, N$. On the reflection-invariant subspace spanned by $f_0 = e_0$ and $f_r = e_r + e_{-r}$ ($1 \leq r \leq N$), the adjacency matrix $\mathcal{A}$ sends $f_0 \mapsto f_1$, $f_1 \mapsto 2f_0 + f_2$, and $f_r \mapsto f_{r-1} + f_{r+1}$ for $r \geq 2$ (omitting f_{N+1}). On D_n , label the two short-arm leaves by a, b and the remaining vertices from the center outward by $1, \dots, N$. On the subspace spanned by $g_0 = e_a + e_b$ and $g_r = e_r$ ($1 \leq r \leq N$), the adjacency matrix of D_n sends $g_0 \mapsto 2g_1$, $g_1 \mapsto g_0 + g_2$, and $g_r \mapsto g_{r-1} + g_{r+1}$ for $r \geq 2$. Since these two $(N+1) \times (N+1)$ matrices are transposes of one another, the recurrence polynomials U_d for D_n and V_d for A_{2N+1} satisfy the corresponding transpose relations. Using the symmetries $(V_d)_{rt} = (V_d)_{tr} = (V_d)_{-r, -t}$, we obtain for $1 \leq r, t \leq N$:

$$(U_d)_{rt} = (V_d)_{r,t} + (V_d)_{r,-t}, \quad (U_d)_{ar} = (U_d)_{br} = (V_d)_{r,0} = (V_d)_{0,r}.$$

On the antisymmetric line $e_a - e_b$, the adjacency matrix of D_n acts by zero, so $U_d(e_a - e_b)$ follows the recurrence $u_{-1} = 0$, $u_0 = 1$, $u_{d+1} = -u_{d-1}$, giving $u_{2q} = (-1)^q$ and $u_{2q+1} = 0$. Since the symmetric leaf coefficient $(V_d)_{00}$ equals 1 for even $d \in [0, 2N]$ and 0 for odd d , writing $e_a = \frac{1}{2}(g_0 + e_a - e_b)$ and $e_b = \frac{1}{2}(g_0 - (e_a - e_b))$ yields the $\{a, b\} \times \{a, b\}$ block of U_d for $0 \leq d \leq 2N$: it vanishes for odd d , and for $d = 2q$ it equals

$$\frac{1}{2} \begin{pmatrix} 1 + (-1)^q & 1 - (-1)^q \\ 1 - (-1)^q & 1 + (-1)^q \end{pmatrix}.$$

Thus $U_0, \dots, U_{2N}$ are entrywise nonnegative integer matrices, and at $d = 2N$ ($q = N$) the matrix U_{2N} is the permutation matrix of ν . The lemma now follows from (43). $\square$

Returning to $\mathbb{Z}D_n / \langle \tau^s \rho \rangle$ with $\rho \in \{\text{id}, \iota\}$, the brick hypothesis forces $N < 2s$: otherwise, deleting the outer $N-2s$ vertices of the long arm leaves $\mathbb{Z}D_{2s+2} / \langle \tau^s \rho \rangle$, in which Lemma B.1 (with $N' = 2s$ even) provides nonzero paths of degree $4s$ fixing every vertex of D_{2s+2} and advancing the repetition coordinate by $2s$, which project to nonzero radical endomorphisms at every vertex for either choice of ρ . Combining $N < 2s$ with (45) yields $1 \leq j-1 \leq N-1 < 2s-1$ and $s \mid (j-1)$, which uniquely forces $j = s+1$ and $d = 2s+2$ (occurring only when $s+1 \leq N < 2s$, with s odd for the trivial twist $\theta = \text{id}$ and s even for $\theta = \iota$).

B.1.3. *Exceptional upper bounds.* For $\Delta = E_6, E_7, E_8$, we retain the presentation $R_m = k\langle x, y \rangle / (x^m, y^3, (x+y)^2)$ from Lemma A.6 for $m = 3, 4, 5$, respectively. In R_5 over a field of characteristic two, set

$$\omega_3 = yx^3y, \quad \omega_4 = yx^4y, \quad \omega_5 = \omega_4x^3y, \quad u = x^2yx^3 + x^3yx^2.$$

Reducing the word identities (42) modulo 2 gives

$$\omega_3x = \omega_4 + x\omega_3 + u, \quad \omega_3y = x\omega_3 + u, \quad \omega_4y = \omega_4x, \quad \omega_4xy = 0, \quad \omega_4x^2y = \omega_4x^3, \quad (46)$$

$$0 = \omega_3x^5 = \sum_{r=0}^4 x^r \omega_4 x^{4-r}, \quad (47)$$

where (47) follows by iterating the first relation of (46) five times and using $x^5 = 0$ to kill all resulting u -terms. Multiplying (47) on the right by y and applying (46) gives $\omega_4x^4y = x\omega_5 + x^2\omega_4x^3 + x^4\omega_4x$. Left-multiplying the last identity of (42) by yx^4 also gives $\omega_5x = \omega_4x^4y$, while expanding $y^2 = xy + yx + x^2$ at the end of ω_5y gives $\omega_5y = 0$; hence

$$\omega_5x = x\omega_5 + x^2\omega_4x^3 + x^4\omega_4x, \quad \omega_5y = 0. \quad (48)$$

From (42) and (46)–(48), one checks immediately that the span of the words

$$\begin{aligned} x^i \ (0 \leq i \leq 4), \quad x^i y x^j \ (0 \leq i, j \leq 4), \quad x^i \omega_3 \ (0 \leq i \leq 4), \\ x^i \omega_4 x^j \ (0 \leq i \leq 4, 0 \leq j \leq 3), \quad x^i \omega_5 \ (0 \leq i \leq 4) \end{aligned} \quad (49)$$

contains 1 and is closed under right multiplication by x and y , so (49) spans R_5 (with maximum word degree 14). Passing to the quotients $R_4 = R_5/(x^4)$ and $R_3 = R_5/(x^3)$ kills ω_4, ω_5 in R_4 (leaving $x^i, x^i y x^j, x^i \omega_3$ with $0 \leq i, j \leq 3$, of maximum word degree 8) and also kills ω_3 in R_3 (leaving $x^i, x^i y x^j$ with $0 \leq i, j \leq 2$, of maximum word degree 5).

Applying Lemma A.6(1) in characteristic two with $a = yx^3$ and $b = yx^4$, the relations $2 = 0$ and $3[ab] = 0$ simplify the block identities to

$$[a^2W] = 0 \ (W \neq 1), \quad [abW] = [baW] \ (W \neq 1), \quad [ab] = 0, \quad [b^q] = 0 \ (q \geq 2).$$

Thus any word in a, b of length at least 3 containing at least two a 's vanishes by commuting the a 's to be adjacent, while a word with a single a reduces to $[ab^r]$, which vanishes for $r = 1$ (as $[ab] = 0$) and for $r \geq 3$ (as $\deg(ab^r) = 4 + 5r \geq 19 > 14$). Among the single- y words, $3[yx^2] = 0$ and $5[yx^4] = 0$ still force $[yx^2] = [yx^4] = 0$ in characteristic two, leaving only the following spanning representatives for the positive ordinary trace:

Dynkin type	spanning trace representatives	path degrees d
E_6	$[yx]$	4
E_7	$[yx], [yx^3], [(yx^3)^2]$	4, 8, 16
E_8	$[yx], [yx^3], [(yx^3)^2], [(yx^3)(yx^4)^2]$	4, 8, 16, 28

Each listed path degree is at most one-dimensional, and all other positive ordinary trace degrees vanish.

For the nontrivial graph automorphism θ of E_6 (which swaps x and y in R_3), the spanning words of R_3 in word degrees 1 through 5 are:

word degree	spanning words in R_3
1	x, y
2	x^2, xy, yx
3	$\xi_1 = x^2y, \xi_2 = xyx, \xi_3 = yx^2$
4	$\xi_4 = x^2yx, \xi_5 = xyx^2$
5	$\xi_6 = x^2yx^2$

The θ -twisted trace relations $[xw] = [wy]$ and $[yw] = [wx]$ identify the words in degree 1 ($[x] = [y]$) and in degree 2 ($[x^2] = [xy] = [yx]$, using $y^2 = xy + yx + x^2$). In degrees 3

and 4, using the algebra expansions $xy^2 = x^2y + xyx = \xi_1 + \xi_2$, $x^2y^2 = x^2yx = \xi_4$, and $y^2x^2 = xyx^2 = \xi_5$ (since $x^3 = 0$), five twisted-trace cuts kill all spanning classes:

$$\begin{aligned} w = x^2 \text{ in } [xw] = [wy] &\implies [\xi_1] = [x^2y] = [x^3] = 0, \\ w = x^2 \text{ in } [yw] = [wx] &\implies [\xi_3] = [yx^2] = [x^3] = 0, \\ w = xy \text{ in } [xw] = [wy] &\implies [\xi_1] = [xy^2] = [\xi_1] + [\xi_2] \implies [\xi_2] = 0, \\ w = x^2y \text{ in } [xw] = [wy] &\implies 0 = [x^3y] = [x^2y^2] = [\xi_4], \\ w = yx^2 \text{ in } [yw] = [wx] &\implies [\xi_5] = [y^2x^2] = [yx^3] = 0. \end{aligned}$$

Degree 5 is spanned by ξ_6 alone, and the fixed long arm contributes no new classes. Thus the positive θ -twisted trace of $\Pi(E_6)$ is at most one-dimensional in path degrees $d = 2, 4, 10$ and vanishes in all other positive degrees.

Applying Lemma A.2 to these candidate degrees:

- For an ordinary trace class ($\theta = \text{id}$) in path degree $d \in \{4, 8, 16, 28\}$, the period s must divide $d/2 - 1 \in \{1, 3, 7, 13\}$. (In the reflected E_6 model $\mathbb{Z}E_6/\langle\tau^s\rho\rangle$, Lemma A.2(1) requires $(d/2 - 1)/s$ to be even for the untwisted trace; for the sole E_6 candidate $d = 4$, $s \mid 1$ forces $s = 1$, for which $(d/2 - 1)/s = 1$ is odd and hence twisted.)
- For a θ -twisted trace class on E_6 in path degree $d \in \{4, 10\}$, Lemma A.2(2) requires $s \mid (d/2 - 1)$ with $(d/2 - 1)/s$ odd, which forces $s = 1$ for $d = 4$ and $s = 4$ for $d = 10$.

Finally, period $s = 1$ is excluded in every case by deleting the vertices outside the central D_4 subquiver and applying Lemma A.5 (noting that the degree-4 closed paths in $\Pi(D_4)$ advance the repetition coordinate by 2 and therefore also close in the reflected quotient $\mathbb{Z}D_4/\langle\tau\rho\rangle$ since $\rho^2 = \text{id}$). Consequently, the only remaining possibilities for a positive correction class of degree $d \geq 3$ in types E_6, E_7, E_8 are:

Dynkin type and action	period s	path degree d
E_6 (with reflection ρ)	4	10
E_7	3, 7	8, 16 respectively
E_8	3, 7, 13	8, 16, 28 respectively

(50)

In every case, each periodic component admits at most one correction degree $d = 2s + 2$, spanned by a single central row path. Taking I to be the set of periodic components for which $\text{coker } D_{2s_i+1}(\mathcal{C}) \neq 0$ completes the proof of Proposition 6.1.

B.2. Correction spaces over the integers modulo four

We now lift the preceding calculations to $\mathcal{O} = \mathbb{Z}/4\mathbb{Z}$ for the integral mesh category E_4 constructed in Proposition 10.1.

Lemma B.2. *Let $\mathcal{O} = \mathbb{Z}/4\mathbb{Z}$, and let E_4 be the finite free mesh category over $\mathcal{O}$ of Proposition 10.1, with mod-2 reduction $\mathcal{C}^{(2)}$ over $\mathbb{F}_2$. Write $D_m(E_4)$ for its mesh derivative with respect to the unit coefficients of that presentation. With I and χ_i as in Proposition 6.1 and (13), there is a degree- and weight-preserving isomorphism*

$$\bigoplus_{m \geq 2} \text{coker } D_m(E_4) \cong \bigoplus_{i \in I} (\mathcal{O}/2\mathcal{O})_{\chi_i},$$

where the summand indexed by $i \in I$ lies in $\text{coker } D_{d_i-1}(E_4)$ ($d_i = 2s_i + 2$) and has internal degree $d_i - 2$.

Proof. Set $Q_m = \text{coker } D_m(E_4)$. By Proposition 10.1, reducing the source, target, and matrix of $D_m(E_4)$ modulo 2 recovers $D_m(\mathcal{C}^{(2)})$, so right exactness of the tensor product gives degree- and weight-preserving identifications

$$Q_m/2Q_m \cong \text{coker } D_m(\mathcal{C}^{(2)}), \quad \text{coker } D_m(\mathcal{C}^{(2)}) \otimes_{\mathbb{F}_2} k \cong \text{coker } D_m(\mathcal{C}).$$

By Proposition 6.1, the retained row paths of path degree $m + 1$ span coker $D_m(\mathcal{C})$ over k and hence already form an $\mathbb{F}_2$ -basis of coker $D_m(\mathcal{C}^{(2)})$. Lifting these retained rows to E_4 and letting $U_m \subseteq Q_m$ denote the $\mathcal{O}$ -submodule they generate, Nakayama's lemma (or directly $Q_m = U_m + 2Q_m = U_m + 2(U_m + 2Q_m) = U_m$ since $4Q_m = 0$) shows that $Q_m = U_m$, and in particular $Q_m = 0$ in all degrees and weights without a retained row. Since the mod-2 reductions of the retained rows are linearly independent over $\mathbb{F}_2$, it suffices to show that twice each retained row lies in $\text{im } D_m(E_4)$, i.e., vanishes in Q_m .

First, any row path that leaves its periodic component passes through a nonperiodic vertex and therefore lies in a finite triangular incidence chain as in the proof of Proposition 5.1; because the mesh coefficients of E_4 are units in $\mathcal{O}$, eliminating from an exposed end expresses any such row path as a derivative over $\mathcal{O}$. Consequently, all terms in a contextual mesh relation that leave the periodic component vanish in Q_m . Likewise, the scalar normalizations of Lemma A.1(1) hold over $\mathcal{O}$, since they divide only by unit mesh coefficients and rescale arrows and rows by units in $\mathcal{O}$. After this normalization to the signed preprojective coefficients ± 1 , each arrow-correction column on a retained periodic component has opposite unit coefficients $\pm \alpha$ in its two row incidences, giving $\alpha[wa] - \alpha[(\tau a)w] = 0$ and hence $[wa] = [(\tau a)w]$ in Q_m . Thus the class of a retained row in Q_m factors through the positive θ -twisted trace of the integral signed preprojective category $\Pi_{\mathcal{O}}(\Delta)$.

Moreover, the star and arm reductions of Lemmas A.3 and A.4 hold verbatim over $\mathcal{O}$, because every replacement step uses only coefficients ± 1 :

- Replacing an outward backtrack $a_j b_j$ along an arm by $\pm b_{j-1} a_{j-1}$ decreases the total arm depth until any closed path at the center is a word in the length-two arm loops, whose signed sum at the center vanishes.
- Along an arm of length L , replacing $b_j a_j$ at internal vertices by $\pm a_{j+1} b_{j+1}$ rewrites $(a_1 b_1)^q$ as $\pm a_1 \cdots a_q b_q \cdots b_1$ for $q \leq L$ and kills $(a_1 b_1)^{L+1}$ via the leaf relation $b_L a_L = 0$.
- In the twisted trace along a θ -fixed arm, rotating a closed word to a vertex of minimal depth and replacing the inward backtrack by the outward backtrack strictly increases the depth sum until the word reaches the leaf and vanishes, showing that positive powers of a fixed-arm loop have zero trace over $\mathcal{O}$.

We now check that 2 annihilates the retained trace classes in each Dynkin type that can contribute to I :

- *Type D_n .* In the central corner quotient over $\mathcal{O}$, the short-arm relations $x^2 = y^2 = 0$ leave the alternating words w_j and w'_j of word degree $j = s + 1$. For the trivial twist $\theta = \text{id}$ (s odd, j even), cyclic rotation gives $[w_j] = [w'_j]$; for the short-arm interchange $\theta = \iota$ (s even, j odd), twisted cyclic rotation $[uv] = [v\iota(u)]$ likewise gives $[w_j] = [w'_j]$. Since the long-arm loop equals $-(x + y)$ in the central mesh relation and its j -th power has zero trace along the fixed long arm, we obtain in both cases

$$0 = [(-x - y)^j] = (-1)^j([w_j] + [w'_j]) = \pm 2[w_j],$$

so $2[w_j] = 0$ over $\mathcal{O}$.

- *Reflected type E_6* ($s = 4$, $d = 10$, word degree 5). In the central corner R_3 over $\mathcal{O}$, the relations $x^3 = y^3 = (x + y)^2 = 0$ give

$$y^2 = -yx - xy - x^2, \quad yxy = xyx, \quad yx^2y = -xyx^2 - x^2yx,$$

where the second identity follows from $0 = y^3 = (-yx - xy - x^2)y = -yxy - xy^2 - x^2y = -yxy - x(-yx - xy - x^2) - x^2y = -yxy + xyx + x^3 = -yxy + xyx$ and the third follows by comparing $(yxy)y = (xyx)y$ with $yx(y^2) = yx(-yx - xy - x^2)$. These three identities eliminate any pair of y 's separated by 0, 1, or 2 factors of x , so R_3 is spanned over $\mathcal{O}$ by x^i and $x^i y x^j$ ($0 \leq i, j \leq 2$), and word degree 5

is spanned by $\xi_6 = x^2yx^2$. Applying the θ -twisted trace relation $[xw] = [wy]$ ($\theta(x) = y$) together with the third identity and $x^3 = 0$ yields

$$[\xi_6] = [x(yx^2)] = [(xyx^2)y] = [x(yx^2y)] = [-x^2yx^2 - x^3yx] = -[\xi_6],$$

so $2[\xi_6] = 0$ over $\mathcal{O}$.

- *Types E_7 and E_8 .* Lemma A.6(1) holds over any commutative ring, and in particular over $\mathcal{O} = \mathbb{Z}/4\mathbb{Z}$. Every spanning class in the positive ordinary trace is annihilated by 2, 3, or 5; since 3 and 5 are units in $\mathbb{Z}/4\mathbb{Z}$, the entire positive trace is annihilated by 2.

Thus $2Q_m = 0$ for all $m \geq 2$, completing the proof. $\square$

APPENDIX C. LOCAL RESTRICTIONS ON RAY PRODUCTS

In this appendix we prove the two local structural results stated in Section 8: the uniqueness of initial and terminal ordinary arrows for one-sided generating rays (Proposition 8.23), and the classification of proper shiftable products and nondeep external returns (Proposition 8.25). Throughout, we work in the ray algebra $A_0 = kP$ of Setting 8.4 over a field of arbitrary characteristic, fix a positive primitive character $\chi \in L$ as in Setting 8.16, and use the endpoint types of Definition 8.17 and the primitive return corners S_x of Definition 8.20. Both proofs rely on the finite faithful supports of Definition 8.7, the finite homogeneous categories of Subsection 8.1, and the quadratic-form criterion of Lemma 8.8.

C.1. One-sided generating rays

We begin by proving Proposition 8.23. Recall from Subsection 8.1 that for any finite collection of graded objects (x_i, w_i) with weights $w_i \in L$, we can construct a representation-finite homogeneous category H whose morphism spaces are spanned by rays $r: x_i \rightarrow x_j$ of weight $\ell(r) = w_j - w_i$. By Lemma 3.1(3), the corner bimodule $A_0(x, z)$ of a generating ray $f: x \rightarrow z$ is cyclic over at least one endpoint ring, so every ray in $A_0(x, z)$ has weight $\ell(f)$ plus a nonnegative multiple of a positive local weight. Since χ is positive, $A_0(x, z)$ contains no ray of weight $\ell(f) - n\chi$ for $n > 0$, and a ray of weight $\ell(f) + \chi$, when it exists, is the proper successor f^+ . Moreover, by Lemma 8.14, every factor of a generating shiftable ray g is itself a generating ray (as the corner bimodule of g has dimension at least 2).

C.1.1. Bimodule retractions from homogeneous categories. To transfer finite representation type from a finite homogeneous category H to the incidence algebra kQ of a finite poset Q , we use a general criterion for when a family of selected rays in H admits a kQ -bimodule retraction, even when H contains additional morphisms outside kQ that do not form an ideal. Write $a \triangleleft b$ when $a < b$ is a cover relation in Q (so no element of Q lies strictly between a and b).

Lemma C.1. *Let H be a finite homogeneous category whose objects are indexed by a finite poset Q . Suppose that for each pair $a \leq b$ in Q we have chosen a nonzero basis morphism $r_{ab} \in H(a, b)$ such that $r_{aa} = 1_a$ and $r_{ab}r_{bc} = r_{ac}$ for all $a \leq b \leq c$. Assume that for every cover relation $a \triangleleft b$ in Q and every homogeneous basis morphism h in H with the appropriate endpoints,*

$$\begin{aligned} r_{ab}h = r_{ac} \quad (a \leq c) &\implies b \leq c, \\ hr_{ab} = r_{cb} \quad (c \leq b) &\implies c \leq a. \end{aligned}$$

Then the linear projection $H \rightarrow kQ$ sending each selected basis morphism r_{ab} to the corresponding poset basis element of kQ and vanishing on all other basis morphisms of H is a kQ -bimodule retraction. In particular, if H is representation-finite, then so is the incidence algebra kQ .

Proof. The relations $r_{aa} = 1_a$ and $r_{ab}r_{bc} = r_{ac}$ embed the incidence algebra kQ as a unital subalgebra of H . We first extend the two cover implications to arbitrary pairs $a \leq b$ in Q by induction on the interval $[a, b]$. If $a < b$, choose a cover $a \triangleleft a_1 \leq b$. Any equality $r_{ab}h = r_{ac}$ with $a \leq c$ can be rewritten as $r_{aa_1}(r_{a_1b}h) = r_{ac}$. Since $r_{ac} \neq 0$, the inner product $r_{a_1b}h$ is a nonzero basis morphism, so the cover assumption for $a \triangleleft a_1$ gives $a_1 \leq c$; because H is thin, $r_{a_1b}h$ must equal r_{a_1c} , and the induction hypothesis on $[a_1, b]$ then yields $b \leq c$. The second implication extends in the same way by factoring out the last cover below b .

It follows that the composition in H of a selected basis morphism r_{ab} with an unselected basis morphism h (on either side) can never equal a selected basis morphism r_{ac} or r_{cb} : otherwise the extended implications would give $b \leq c$ (respectively, $c \leq a$), forcing $r_{ab}h = r_{ab}r_{bc}$ and hence $h = r_{bc}$ (respectively, $h = r_{ca}$) by thinness of H . Thus the linear projection $H \rightarrow kQ$ onto the span of the selected morphisms $\{r_{ab}\}_{a \leq b}$ is a kQ -bimodule

homomorphism restricting to the identity on kQ . The final assertion then follows from Lemma 4.6. $\square$

We will apply Lemma C.1 when Q is the reachability poset of an oriented tree and the selected morphisms r_{ab} are the evaluations of the unique oriented paths in the tree. Since the cover relations of Q are precisely the arrows of the tree, it suffices to verify the two implications of Lemma C.1 at each tree arrow (and any candidate factor h whose required weight has negative categorical degree $\deg_{\mathcal{J}}(h) < 0$ is automatically absent).

The first tree we will use is the six-vertex tree with two adjacent trivalent vertices:

$$\begin{array}{ccccc}
 v_1 & \xrightarrow{\gamma} & v_2 & \xrightarrow{\alpha} & v_3 \\
 & & \uparrow \lambda & & \\
 v_4 & \xrightarrow{\alpha} & v_5 & \xleftarrow{\beta} & v_6
 \end{array} \tag{51}$$

In our application below, the two arrows labeled α will be lifts of the same ray α , with (v_4, v_5) obtained from (v_2, v_3) by shifting both object weights by $-\chi$ so that the six graded objects are pairwise distinct. Assigning dimension 2 to the trivalent vertices v_2, v_5 and dimension 1 to the four leaves gives $N = D = 12$, so Lemma 8.8 rules out every orientation of (51). The second tree we will use is the eight-vertex tree of type $\tilde{E}_7$ with arm lengths $(3, 3, 1)$, which is likewise representation-infinite in every orientation by the dimension vector recorded after Lemma 8.8.

C.1.2. Two distinct first arrows are impossible. Suppose for a contradiction that a generating left-sided ray $f: x \rightarrow z$ can be represented by ordinary paths starting with two distinct ordinary arrows $a_1: x \rightarrow y_1$ and $a_2: x \rightarrow y_2$. Write $f = a_1 w_1 = a_2 w_2$; since $a_1 \neq a_2$ are ordinary arrows, they are incomparable under right factorization ($a_1 h = a_2$ with $\deg_{\mathcal{J}}(h) > 0$ would contradict the ordinarity of a_2 , while $\deg_{\mathcal{J}}(h) = 0$ would force $a_1 = a_2$), and both suffixes w_1, w_2 are nonidentity generating rays. Because $\hat{q}_x f = \hat{q}_x a_i w_i \neq 0$, both $\hat{q}_x a_1$ and $\hat{q}_x a_2$ are nonzero.

We first show that one of a_1, a_2 is left-sided and the other is two-sided:

- *Both a_1 and a_2 cannot be two-sided.* If both were two-sided, then $\hat{q}_x a_i = a_i \hat{q}_{y_i}$, so $\hat{q}_x f = a_i(\hat{q}_{y_i} w_i)$ would be right-divisible by both a_1 and a_2 , as is $f = a_i w_i$. In the faithful projective support at x , the four rays $\{a_1, a_2, f, \hat{q}_x f\}$ would then form a four-point crown: a_1 and a_2 are incomparable; f and $\hat{q}_x f$ are incomparable because f is generating and left-sided ($f \hat{q}_z = 0$); and neither f nor $\hat{q}_x f$ can precede a_1 or a_2 for degree reasons. This contradicts Lemma 8.8.
- *Both a_1 and a_2 cannot be left-sided.* If both were left-sided, consider the six rays

$$1_x, \quad \hat{q}_x, \quad a_1, \quad a_2, \quad \hat{q}_x a_1, \quad \hat{q}_x a_2$$

in the faithful projective support at x . We claim that their induced poset has only the comparisons generated by

$$1_x \longrightarrow \hat{q}_x \longrightarrow \hat{q}_x a_1, \hat{q}_x a_2, \quad 1_x \longrightarrow a_1, a_2.$$

Indeed, $\hat{q}_x$ cannot precede a_i (as a_i is ordinary) and a_i cannot precede $\hat{q}_x$ (as $a_i h = \hat{q}_x$ would place $y_i \in S_x$ by Lemma 8.21(3) and make a_i two-sided); a_1 and a_2 (and hence $\hat{q}_x a_1$ and $\hat{q}_x a_2$) are incomparable; and any cross comparison $a_i h = \hat{q}_x a_j$ ($i \neq j$) would multiply on the right by w_j to give $a_i(h w_j) = \hat{q}_x a_j w_j = \hat{q}_x f \neq 0$, forcing $h w_j$ to be the unique ray w_i^+ in $A_0(y_i, z)$ of weight $\ell(w_i) + \chi$, which contradicts $a_i w_i^+ = 0$ from the $(L, *)$ row of (17). All reverse comparisons are ruled out because a_1, a_2 are generating or by degree. The resulting six-element poset is the orientation of (51) with two outgoing leaves at each trivalent vertex, which is excluded by Lemma 8.8.

Thus one of the two initial arrows of a generating left-sided ray must have type B and the other type L; dually, if a generating right-sided ray had two distinct terminal arrows, one would have type B and the other type R. Passing to the opposite category A_0^{op} , it therefore suffices to rule out a generating right-sided ray $f: x \rightarrow z$ admitting two factorizations

$$f = \gamma\alpha = \eta\beta, \quad \alpha: y \rightarrow z \text{ ordinary of type B, } \beta: t \rightarrow z \text{ ordinary of type R.}$$

Here the prefixes $\gamma: x \rightarrow y$ and $\eta: x \rightarrow t$ are nonidentity generating rays by Lemma 8.14, and are incomparable (for instance, $\gamma h = \eta$ would imply $\gamma(h\beta) = \gamma\alpha = f \neq 0$, so $h\beta = \alpha$ because both lie in $A_0(y, z)$ with the same weight, contradicting the fact that α and β are distinct ordinary arrows). Using the local generators $\hat{q}_y$ and $\hat{q}_z$, we have $\gamma\hat{q}_y\alpha = \gamma\alpha\hat{q}_z = f\hat{q}_z \neq 0$, so γ has type R (since f has type R, $\hat{q}_x\gamma = 0$). Moreover, if a local ray $u \in R_x$ has weight χ , then primitivity of χ gives $u = \hat{q}_x$, and hence $uf = \hat{q}_x f = 0 \neq f\hat{q}_z$ because f is right-sided; thus

$$uf \neq f\hat{q}_z \quad \text{for every local ray } u \in R_x. \quad (52)$$

We now distinguish two cases according to whether α admits a primitive reverse.

Case 1: α has a primitive reverse $\lambda: z \rightarrow y$. Here $\alpha\lambda = \hat{q}_y$ and $\lambda\alpha = \hat{q}_z$. Set $\zeta = \beta\lambda: t \rightarrow y$, which is nonzero because $\zeta\alpha = \beta\hat{q}_z \neq 0$. In a finite homogeneous category H , choose the six graded objects

vertex	v_1	v_2	v_3	v_4	v_5	v_6
object	x	y	z	y	z	t
weight	$-\ell(\gamma)$	0	$\ell(\alpha)$	$-\chi$	$\ell(\alpha) - \chi$	$\ell(\alpha) - \chi - \ell(\beta)$

(53)

and select the tree (51), whose composite oriented paths evaluate to the nonzero rays $\hat{q}_y, \zeta, \hat{q}_z, \alpha\hat{q}_z, \beta\hat{q}_z, f$. To verify the cover conditions of Lemma C.1, note that every left-multiplication check $r_{ab}h = r_{ac}$ with $b \not\leq c$ requires $\deg_{\mathcal{J}}(h) < 0$, while the right-multiplication checks $hr_{ab} = r_{cb}$ with $c \not\leq a$ and $\deg_{\mathcal{J}}(h) \geq 0$ reduce to the following forbidden factorizations:

forbidden factorization	reason for exclusion
$h\gamma = \hat{q}_y$	$(\gamma h)f = \gamma\hat{q}_y\alpha = f\hat{q}_z$, contradicting (52)
$h\gamma = \lambda$	$hf = h\gamma\alpha = \hat{q}_z \implies (fh)f = f\hat{q}_z$, contradicting (52)
$h\gamma = \zeta$	$(\eta h)f = \eta\beta\lambda\alpha = f\hat{q}_z$, contradicting (52)
$h\alpha = \beta$ or $h\beta = \alpha$	α, β are distinct ordinary arrows
$h\lambda = \gamma$	$\ell(h) = \ell(\gamma) - \ell(\lambda) = \ell(f) - \chi$ lies below the generator f

Thus Lemma C.1 provides a bimodule retraction onto the incidence algebra of (51), contradicting Lemma 8.8. Consequently, α cannot admit a primitive reverse.

Case 2: α has no primitive reverse. We first note that no ray $h: t \rightarrow y$ can satisfy $\beta\hat{q}_z = h\alpha$. Otherwise, in the faithful injective support at z , the four rays $\{\hat{q}_z, \alpha, \alpha\hat{q}_z, \beta\hat{q}_z\}$ would form a four-point crown: $\hat{q}_z$ and α are incomparable because α is ordinary and has no primitive reverse; $\alpha\hat{q}_z$ and $\beta\hat{q}_z$ are incomparable because $u\alpha\hat{q}_z = \beta\hat{q}_z$ or $u\beta\hat{q}_z = \alpha\hat{q}_z$ cancels $\hat{q}_z$ to $u\alpha = \beta$ or $u\beta = \alpha$; and no upper ray $\alpha\hat{q}_z, \beta\hat{q}_z$ can precede $\hat{q}_z$ or α (the only nonnegative-degree possibility $u\beta\hat{q}_z = \alpha$ would factor the ordinary arrow α through $\hat{q}_z$).

Now choose the eight graded objects in H :

vertex	v_1	v_2	v_3	v_4
object	x	x	y	y
weight	$-\ell(\gamma)$	$\chi - \ell(\gamma)$	0	χ

vertex	v_5	v_6	v_7	v_8
object	z	z	t	t
weight	$\ell(\alpha)$	$\ell(\alpha) - \chi$	$\ell(\alpha) + \chi - \ell(\beta)$	$\ell(\alpha) - \chi - \ell(\beta)$

(54)

and select the oriented tree of type $\tilde{E}_7$:

$$\begin{array}{ccccccccccc}
 & & & & v_1 & & & & & & \\
 & & & & \downarrow \gamma & & & & & & \\
 v_7 & \xleftarrow{\eta} & v_2 & \xrightarrow{\gamma} & v_4 & \xleftarrow{\hat{q}_y} & v_3 & \xrightarrow{\alpha} & v_5 & \xleftarrow{\hat{q}_z} & v_6 & \xleftarrow{\beta} & v_8
 \end{array} \tag{55}$$

Its composite oriented paths evaluate to the nonzero rays $\gamma\hat{q}_y$, $f = \gamma\alpha$, and $\beta\hat{q}_z$. After discarding negative-degree factors, the cover conditions of Lemma C.1 across the seven arrows of (55) reduce to the following forbidden factorizations:

forbidden factorization	reason for exclusion
$\hat{q}_y h = \alpha$ or $h\hat{q}_z = \alpha$	α is an ordinary arrow
$\alpha h = \hat{q}_y$ or $h\alpha = \hat{q}_z$	α has no primitive reverse
$\gamma h = \eta$ or $\eta h = \gamma$	γ and η are incomparable
$h\hat{q}_y = \gamma$	$\ell(h) = \ell(\gamma) - \chi$ lies below the generator γ
$h\alpha = \beta\hat{q}_z$	excluded by the injective-support crown above
$h\gamma = \gamma\hat{q}_y$	$hf = h\gamma\alpha = \gamma\hat{q}_y\alpha = f\hat{q}_z$, contradicting (52)
$h\gamma = \hat{q}_y$	$(\gamma h)f = \gamma\hat{q}_y\alpha = f\hat{q}_z$, contradicting (52)
$h\hat{q}_z = f$	$\ell(h) = \ell(f) - \chi$ lies below the generator f

(In the second row, note that $\alpha h = \hat{q}_y$ forces $\alpha(h\alpha) = \hat{q}_y\alpha = \alpha\hat{q}_z \neq 0$ and hence $h\alpha = \hat{q}_z$, and dually $h\alpha = \hat{q}_z$ forces $\alpha h = \hat{q}_y$, so either equality would make h a primitive reverse of α .) Lemma C.1 therefore yields a bimodule retraction onto the incidence algebra of (55). Assigning dimension 4 to v_3 , dimensions 3, 2, 1 along the two arms of length 3, and dimension 2 to v_1 gives $N = D = 48$, contradicting Lemma 8.8. This completes the proof of Proposition 8.23.

C.2. Proper products and external returns

We now prove Proposition 8.25. Recall that the finite homogeneous categories of Subsection 8.1 are representation-finite by Lemma 4.7, as are their corner subcategories and object quotients, and in particular the incidence algebras of the faithful projective and injective supports of Definition 8.7.

C.2.1. One-sided lengths and the two-object return. We begin by noting a mild strengthening of Case 1 in the proof of Lemma 8.11: if $f: u \rightarrow c$ and $g: c \rightarrow v$ are generating rays whose corner bimodules are generated only at c , and if the radical generator $q = q_c$ of R_c satisfies $fq \neq 0$, $qg \neq 0$, and $q^2 \neq 0$, then

$$fg = 0. \tag{56}$$

Indeed, Case 1 of the proof of Lemma 8.11 uses only the four nonzero composite rays fg, fq, qg, q^2 together with the fact that f and g are generating rays and satisfy the radical inclusions (15); it does not require fq^2 or q^2g to be nonzero.

We now prove the dimension bound $\dim A_0(b, y) \leq 3$ (with equality only when $N_b = 3$) for a generating left-sided ray $f: b \rightarrow y$ in Proposition 8.25(1). If $\hat{q}_b = 0$ or $\hat{q}_b f = 0$, then $\dim A_0(b, y) = 1$; if $\hat{q}_b f \neq 0$, then $A_0(b, y) = R_b f$ by Lemma 8.18, so $\dim A_0(b, y) \leq N_b$, which already gives the bound when $N_b \leq 3$. Suppose for a contradiction that $N_b \geq 4$ and $\hat{q}_b^2 f \neq 0$ (so $\dim A_0(b, y) \geq 3$). We reach a contradiction in three steps:

Step 1: Support points and forward comparisons. In the faithful projective support at b , retain the six rays

$$Q_i = \hat{q}_b^i \quad (0 \leq i \leq 3), \quad Y_j = \hat{q}_b^j f \quad (j = 1, 2).$$

Clearly $Q_0 < Q_1 < Q_2 < Q_3$, and $Q_i < Y_j$ whenever $i \leq j$. Conversely, if $i > j$, an inequality $Q_i \leq Y_j$ would require a ray $h \in A_0(b, y)$ of weight $\ell(f) - (i - j)\chi$, which is impossible because $A_0(b, y) = R_b f$ has only weights $\ell(f) + a\chi$ with $a \geq 0$. Moreover, Y_1

and Y_2 are incomparable: any comparison between them would produce a local ray at y of weight χ , which by primitivity of χ would force $f\hat{q}_y \neq 0$ and make f two-sided.

Step 2: Excluding reverse comparisons. Suppose that $Y_j < Q_i$ for some $j \in \{1, 2\}$ and $i \in \{0, 1, 2, 3\}$. Left cancellation of $\hat{q}_b^j$ gives $fh = \hat{q}_b^a$ for a ray $h: y \rightarrow b$ and $a = i - j \in \{1, 2\}$. Since $a \leq 2$ and $N_b \geq 4$, we have $fh\hat{q}_b = \hat{q}_b^{a+1} \neq 0$, so $h\hat{q}_b \neq 0$. By the dual of Lemma 8.18, the corner bimodule $A_0(y, b)$ is generated at b : writing $h = l\hat{q}_b^u$ for the generating ray $l: y \rightarrow b$ gives $fl = \hat{q}_b^m$ with $1 \leq m \leq a \leq 2$. Note that $y \notin S_b$ because f is one-sided, and both lf and $l\hat{q}_b$ are nonzero because $f(lf) = \hat{q}_b^m f \neq 0$ and $f(l\hat{q}_b) = \hat{q}_b^{m+1} \neq 0$. If $A_0(y, b)$ were also cyclic at y , Lemma 8.18 would give $q_y l = l\hat{q}_b \neq 0$ with $\ell(q_y) = \chi$, and Lemma 8.21(1) applied to l and f would yield $lf = q_y$ and $fl = \hat{q}_b$, placing $y \in S_b$, a contradiction. Thus both $l: y \rightarrow b$ and $f: b \rightarrow y$ are generated only at b . Since $l\hat{q}_b \neq 0$, $\hat{q}_b f \neq 0$, and $\hat{q}_b^2 \neq 0$, the strengthened star identity (56) gives $lf = 0$, contradicting $f(lf) = \hat{q}_b^m f \neq 0$.

Step 3: The dimension contradiction. By Steps 1 and 2, the induced poset on $\{Q_0, Q_1, Q_2, Q_3, Y_1, Y_2\}$ has cover diagram

$$Q_0 \longrightarrow Q_1 \longrightarrow Q_2 \longrightarrow Q_3, \quad Q_1 \longrightarrow Y_1, \quad Q_2 \longrightarrow Y_2,$$

which is an orientation of the representation-infinite tree (51) (with $N = D = 12$ for dimensions 2 on Q_1, Q_2 and 1 on the four leaves), contradicting Lemma 8.8. Thus $\hat{q}_b^2 f = 0$ and $\dim A_0(b, y) \leq 2$ whenever $N_b \geq 4$; the dual argument gives the same bound for a generating right-sided ray.

Before proving the second part of Proposition 8.25(1), we isolate the two-object return calculation used in Step 2 above.

Lemma C.2. *Work in Settings 8.4 and 8.16. Let b be a supporting vertex, let $y \notin S_b$, and let $e: b \rightarrow y$ and $l: y \rightarrow b$ be generating rays satisfying*

$$el = \hat{q}_b^a \quad \text{with} \quad \hat{q}_b^{a+1} \neq 0. \quad (57)$$

Then e has type L, l has type R, and

$$N_b = 4, \quad el = \hat{q}_b^2, \quad le = 0. \quad (58)$$

Proof. Since $el\hat{q}_b = \hat{q}_b^{a+1} \neq 0$, we have $\hat{q}_b e \neq 0$ and $l\hat{q}_b \neq 0$, so both $A_0(b, y)$ and $A_0(y, b)$ are generated at b by Lemma 8.18. Neither $A_0(b, y)$ nor $A_0(y, b)$ can be cyclic at y : if $A_0(b, y)$ were cyclic at y , Lemma 8.18 and Lemma 8.21(1) would give $el = \hat{q}_b$ and $le = q_y$, placing $y \in S_b$; if $A_0(y, b)$ were cyclic at y , then $q_y l = l\hat{q}_b \neq 0$ would imply $(eq_y)l = el\hat{q}_b = \hat{q}_b^{a+1} \neq 0$, so $eq_y = \hat{q}_b e \neq 0$ and $A_0(b, y)$ would be cyclic at y as well. Thus e has type L, l has type R, and both corner bimodules are generated only at b . By (56) applied to $l: y \rightarrow b$ and $e: b \rightarrow y$, we obtain $le = 0$. Moreover, $a \geq 2$ because $a = 1$ would place $y \in S_b$ by Lemma 8.21(3). Note also that $\hat{q}_b^a e = e(le) = 0$ and $l\hat{q}_b^a = (le)l = 0$.

To rule out $a \geq 3$, set $E = \ell(e)$ and consider in a finite homogeneous category H the six graded objects

$$B_i = (b, i\chi) \quad (0 \leq i \leq 3), \quad Y_+ = (y, E + \chi), \quad Y_- = (y, E + (2 - a)\chi),$$

together with the selected morphisms along the oriented tree

$$B_0 \xrightarrow{\hat{q}_b} B_1 \xrightarrow{\hat{q}_b} B_2 \xrightarrow{\hat{q}_b} B_3, \quad B_1 \xrightarrow{e} Y_+, \quad Y_- \xrightarrow{l} B_2. \quad (59)$$

Every oriented path in (59) evaluates to a nonzero ray in H because $\hat{q}_b^3$ (as $a \geq 2$ and $\hat{q}_b^{a+1} \neq 0$), $\hat{q}_b e$, and $l\hat{q}_b$ are all nonzero. Writing $\ell(l) = a\chi - E$, any ray in $A_0(b, y) = R_b e$ has weight $E + m\chi$ with $0 \leq m < a$, and any ray in $A_0(y, b) = l R_b$ has weight $\ell(l) + m\chi$

with $0 \leq m < a$. Relative to the generating weights $E = \ell(e)$ (for $B_i \rightarrow Y_\pm$) and $\ell(l)$ (for $Y_\pm \rightarrow B_i$), the required exponent m of χ in each cross morphism space is:

morphism space	$B_i \rightarrow Y_+$	$Y_- \rightarrow B_i$	$B_i \rightarrow Y_-$	$Y_+ \rightarrow B_i$
exponent m of χ	$1 - i$	$i - 2$	$2 - a - i$	$i - a - 1$

When $a \geq 3$, the exponents in the last two columns are negative for all $0 \leq i \leq 3$, while the exponents in the first two columns lie in $\{0, \dots, a - 1\}$ only for the selected paths $B_0, B_1 \rightarrow Y_+$ and $Y_- \rightarrow B_2, B_3$. The only remaining unselected morphism space between distinct objects of (59) is $H(Y_-, Y_+)$ (of weight $(a - 1)\chi$), which cannot compose on either side with any nonidentity selected morphism because Y_- is a source and Y_+ is a sink in (59). Thus Lemma C.1 gives a bimodule retraction onto the incidence algebra of (59), contradicting Lemma 8.8 (with $N = D = 12$). Hence $a = 2$, and therefore $\hat{q}_b^4 = (el)^2 = e(le)l = 0$ while $\hat{q}_b^3 \neq 0$, so $N_b = 4$. $\square$

We now complete the proof of Proposition 8.25(1) by showing that if $rs \neq 0$ and r both have type L, then rs and r have the same endpoint layer (and dually for type R).

- *Generating first factor.* Suppose that $f: b \rightarrow y$ is a generating ray of type L and that fv is a proper shiftable ray of type L, so that $fv = \hat{q}_b^i h$ for a generating ray $h: b \rightarrow z$ of type L and $i \geq 1$. By the dimension bound just proved, $A_0(b, z)$ has dimension at most 3 and fv is shiftable, so $i = 1$ and $fv = \hat{q}_b h$. Since f and h are one-sided, $y, z \notin S_b$, so Lemma 8.22 implies that f and h are comparable under factorization:
 - If $fc = h$ for a nonidentity ray $c: y \rightarrow z$, then $fv = \hat{q}_b fc = f^+c$ shows that $\ell(v) = \ell(c) + \chi$, so $v = c^+$. Applying the $(L, *)$ row of (17) to the nonzero product $fc = h$ of type L forces the second substitution $fc^+ = fv$ to vanish, contradicting $fv = \hat{q}_b h \neq 0$.
 - If $y = z$ and $f = h$, then $\ell(v) = \chi$, so $v = \hat{q}_y$ by primitivity, making $f\hat{q}_y = fv \neq 0$ and contradicting the hypothesis that f has type L.
 - If $hc = f$ for a ray $c: z \rightarrow y$, then $\hat{q}_b h = fv = h(cv)$, so $cv \in R_z$ has weight χ and therefore equals $\hat{q}_z$, making $h\hat{q}_z = \hat{q}_b h \neq 0$ and contradicting the hypothesis that h has type L.

Thus a generating ray f of type L cannot have a proper shiftable product fv of type L.

- *Proper first factor.* Now let $r = \hat{q}_b^i r_0$ ($i \geq 1$, r_0 generating of type L) be a proper ray of type L such that $rs = \hat{q}_b^i(r_0s)$ has type L. Then $\hat{q}_b(r_0s)$ is a factor of $\hat{q}_b^{i+1}(r_0s) = \hat{q}_b(rs) \neq 0$, so r_0s is shiftable of type L or B. If r_0s had type B, then $(rs)\hat{q}_z = \hat{q}_b^i(r_0s\hat{q}_z) = \hat{q}_b^{i+1}(r_0s) = \hat{q}_b(rs) \neq 0$, contradicting the hypothesis that rs has type L. Thus r_0s has type L, so by the generating case just proved r_0s has endpoint layer 0, and hence $rs = \hat{q}_b^i(r_0s)$ has endpoint layer i , matching the endpoint layer of r . The statement for type R follows by duality.

C.2.2. Absence of external two-sided action at the return base. We next prove Proposition 8.25(4). Let (e, l) be a nondeep external return at b through $y \notin S_b$ (so $el = \hat{q}_b^2$, $\hat{q}_b^3 \neq 0$, $N_b = 4$, and $le = 0$ by Lemma C.2), and suppose for a contradiction that there exists a generating two-sided ray $f: b \rightarrow c$ with $c \notin S_b$ and $\hat{q}_b f \neq 0$.

First observe that no ray $v: c \rightarrow b$ can satisfy $fv = \hat{q}_b^2$. Indeed, since f is generating and two-sided, $A_0(b, c)$ is cyclic at both endpoints by Lemma 8.18; if $fv = \hat{q}_b^2 \neq 0$, then Lemma 8.21(1) supplies a generating ray $h': c \rightarrow b$ with $fh' = \hat{q}_b$ and $h'f = q_c$, which places $c \in S_b$ and contradicts $c \notin S_b$.

Now apply Lemma 8.22 at b to compare the outgoing generating rays $e: b \rightarrow y$ and $f: b \rightarrow c$. We cannot have $fa = e$ for a ray $a: c \rightarrow y$, as this would give $f(al) = el = \hat{q}_b^2$, which was just ruled out. Hence $eh = f$ for some ray $h: y \rightarrow c$. In the faithful projective

support at b , consider the four rays

$$e, \quad \widehat{q}_b e; \quad \widehat{q}_b^3, \quad \widehat{q}_b f.$$

All four upward comparisons from $\{e, \widehat{q}_b e\}$ to $\{\widehat{q}_b^3, \widehat{q}_b f\}$ hold via right multiplication by $l\widehat{q}_b$, $l, h\widehat{q}_c$ (using $\widehat{q}_b f = f\widehat{q}_c = eh\widehat{q}_c$), and h (using $\widehat{q}_b e \cdot h = \widehat{q}_b f$). On the other hand:

- e and $\widehat{q}_b e$ are incomparable because e is left-sided;
- $\widehat{q}_b^3$ and $\widehat{q}_b f$ are incomparable: $\widehat{q}_b^3 u = \widehat{q}_b f$ cancels to $\widehat{q}_b^2 u = f$, contradicting the fact that f is generating, while $(\widehat{q}_b f)v = \widehat{q}_b^3$ cancels to $fv = \widehat{q}_b^2$, which was ruled out above;
- no upper ray can precede a lower ray: $\widehat{q}_b^3 u = \widehat{q}_b^t e$ ($t \in \{0, 1\}$) would require a ray in $A_0(b, y) = R_b e$ of weight $\ell(e) - (3 - t)\chi < \ell(e)$; $(\widehat{q}_b f)u = e$ would contradict the fact that e is generating; and $(\widehat{q}_b f)u = \widehat{q}_b e \neq 0$ cancels to $fu = e$, which was already ruled out.

Thus $\{e, \widehat{q}_b e, \widehat{q}_b^3, \widehat{q}_b f\}$ forms a forbidden four-point crown in the faithful projective support at b , contradicting Lemma 8.8. The incoming assertion for $g: z \rightarrow b$ follows by passing to A_0^{op} .

C.2.3. Products with one-sided factors. We now prove Proposition 8.25(2) and begin the proof of Proposition 8.25(3). By Lemma 8.19 and the layer-preservation property of Proposition 8.25(1), if $r: x \rightarrow y$ and $s: y \rightarrow z$ are generating rays whose product rs is proper and shiftable, the pair of endpoint types of (r, s) must be one of

$$(B, B), \quad (B, L), \quad (R, B), \quad (L, R),$$

and when rs has type L or R , Proposition 8.25(1) forces rs to have endpoint layer 1 inside a three-dimensional corner bimodule $A_0(x, z)$ with $N_x = 3$ (respectively, $N_z = 3$).

For Proposition 8.25(2), let $r: x \rightarrow y$ of type B and $s: y \rightarrow z$ of type L be generating rays such that rs is proper and shiftable. By (17), rs has type L , so by the preceding paragraph $rs = \widehat{q}_x f$ where $f: x \rightarrow z$ is the generating ray of $A_0(x, z)$ (of type L) and $N_x = 3$. Suppose for a contradiction that $y \notin S_x$. Since $z \notin S_x$ (as f is one-sided), Lemma 8.22 compares r and f :

- If $ra = f$ for a ray $a: y \rightarrow z$, then $rs = \widehat{q}_x ra = r\widehat{q}_y a$ cancels to $s = \widehat{q}_y a$, contradicting the fact that s is generating.
- If $fa = r$ for a ray $a: z \rightarrow y$, then $f(as) = rs = \widehat{q}_x f$, so $as \in R_z$ has weight χ and equals $\widehat{q}_z$, making $f\widehat{q}_z = \widehat{q}_x f \neq 0$ and contradicting the fact that f has type L .

Thus $y \in S_x$, so r admits a primitive reverse. The case (R, B) follows by duality, proving Proposition 8.25(2).

Now consider Proposition 8.25(3): let $r: x \rightarrow y$ of type L and $s: y \rightarrow z$ of type R be generating rays whose product rs is proper and shiftable. By (17), rs has type B , so $rs = \widehat{q}_x^j f = f\widehat{q}_z^j$ for the generating ray $f: x \rightarrow z$ and some layer $j \geq 1$ with $\widehat{q}_x^{j+1} f = f\widehat{q}_z^{j+1} \neq 0$. If $x = z$, then $f = 1_x$ and Lemma C.2 immediately gives $j = 2$, $N_x = 4$, and that (r, s) is a nondeep external return at x , proving (3)(a).

Assume henceforth that $x \neq z$. Then $f: x \rightarrow z$ is a generating ray of type B , while $y \notin S_x$ and $y \notin S_z$ because r and s are one-sided. We claim that $z \in S_x$ (so $S_x = S_z$). Suppose to the contrary that $z \notin S_x$. Then Lemma 8.22 at x compares $r: x \rightarrow y$ and $f: x \rightarrow z$. We cannot have $ra = f$ (as $rs = f\widehat{q}_z^j = ra\widehat{q}_z^j$ would cancel to $s = a\widehat{q}_z^j$, contradicting the fact that s is generating), so $fa = r$ for a generating ray $a: z \rightarrow y$. Canceling f from $f(as) = rs = f\widehat{q}_z^j$ gives $as = \widehat{q}_z^j$ with $\widehat{q}_z^{j+1} \neq 0$, so Lemma C.2 implies that (a, s) is a nondeep external return at z (with $j = 2$). However, $f: x \rightarrow z$ is an incoming generating two-sided ray at z with $x \notin S_z$ and $f\widehat{q}_z \neq 0$, which contradicts Proposition 8.25(4)! Therefore $z \in S_x$.

C.2.4. *A complementary path forces an endpoint return.* To finish the proof of Proposition 8.25(3)(b), continue with $x \neq z$ and $z \in S_x$. Since S_x is a primitive return corner, the generating ray $f: x \rightarrow z$ admits a primitive reverse $g: z \rightarrow x$ satisfying $fg = \hat{q}_x$ and $gf = \hat{q}_z$. Note that

$$r(sg) = rsg = \hat{q}_x^j fg = \hat{q}_x^{j+1} \neq 0, \quad (gr)s = grs = gf\hat{q}_z^j = \hat{q}_z^{j+1} \neq 0, \quad (60)$$

so both $sg: y \rightarrow x$ and $gr: z \rightarrow y$ are nonzero rays, as is $gfg = \hat{q}_z g$ (since $\hat{q}_z^2 = gfgf \neq 0$ from $j \geq 1$ and $\hat{q}_z^{j+1} \neq 0$).

Set $F = \ell(f)$ and $E = \ell(r)$. In a finite homogeneous category H , retain the six graded objects

$$\begin{aligned} B_0 &= (z, F), & B_1 &= (x, \chi), & B_2 &= (z, F + \chi), & B_3 &= (x, 2\chi), \\ Y_- &= (y, E + (1 - j)\chi), & Y_+ &= (y, E + \chi), \end{aligned}$$

and select the oriented tree with spine $B_0 \xrightarrow{g} B_1 \xrightarrow{f} B_2 \xrightarrow{g} B_3$ and branches $B_1 \xrightarrow{r} Y_+$ and $Y_- \xrightarrow{s} B_2$. By (60), every oriented path in this six-vertex tree evaluates to a nonzero ray in H . After discarding negative-degree factors, checking the cover conditions of Lemma C.1 at the five arrows of this tree reduces to the following six possibilities for an unselected factor h :

candidate factor or weight	consequence
$fh = r$	$hs = \hat{q}_z^j$ (canceling f from $fhs = f\hat{q}_z^j$)
$\ell(r) + \ell(h) = \ell(f)$	$\ell(h) = \ell(s) - j\chi < \ell(s)$ (impossible as s is generating)
$rh = \hat{q}_x$	$y \in S_x$ by Lemma 8.21(3) (impossible)
$hf = s$	$rh = \hat{q}_x^j$ (canceling f from $rhf = \hat{q}_x^j f$)
$\ell(h) + \ell(s) = \ell(f)$	$\ell(h) = \ell(r) - j\chi < \ell(r)$ (impossible as r is generating)
$hs = \hat{q}_z$	$y \in S_z$ by Lemma 8.21(3) (impossible)

The second, third, fifth, and sixth rows are impossible in all cases. In the first row, any ray $h: z \rightarrow y$ satisfying $fh = r$ is a factor of the generating shiftable ray r and is therefore generating by Lemma 8.14, yielding a return $hs = \hat{q}_z^j$ at z with $\hat{q}_z^{j+1} \neq 0$. Similarly, in the fourth row, any ray $h: y \rightarrow x$ satisfying $hf = s$ is generating and yields a return $rh = \hat{q}_x^j$ at x with $\hat{q}_x^{j+1} \neq 0$.

If neither the first nor the fourth row occurs, then all six rows are excluded, so Lemma C.1 gives a bimodule retraction onto the incidence algebra of the six-vertex tree, contradicting Lemma 8.8 ($N = D = 12$). Thus at least one of the two factorizations $fh = r$ (with $hs = \hat{q}_z^j$) or $hf = s$ (with $rh = \hat{q}_x^j$) must hold. Applying Lemma C.2 to the resulting return (h, s) at z or (r, h) at x shows that $j = 2$ and that (h, s) (respectively, (r, h)) is a nondeep external return at z (respectively, at x). This completes the proof of Proposition 8.25(3)(b), and hence of Proposition 8.25.

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