CRYSTALS AND PREPROJECTIVE ALGEBRA MODULES

Based on arXiv: 2202.02490
with Elek, Kamnitzer, and Morton-Ferguson

Plan

I) see patterns, mostly blackboxing the objects involved; state generalization; literature review

II) give precise definitions uncovering the key ingredients in our generalizations.

$$\Pi(Q), \quad \Lambda, \quad Gr(M), \ldots$$

III) RPP combinatorics coming from

(a) tensor product quiver varieties via quiver grassmannians

(b) nilpotent filtrations of Π-modules.

IV) Open questions related to canonical bases and cluster algebras.

Patterns (Demonstrating cool connections!)

Consider the set $SSYT_4(1^2)$ of semistd Young tableaux in $\{1,2,3,4\}$ of shape $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$.

This set is in interesting bijection bijection with several other (po)sets.

① $\mathrm{Irr}\, F_4(1^2)$ where $F_4(1^2)$ is the 4-step Springer fibre preserved by the Jordan type $(1^2)^t = (2)$ normal form: $A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$

$$\{(0 \overset{1}{\subseteq} V_1 \overset{0}{\subseteq} V_2 \overset{1}{\subseteq} V_3 \overset{0}{\subseteq} \mathbb{C}^2) : AV_i \subseteq V_{i-1}\}$$

Exer This set is size $\frac{4!}{4} := \frac{|S_4|}{|S_2 \times S_2|}$

$\boxed{\text{Eg}}$ $F_4(1^2)_{\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array}} = \{V_\bullet \in F_4(1^2) : \dim V_i/V_{i-1} = \mu_i\}$

$\mu = wt\left(\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array}\right)$

$wt\left(\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array}\right) = (1010)$

$$AV_1 \subseteq 0 \Rightarrow \ker A \supseteq V_1 \Rightarrow V_1 = \langle e_1\rangle$$

... This irreducible component is the point $\{0 \subseteq \langle e_1\rangle \subseteq \langle e_1\rangle \subseteq \mathbb{C}^2 \subseteq \mathbb{C}^2\}$

② $GT_4(1^2)$ the set of Gelfand-Tsetlin patterns of shape $(1,1,0,0)$

i.e. arrays (g^i_j)

$$\begin{array}{ccccccccccccc} & & & & & & g^1_1 & & & & & & \\ & & & & & \geq & & \geq & & & & & \\ & & & & g^2_1 & & & & g^2_2 & & & & \\ & & & \geq & & \geq & & \geq & & \geq & & & \\ & & g^3_1 & & & & g^3_2 & & & & g^3_3 & & \\ & \geq & & \geq & & & & \geq & & & & \geq & \\ g^4_1=1 & & & & g^4_2=1 & & & & g^4_3=0 & & & & g^4_4=0 \end{array}$$

s.t.

$$\begin{array}{ccccc} & & g^i_k & & \\ & \geq & & \geq & \\ g^{i+1}_k & & & & g^{i+1}_{k+1} \end{array}$$

Eg $\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array} \rightsquigarrow g = \begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 0 & & \\ & 1 & & 1 & & 0 & \\ 1 & & 1 & & 0 & & 0 \end{array}$

In general, a tableax τ defines

$$(g^i_j) = \text{shape of } \tau|_{\{1\dots i\}}$$

Spoiler: GT patterns will give reverse plane partitions!

③ $\boxed{J}(\boxed{H}(2132))$ $\boxed{\text{order ideals}}$ in the $\boxed{\text{heap}}$ of $s_2 s_1 s_3 s_2$

$\underline{w} = (2,1,3,2) = (i_1, i_2, i_3, i_4)$

Then $H(\underline{w})$ is the poset obtained by taking transitive closure the relation

$$a \prec b \quad \text{if} \quad a > b \text{ and } s_{i_a} s_{i_b} \neq s_{i_b} s_{i_a}$$

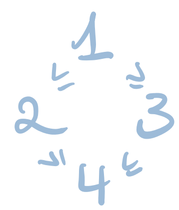
on $\{1,2,3,4\}$ (more generally, $\{1,2,\dots l\}$ where $l = \mathrm{len}(w)$)

In this example $\underline{w} = \begin{pmatrix} \underline{2} & \underline{1} & \underline{3} & \underline{2} \\ \underline{1} & \underline{2} & \underline{3} & \underline{4} \end{pmatrix}$

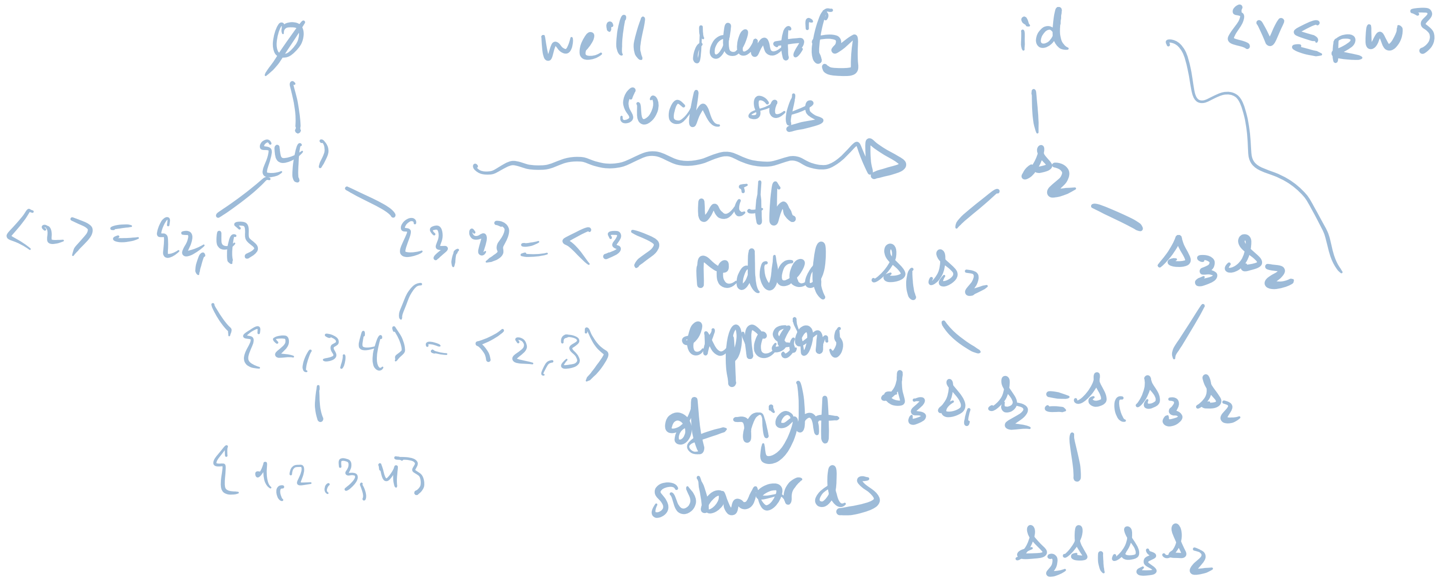
Can compare

$1,2;\ 1,3;\ 3,4;\ 2,4$

So

$H(\underline{w}) =$ 

This heap has order ideal poset:



For example $\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array}$, of weight (1010),

is associated with $s_2 \in J(H(\underline{w}))$

because $s_2(1100) = (1010)$

and s_2 is minimal for this requirement.

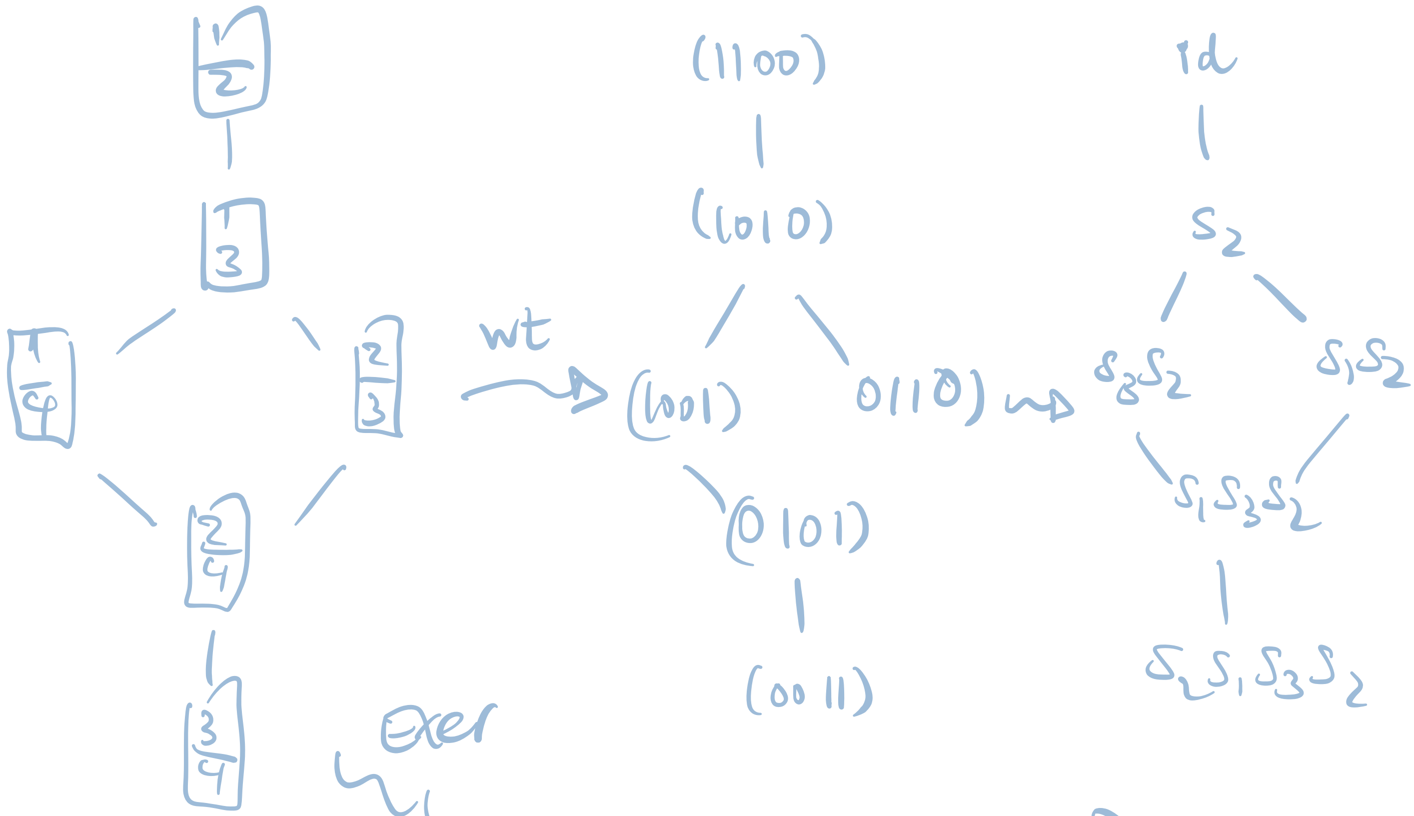
Remark $\underline{w} = (2\,1\,3\,2)$ is fully commutative, which means that every reduced word of $s_2 s_1 s_3 s_2$ can be obtained by swapping adjacent commuting pairs of generators.

This attribute allows us to write unambiguously $H(w)$ for $w \in S_4$ because w is fully commutative.

So any choice of reduced word will result in the same heap.

4 The orbit set $S_4 \cdot (1100)$

Summary



Exer ⇝ poset of of patterns??

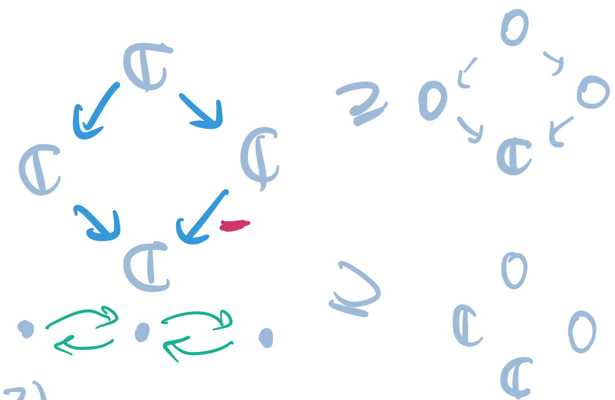
⑤ Irr $\underbrace{Gr(I(1^2))}$

quiver grassmannian of modules for $\Pi(A_3)$ which are submodules of $I(1^2)$ the injective hull of $S(2)$ the simple over vertex 2.

$I(2) = I(1^2) =$

$\uparrow$

ω_2



Exer..

$I(3) = I(\omega_3) \leftarrow S(3)$

These posets are actually in crystal bijection.

Eg $X < Y$ in one of these posets

$\Leftrightarrow f_i Y = X$ where f_i denotes a lowering operator in the crystal structure intrinsic to the poset

Compatibility $\quad \tau = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array} \xrightarrow{f_1} \tau' = \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array}$

[Hong, Kang]

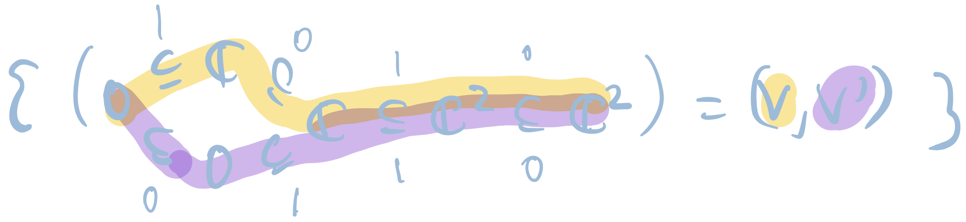
① $f_1 F_4(1^2)_\tau = F_4(1^2)_{\tau'}$

indeed in this context crystal operators f_i are defined using correspondences.

$$\left\{ (V, V', A) : \begin{array}{l} V_k = V'_k \quad k \neq i \\ V'_i \subseteq V_i \quad \dim V_i = \dim V'_i + 1 \end{array} \right\}$$

$\{\cdot\} \overset{Ex}{\cong} F(\lambda)_\mu$

$F(\lambda)_{\mu - \alpha_i} = \{\cdot\}$

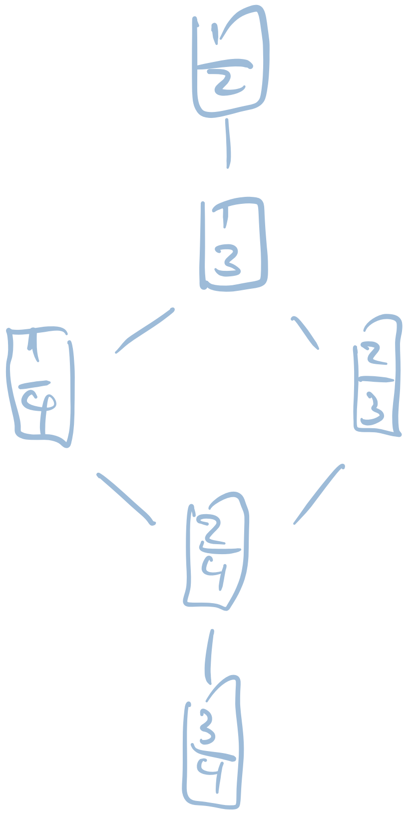


Indeed $SSYT_4\left(\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}\right)_{0110} = \left\{ \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array} \right\}$.

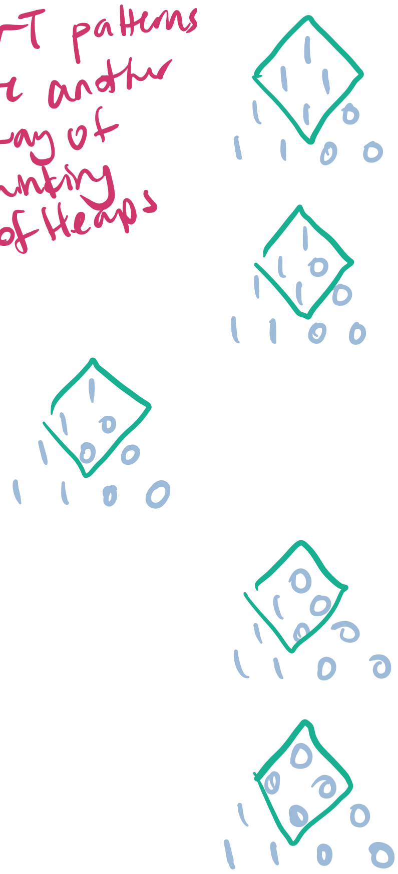
→ Similarly, in the heap model f_i will act on a right subword by premultiply by s_i

→ And, in $S_4 \cdot (1100)$, f_i will act by permuting μ by s_i

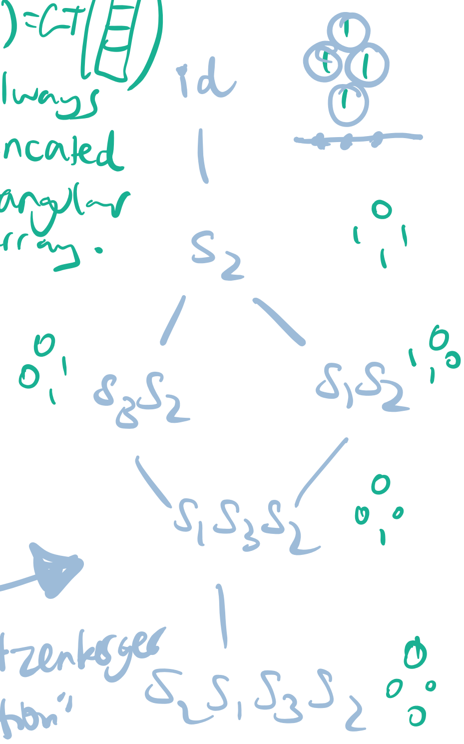
Generalization



GT patterns are another way of thinking of Heaps



$GT(\omega_n) = GT(\text{column})$ can always be truncated to rectangular subarray.



"Schutzenberger involution"

rotate the

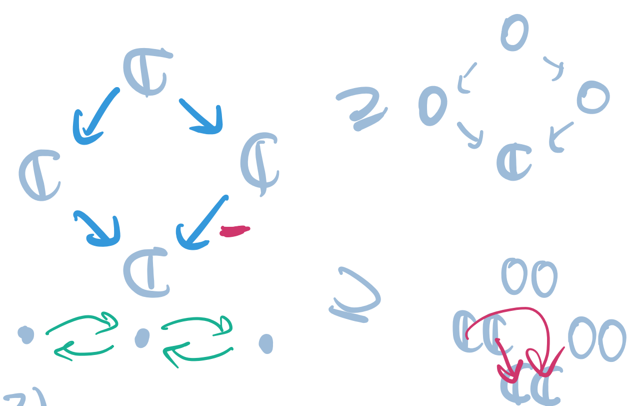
90° and flip in a vertical axis

⑤ Irr $\underbrace{Gr(I(1^2))}$

quiver grassmannian of modules for $\Pi(A_3)$ which are submodules of $I(1^2)$ the injective hull of $S(2)$ the simple over vertex 2.

$I(2) = I(1^2) =$

ω_2



Exer..

$I(3) = I(\omega_3) \leftarrow S(3)$

False: $\forall\ M \subseteq I(i)^{\oplus n}$, $M = \oplus N$, $N \subseteq I(i)$.

We are witnessing an iso of crystals

$$\begin{tikzcd}
\operatorname{Irr} F_m(\lambda) \arrow[rr] & & SSYT_m(\lambda) & & J(H(w)) \arrow[ll] \\
\operatorname{Irr} Gr_{\pi(A_{m-1})}(I(\lambda)) \arrow[urr] & & GT_m(\lambda) \arrow[u] & & W\cdot\lambda \arrow[ull]
\end{tikzcd}$$

where w is minimal for $w\lambda = w_0\lambda$

when λ is minuscule.

Our Goal Generalize

$$\operatorname{Irr} F_m(\lambda) \to SSYT_m(\lambda)$$

in a type-independent way, by using the geometry of quiver grassmannians

We replace $F_m(\lambda)$ by $Gr_{\pi(A_{m-1})}(I(\lambda))$

for $\lambda = \sum \lambda_i \omega_i$

$$I(\lambda) := \oplus \underbrace{I(i)}_{\text{injective hull of simple } S(i).}{}^{\oplus \lambda_i}$$

We replace $SSYT_m(\lambda)$ by (disjoint unions of) reverse plane partitions of shape $H(w_0^i)$ where $\forall i$, w_0^i is minimal for $w_0^i \omega_i = w_0 \omega_i$

Theorem A There is a crystal isomorphism (up to Schützenberger involution) of $SSYT(n\omega_i)$ and $RPP(w_0^i, n)$.

Theorem B There is a compatible isomorphism

$$Gr(\mathcal{I}(i)^{\oplus n}) \longrightarrow F(n\omega_i)$$

of varieties!

Type ADE

Let λ be dominant minuscule weight. (W acts transitively on weights of $V(\lambda)$.) Let w be minimal $w\lambda = w_0\lambda$. Call w λ-minuscule. (Stembridge.)

To a λ-min. elt. w is associated a poset

$H(w)$ "heap of w".

If $\underline{w} = (i_1 \dots i_\ell)$ is a reduced word then $H(\underline{w}) = (\underbrace{\{1, \dots, \ell\}}_{=[\ell]}, \bar{\prec})$ where

$\bar{\prec}$ is the trans. clos. of

$\forall a, b \in [\ell],\ a \prec b$ iff $a > b$ and $s_{i_a} s_{i_b} \neq s_{i_b} s_{i_a}$

Fact/Exer in type A heaps are Young diagrams.

An RPP of shape $H(w)$ and height n is an order reversing map

$$\Phi : H(w) \longrightarrow \{1, \dots, n\} \sqcup \{0\} = [n] \cup \{0\}.$$

$$a \leq b \rightsquigarrow \Phi(a) \geq \Phi(b)$$ (in the usual increasing order on $[n] \sqcup \{0\}$)

We denote the set of all such RPPs by

$$\mathrm{RPP}(H(w), n) \quad \text{or} \quad R(w,n).$$

Remark / Exer when $n=1$.

$$R(w, 1) = J(H(w))$$

Theorem There is a crystal bijection

$$\begin{tikzcd} R(w,n) \arrow[r] \arrow[d] & B(n\lambda) \arrow[d] \\ J(H(w))^{\times n} \arrow[r] & B(\lambda)^{\otimes n} \end{tikzcd}$$

and this diagram commute:

$$R(w,n) \to J(H(w))^{\times n}$$
$$\Phi \mapsto (\phi_1, \dots, \phi_n)$$

where $\phi_k = \Phi^{-1}(\{n-k+1, \dots, n\})$.

Eg $\underline{w} = (1\,3\,2)$

$H(\underline{w}) =$ 

A_3

$$R(\underline{w}, 2) = \left\{ {}^{0}{}_{0}{}^{0},\ {}^{0}{}_{1}{}^{0},\ {}^{0}{}_{2}{}^{0},\ {}^{0}{}_{2}{}^{2},\ {}^{1}{}_{1}{}^{0},\ {}^{1}{}_{2}{}^{0},\ {}^{2}{}_{2}{}^{0},\ {}^{0}{}_{1}{}^{1},\ {}^{0}{}_{2}{}^{1},\ {}^{2}{}_{2}{}^{2},\ {}^{1}{}_{1}{}^{1},\ {}^{1}{}_{2}{}^{1} \right\}$$

Eg $\Phi = {}^{1}{}_{2}{}^{0} \longrightarrow ({}^{1}{}_{1}{}^{0},\ {}^{0}{}_{1}{}^{0})$

The result will be ordered wrt the order on $J(H(w))$.

Crystals Let $\mathfrak{g}$ be ADE. simple. P-weight.
$\{\alpha_i\}$ simple roots

Def The data $(B, wt, \varepsilon_i, \varphi_i, e_i, f_i)$

defines a $\mathfrak{g}$-crystal which is <u>upper</u> semi-normal.

or highest weight if (1) it's a crystal, i.e.

- $wt : B \to P$
- $\varepsilon_i, \varphi_i : B \to \mathbb{N}$
- $e_i, f_i : B \to B \sqcup \{0\}$

Satisfying

- $\varphi_i(b) = \varepsilon_i(b) + wt(b)(\alpha_i^\vee)$
- $wt(e_i b) = wt(b) + \alpha_i$

  $wt(f_i b) = wt(b) - \alpha_i$ if $e_i b, f_i b \in B$
- $f_i(b_2) = b_1$ iff $e_i(b_1) = b_2$

(2) it is seminormal for the raising operator e_i

$$\varepsilon_i(b) = \max\{n \geq 0 : e_i^n(b) \neq 0\}$$

In other words, there is a distinguished highest weight elt. b_+ which we can reach to from every $b \in B$ by applying $e_{\underline{i}} = e_{i_1} \cdots e_{i_N}$

for some $\underline{i} \in I^N$ for some n. : $b_t = e_{\underline{i}} b$

vertex set of Γ - dynkin graph.

Example $W\text{-}\omega_2$

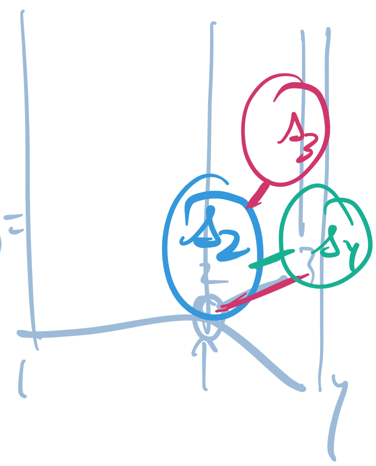
There is a classification of

minuscule fundamentals in all types

$\omega_i \leftarrow$ vertex i of the Dynkin

diagram can support a

glass bead game.

$H(\underline{w}) =$  $\not\rightarrow$ Π-module.

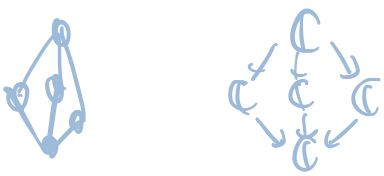
$B_{\underline{w}}(\lambda)$

· non minuscule

· non-simply laced.

too many edges

for nice module structure.



$\varepsilon : Q_1 \to \{\pm 1\}$.

$\underline{w} = 21432$ when $\Gamma = D_4$ is not minuscule.

Next time we'll start with recalling the $\otimes$ rule on crystals.

$$R(\mathfrak{gl}_n) \longrightarrow \mathcal{J}(H(\omega))^{\times n}.$$