# ŌSAKA 2022 Lecture III

## HOUSEKEEPING

$<_L$ instead of $<_R$

the glass bead game

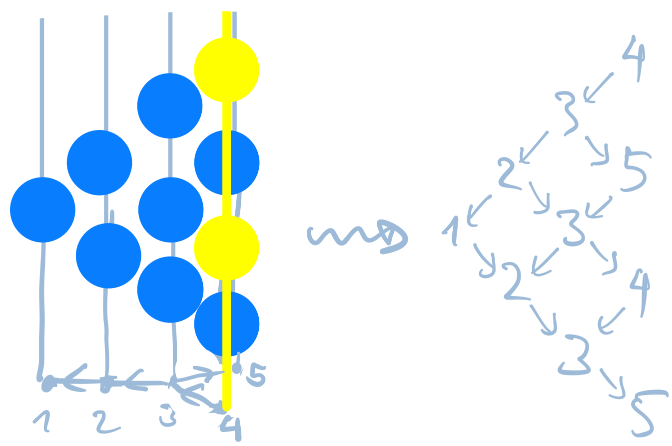
other influences: TODO

$\leq_L$ instead of $\leq_R$ : notation for "is right subword"
s/b $\leq_L$ as its std to define left weak order
by $u \leq_L w$ if $w = s_1 \cdots s_k u$ ...

$H(\underline{w})$ as a bead game or Mikado: we can visualizes heaps of words (après Stembridge, Viennot) as configurations of beads on a $\Gamma \times \mathbb{R}_{\geq 0}$ abacus where $\Gamma$ is the relevant Dynkin diagram

Eg

$\Gamma = D_5$
$\underline{w} = (5\,3\,2\,4\,1\,3\,2\,5\,3\,4)$



## Tensor product crystals

Given two $\mathfrak{g}$-crystals $A$, $B$, can endow $A \otimes B$ with crystal str with raising and lowering operators defined as:

$$e_i(a \otimes b) = \begin{cases} a \otimes e_i(b) & \text{if } \varepsilon_i(a) \leq \varphi_i(b) \\ e_i(a) \otimes b & \text{else.} \end{cases}$$

$$f_i(a \otimes b) = \Big\{ \quad \text{Similar rule.}$$

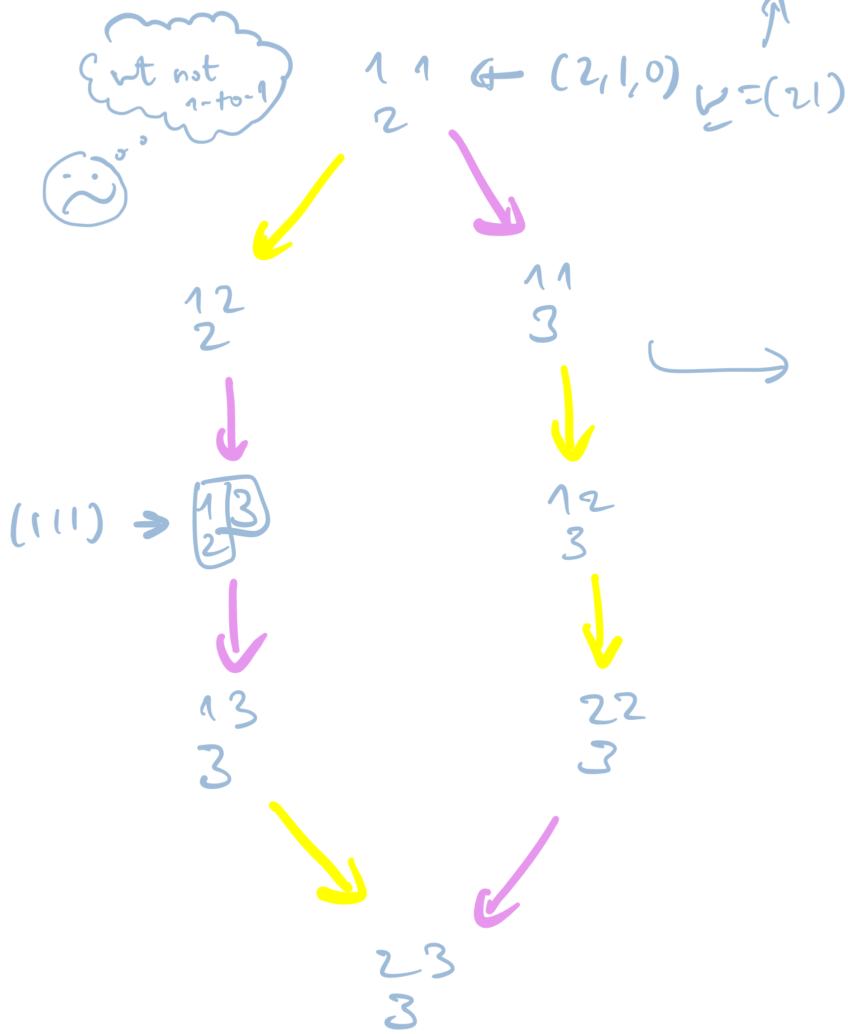
In practice the sign pattern (signature rule) rewording of this rule to determine $e_i$ or $f_i$ of a $\otimes$ crictor.

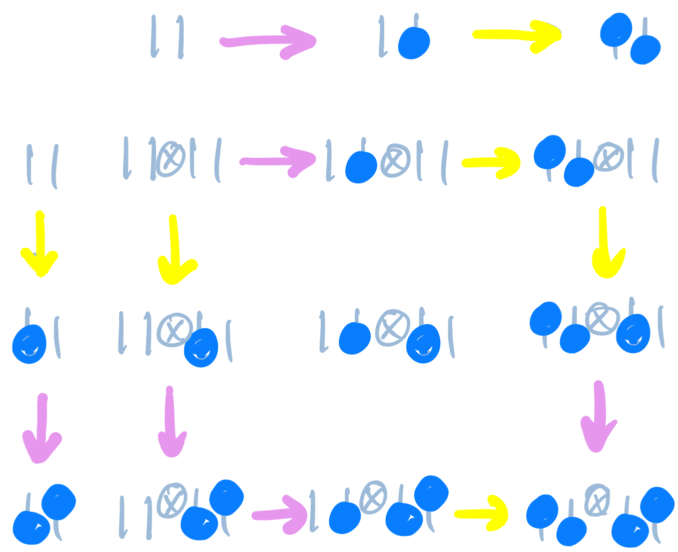
**Lem** (Tingley) Given $b \in B$, denote by $S_i(b)$ the string made up of $\varphi_i(b)$ many $+$ signs followed by $\varepsilon_i(b)$ many $-$ signs. Call $S_i(b)$ the sign pattern

of b. On tensor crictors define $S_i(a\otimes b) = S_i(a)\, S_i(b)$.
The action of $f_i$ is found by first cancelling in $S_i(a\otimes b)$ all $-+$ pairs. The result is a sequence of the form $+ \_ + - \ldots -$. The signature rule says that $f_i$ will act on the elt contributing the rightmost $+$ if it exists, and by zero otherwise.

$\Gamma = A_3$

Eg $B(\,\square\,)$ $\qquad B(\square)\otimes B(\square)$ $\leftarrow \epsilon(12)=\nu$



Exr
use the sign rule to verify this $\otimes$.

Question can we avoid $\otimes$ combinatorics when

$$\lambda = \sum a_i \omega_i$$

Example $\Gamma = A_3$ $\quad \underline{w} = (2\,3\,1\,2)$ $\qquad \phi_1 \otimes \phi_2 \otimes \phi_3 \otimes \phi_4$

Use the signature rule to compute $f_2(\Phi)$, $e_2(\Phi)$

$$S_2(\phi_1 \otimes \cdots \otimes \phi_4) = - \ \square\ + - \leadsto -$$

So $f_2 \underline{\Phi} = 0$.

## Preprojective algebras

Let $Q = (Q_0, Q_1, h, t : Q_1 \to Q_0)$ be quiver with underlying graph $\Gamma = (Q_0, Q_1)$ or $(I, E)$ among $(A_n, D_n, E_\bullet)$

Double $Q$ by adding $j \xrightarrow{a^-} i$ for each $i \xrightarrow{a^+} j$ in $Q_1$ to get the associated doubled quiver $\bar{Q} = (Q_0, \bar{Q}_1, h, t : \bar{Q}_1 \to Q_0)$. Define $\bar{Q}_1 \xrightarrow{*} \bar{Q}_1$ by $a^{\pm} \mapsto a^{\mp}$ and charge $\bar{Q}$ by defining

$$c : \bar{Q}_1 \to \{-1, +1\} : a^{\pm} \mapsto \pm 1$$

Notice $c(a^*) = -c(a)$ for all $a \in \bar{Q}_1$

**Def** The preprojective algebra $\Pi(Q)$ over $\mathbb{C}$ is

$$\mathbb{C}\bar{Q} \Big/ \sum_{a \in \bar{Q}_1} c(a) a a^*$$ Preprojective relation.

## Varieties of modules

modules for $\Pi(Q)$

- $Q_0$-grading so that $\underrightarrow{\dim}\, M \in \mathbb{N}^{Q_0}$

- $M_a : M_{\underset{t(a)}{i}} \to M_{\underset{h(a)}{j}}$ whenever $i \xrightarrow{a} j \in \overline{Q}_1$
- $\displaystyle\sum_{\substack{a\in\overline{Q}_1 \\ h(a)=i}} M_a M_{a^*} c(a) = 0$ at each $i \in Q_0$

Given $\vec{v} = (v_i) \in \mathbb{N}^{Q_0}$, consider $\Lambda(\vec{v})$ the variety of $\Pi(Q)$-module structures on $\mathbb{C}^{v_1} \oplus \cdots \oplus \mathbb{C}^{v_n}$ ($|I| = n$)

$\Lambda = \bigsqcup \Lambda(\vec{v})$ is Lusztig's nilpotent variety.

The simple $\Pi(Q)$-mods are 1d. $S(i)$, $\underline{\dim}\, S(i) = e_i$
$(0..010\ 0)$

**Def** The socle of $M$ is the maximal semisimple submodule of $M$.

Let $T(i)$ denote the injective hull of $S(i)$.

$\underline{\dim}\, T(i) = \vec{v} = (v_i)$ s.t. $\omega_i - w_0\omega_i = \sum v_i \alpha_i$

$\operatorname{Soc} T(i) = S(i)$.

<u>Thm</u> (Baumann-Kamnitzer) $\forall w \in W$, $\exists!\, T(i,w)$.

with $\underline{\dim} = (v_i)$ s.t. $\omega_i - w\omega_i = \sum v_i \alpha_i$

and $\operatorname{Soc} T(i,w) = S(i)$. **Maya modules.**

If $w\omega_i = \omega_i$, $T(i,w) = 0$. <u>Exercise</u> $T(i,w) \subseteq T(i)$.

More generally if $\lambda = \sum d_i \omega_i$

$$T(\lambda) := \bigoplus T(i)^{\oplus d_i} \qquad \lambda - w_0\lambda = \sum v_i \alpha_i$$

$$T(\lambda, w) := \bigoplus T(i, w)^{\oplus \lambda_i} \qquad \lambda - w\lambda := \sum v_i \alpha_i.$$

Exr $\operatorname{Soc} T(\lambda, w)$?

## Quiver grassmannians

Define

$$Gr(T(\lambda)) = \{ M \overset{\pi}{\subseteq} T(\lambda) : M \in \Lambda \}$$

$Gr(T(\lambda, w))$ defined analog.

The connected components are

$$Gr(\vec{v}, \_) = \{ M \subseteq \_ : \underset{\to}{\dim} M = \vec{v} \}.$$

**Thm** (Saito, Savage, Savage-Tingley)

① $\operatorname{Irr} Gr(T(\lambda)) \cong B(\lambda)$

② $\operatorname{Irr} Gr(T(\lambda, w)) \cong B_w(\lambda)$

## From heaps to modules

Apparent in the glass bead visualization.
And can be derived from the def of a heap.

Heaps are equipped a map

$\pi : H(w) \to I$ with fibres $\pi^{-1}(i) =: H(w)_i$

Note $H(w)_i$ are totally ordered $\rightsquigarrow$ filtration -
(Later)

We can give $I$-graded vector space.

$$\mathbb{C}H_w := \mathrm{Span}_{\mathbb{C}}(H(w))$$

$$= \bigoplus_{i \in I} \underbrace{\mathbb{C}H(w)_i}_{\mathrm{Span}\, H(w)_i}$$

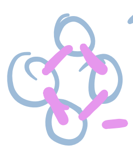
Promote this vectsp. to a $\Pi(Q)$-module.

by defining $\forall x \in \mathbb{C}H_w$, $\forall a \in \bar{Q}_1$

$$a \cdot x = \pm y$$

if $x > y$ are adjacent in $H(w)$.

$\pi(x) = t(a)$
$\pi(y) = h(a)$.

Eg  $= H(2132)$ $\rightsquigarrow$ $\mathbb{C}H_w$

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"] & \mathbb{C}^2 \arrow[l, bend left, "{[1\ 0]}"] \arrow[r, bend left, "{[1\ 0]}"] & \mathbb{C} \arrow[l, bend left, "{\left[\begin{smallmatrix}0\\-1\end{smallmatrix}\right]}"] \end{tikzcd}$$

**Prop** • If $w$ is minuscule, then $\mathbb{C}H(w)$ is a module for $\Pi(Q)$.

• If $w$ is dominant $\lambda$-minuscule, $\mathbb{C}H(w) \cong T(\lambda, w)$.

In particular, when $\lambda$ is minuscule,

$$\mathbb{C}H(w_0^J) \cong T(\lambda)$$

where $w_0^J$ is the smallest rep of $w_0 W_J$

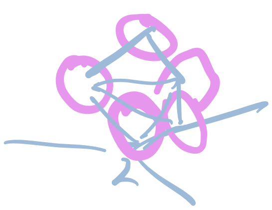
where $W_J = \langle s_j : j \in J \rangle$

and $J = \{ j : s_j \lambda = \lambda \} \subseteq I$.

Eg (non) $\Gamma = D_4$

$$H(2,1,3,4,2) \rightsquigarrow \begin{tikzcd} & \mathbb{C} \arrow[dl] \arrow[d] \arrow[dr] & \\ \mathbb{C} \arrow[dr] & \mathbb{C} \arrow[d] & \mathbb{C} \arrow[dl] \\ & \mathbb{C} & \end{tikzcd}$$

not a module for $\Pi(Q)$ for any choice of $\pm$ in def of $\mathbb{C}\overline{Q} \circlearrowright H(w)$.




$w$ is assumed minuscule.

**Prop** $\forall\, \phi \in J(H(w))$, $\mathbb{C}\phi \overset{\Pi}{\subseteq} \mathbb{C}H(w)$

And $\phi \mapsto \mathbb{C}\phi$ yields a bij.

$$\mathcal{J}(H(w)) \longrightarrow \operatorname{Irr} Gr(\mathbb{C}H(w))$$

~~Study Nakajima tensor product varieties~~.

Goal: $(X_1 \ldots X_r) \in \operatorname{Irr} Gr(T(\lambda^1)) \times \cdots \times \operatorname{Irr} Gr(T(\lambda^r))$

$$\Rightarrow X_1 \otimes - \otimes X_r \in B(\lambda)$$

$$\downarrow$$

$$\otimes B(\lambda^i) \ni Z(X_1, \ldots, X_r) := \overline{\{ M \in Gr(T(\lambda)) : M^k \in X_k \ \forall k = 1..r \}}$$

$\mathbb{C}\phi_k$

$\lambda = \lambda^1 + \cdots + \lambda^r$ is a composition.

$\phi_1 \otimes \cdots \otimes \phi_r$ is a filtration of an rpp $\Phi$ by order ideals.

$T(\lambda, w)$ are $\tau$-rigid?