Generalized preprojective algebras (1)

Plan for today :

1 Introduction : classical preprojective algebras

2 Graded generalized preprojective algebras

1 Introduction

vertices arrows

. Q : finite connected acyclic quiver : $Q = (Q_0, Q_1)$

. K : field . $\rightsquigarrow$ KQ : path algebra.

. $\overline{Q}$: double quiver : $\forall \alpha : i \to j \in Q_1$ add $\alpha^* : j \to i \in \overline{Q}_1$

. $\rho = \sum_{\alpha \in Q_1} [\alpha, \alpha^*] \in K\overline{Q}$.

Definition : (Gelfand - Ponomarev 1979)

$$\pi(Q) := K\overline{Q}/(\rho).$$

. $\dim \pi(Q) < +\infty \iff Q$ of type A, D, E

in this case $\pi(Q)$ is self injective.

. In general $\pi(Q)|_{KQ} \cong \bigoplus$ "indecomposable preprojective modules" preprojective : $\tau^{-k} P$

τ: A-R transl.

P: indec. proj KQ-module

. Why is $\pi(Q)$ important ?
relations with Lie theory.

<u>Fix $K = \mathbb{C}$.</u>

. 1990 Lusztig : "nilpotent varieties" = representation varieties of <u>nilpotent</u> modules :

all composition factors are 1-dimensional.

Let $\mathfrak{g}$ be the $\underbrace{\text{Kac-Moody}}_{\text{symmetric}}$ algebra associated with Q

Let $U_q(\mathfrak{g})$ be the corresp. quantum group.

<u>Thm</u> Lusztig 1991 - Kashiwara-Saito 1997
2000

. Nilpotent varieties $\pi(Q)^{nil}_{\underline{d}}$ are of pure dimension :

$\underline{d}$: dim vector

all irred. components have the same dimension :

$\dim(\mathrm{rep}(Q)_{\underline{d}})$ ← one of the irreducible components.

$$\# \mathrm{Irr}\left(\pi(Q)^{nil}_{\underline{d}}\right) = \dim U_q^+(\mathfrak{g})_{\underline{d}}$$

$$\mathrm{Irr} := \bigsqcup_{\underline{d}} \mathrm{Irr}\left(\pi(Q)^{nil}_{\underline{d}}\right)$$ is a labelling set for vertices of $\mathcal{B}(-\infty)$

$\rightsquigarrow$ geometric description of $\mathcal{B}(-\infty)$

- There is an associative algebra of constructible functions on nilpotent varieties isomorphic to $U^+(\mathfrak{g})$.

$\rightsquigarrow$ semicanonical basis of $U^+(\mathfrak{g})$

<u>Nakajima varieties</u>: V, W Q_0-graded vector spaces

$$\begin{array}{ccc} \mathfrak{M}(V,W) & \supset & \mathcal{L}(V,W) \\ \downarrow \pi & & \downarrow \pi \\ \mathfrak{M}_0(V,W) & \ni & \{0\} \end{array}$$

<u>Lusztig</u>: In type A-D-E

$$\mathcal{L}(V,W) \cong \mathrm{Grass}_{\underline{e}_V}(I_W)$$

injective $\mathbb{C}(Q)$ module

dimension vector

Let $L(W) = L(\lambda_W)$ highest-weight simple $\mathfrak{g}$-module:

$$\lambda_W = \sum_{i \in Q_0} \dim(W_i)\, \varpi_i$$

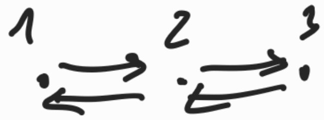
<u>Thm</u> (Nakajima)

$$\mathrm{ch}(L(W)) = \sum_V \dim\left(H^{\mathrm{top}}\left(\mathrm{Grass}_{\underline{e}_V}(I_W)\right)\right) e^{\lambda_W - \alpha_V}$$

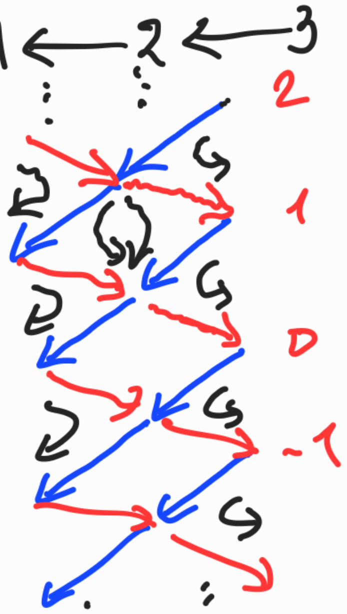
<u>graded setting</u>: $Q \rightsquigarrow \mathbb{Z}Q$ (repetition quiver)

$(\mathbb{Z}Q)_0 \simeq Q_0 \times \mathbb{Z}$

Ex: $Q: \; 1 \leftarrow 2 \leftarrow 3$



$\mathbb{Z}Q$:



Nakajima theory: ($Q = A - D - E$)

$W^\bullet, V^\bullet$ $\mathbb{Z}Q_0$-graded f.d. vector spaces

$$\mathcal{L}^\bullet(V^\bullet, W^\bullet) \cong \mathrm{Grass}_{\underline{e}_{V^\bullet}}(I_{W^\bullet})$$

(Savage Tingley)

injective modules over $\Pi^\bullet(Q)$

$$W^\bullet \rightsquigarrow \sum_{(i,r)\in(\mathbb{Z}Q)_0} \dim W^\bullet_{(i,r)} \cdot \varpi_{i,r} = \lambda_{W^\bullet}$$

"highest loop weight"

irreducible finite dim repr. of $U_q(L\mathfrak{g})$

$L(\lambda_{W^\bullet})$

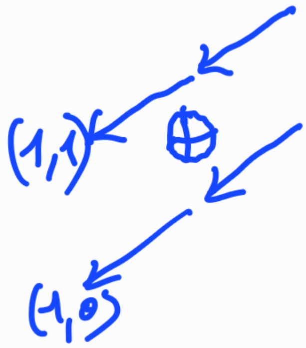
Nakajima: the homology of $\mathcal{L}^\bullet(W^\bullet)$ gives the character of the standard module

$$M(\lambda, W^\bullet) \twoheadrightarrow L(W^\bullet)$$

Ex: $$W^\bullet = \underset{\dim 1}{W_{1,0}} \oplus \underset{\dim 1}{W_{1,1}}$$

$I_{W^\bullet}$

$\cong$



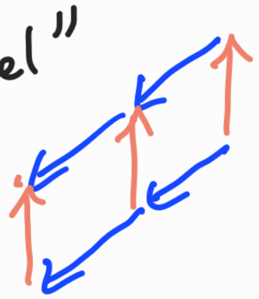
$$M(\varpi_{1,0} + \varpi_{1,1}) \cong L(\varpi_{1,0}) \otimes L(\varpi_{1,1}) \quad \dim 16$$

$$\Downarrow$$

$$L(\varpi_{1,0} + \varpi_{1,1}) \quad \dim 10$$

If we replace $I_{W^\bullet}$ by:

"generic kernel" $\longrightarrow$



can check this gives the correct character for the simple module.

2 – Graded generalized preprojective algebras (jt. w. D. Hernandez)

- $C = (c_{ij})_{i,j \in I}$ indecomposable $n \times n$ Cartan matrix (finite type: $A_n, B_n, \dots, F_4, G_2$)

- $B = DC$ is symmetric $= (b_{ij})$ $\qquad D = (d_i)_{i \in I}$ diagonal $d_i \in \mathbb{Z}_{>0}$, $\min(d_i) = 1$

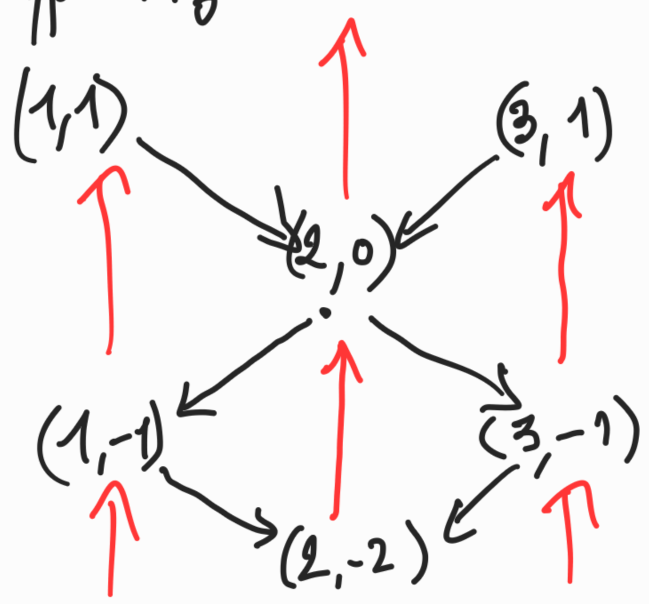
- $\tilde{\Gamma}$ infinite quiver with vertex set: $\tilde{V} = I \times \mathbb{Z}$

  arrows: $(i,r) \longrightarrow (j,s) \Longleftrightarrow \left(\begin{array}{l} b_{ij} \neq 0 \\ \text{and} \\ s = r + b_{ij} \end{array} \right)$

$\tilde{\Gamma}$ has two isomorphic connected components.

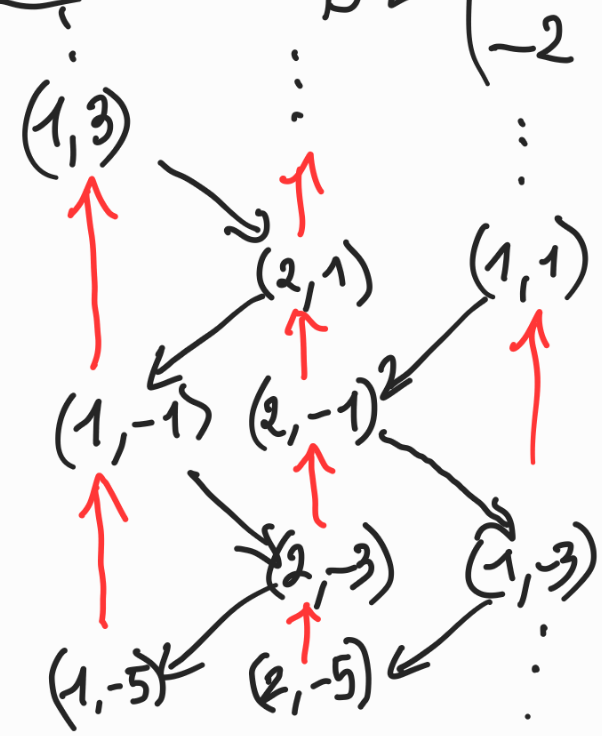
Pick one and call it Γ.

Ex: type A_3, $\qquad C = B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

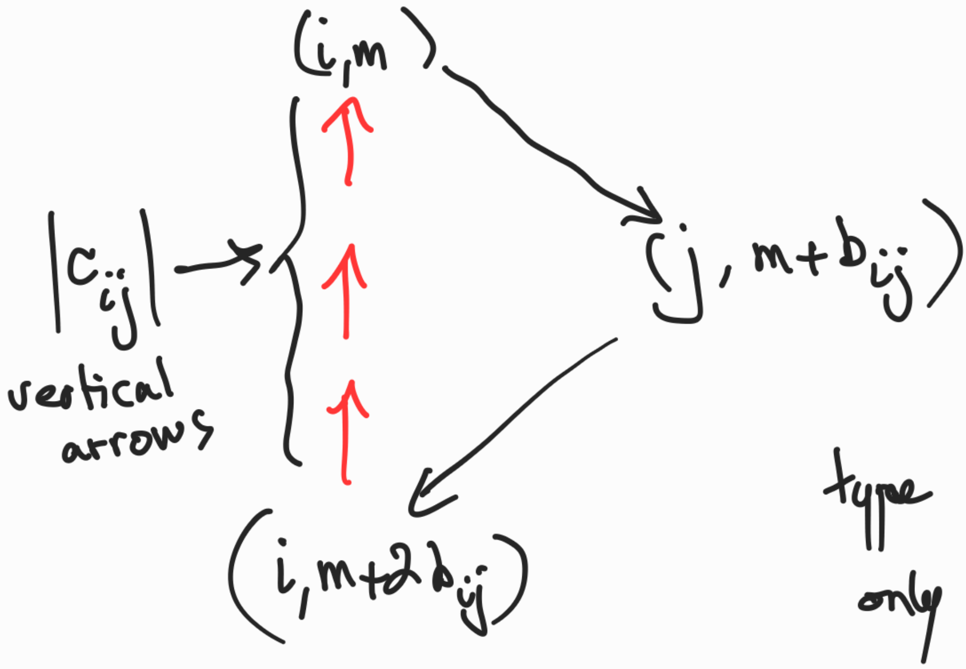


Type B2:

$B=\begin{pmatrix}4&-2\\-2&2\end{pmatrix}$ $\quad C=\begin{pmatrix}2&-1\\-2&2\end{pmatrix}$ $\quad D=\begin{pmatrix}2&0\\0&1\end{pmatrix}$



Relations: For every $i\neq j$ s.t. $c_{ij}\neq 0$. For every $(i,m)\in V$ there is an oriented cycle:



type ADE only triangles

Potential: S = formal sum of all these cycles.

Relations: all cyclic derivatives $\partial_\alpha S$ (α arrow of Γ!) $=0$.

Definition: (Hernandez-L 2016)

$$\Pi^{\bullet}(C) := K\Gamma \Big/ \left(\partial_\alpha S,\ \alpha \in \{\text{arrows of } \Gamma\}\right)$$

- Let $(i,m) \in V$. Let $k \in \mathbb{Z}_{>0}$.

$(i,m) \rightsquigarrow S_{i,m}$ 1-dim simple $\Pi^{\bullet}(C)$-mod.

$\rightsquigarrow I_{i,m}$ injective hull of $S_{i,m}$.

generic homomorphism

$$K^{(i)}_{k,m} \hookrightarrow I_{(i,m)} \twoheadrightarrow I_{(i,\, m-kb_{ii})}$$

generic kernel (finite-dimensional)

Thm: Let $U_q(L\mathfrak{g})$ be the quantum loop algebra associated with C and (q <u>not</u> a root of unity)).

$$i, m, k \rightsquigarrow L\left(\sum_{s=1}^{k} \varpi_{i,\, m-(2s-1)d_i}\right)$$

Kirillov-Reshetikhin modules.

- The q-character of this module is equal to the highest monomial times a Laurent polynomial equal (up to some explicit monomial change of variables) to the $\mathcal{F}$-polynomial of $K^{(c)}_{k,m}$.

Ideas of the proof: Very indirect.

(1) Introduce the cluster algebra $\mathcal{A}$ with initial seed Γ.

(2) Prove that (truncations) of q-characters of KR-modules are "given" by certain cluster variables of $\mathcal{A}$.

(3) Derksen-Weyman-Zelevinsky theory.

Remarks: (1) Thm extends to tensor products of KR-modules and direct sum of generic kernels.

(2) In particular obtain a formula the q-char. of standard modules. In type ADE this recovers the formulas of Nakajima (using Lusztig, Savage-Tingley)

But our formula works also for BCFG.

$\rightsquigarrow$ "Nakajima type varieties" for BCFG ?

(3) There are many more cluster variables in $\mathcal{A}$!!

Conjecture (HL 2016)

m "cluster monomial" of $\mathcal{A}$ $\xrightarrow{DWZ}$ $\dot{\Pi}(C)_M$-mod

affine highest-weight
of an irreducible
$U_q(L\mathfrak{g})$-module L

same connection

- Recently proved by Kashiwara-Kim-Oh-Park (2021)

<u>Example:</u> type A_3 $\quad L(\varpi_{1,-6} + \varpi_{2,-3})$

The corresponding $\pi^\bullet(C)$-module:

$$M = \begin{tikzcd}
 & (2,-1) \arrow[dl] \arrow[dr] & \\
(1,-2) \arrow[dr] & & (3,-2) \arrow[dl] \\
 & (2,-3) & \\
 & & (3,-4) \arrow[uu] \arrow[dl] \\
 & (2,-5) \arrow[uu] \arrow[dl] & \\
(1,-6) & &
\end{tikzcd}$$