Generalized preprojective algebras (2)

Ungraded setting

Representation theory

Representation varieties and crystals

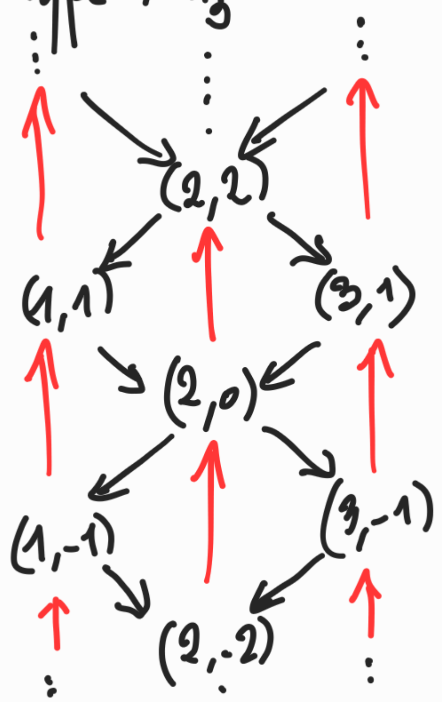
3 Ungraded setting (jt. w. Geiss – Schröer)

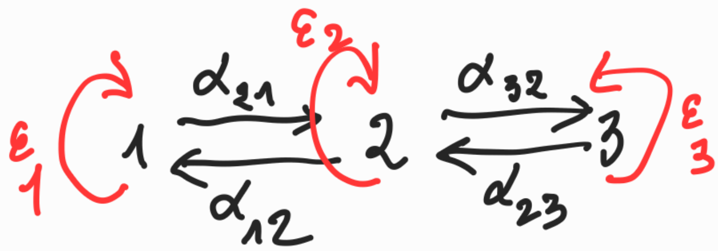
The quiver Γ has a natural $\mathbb{Z}$-action generated by:

$$V \longrightarrow V$$
$$(i,r) \longmapsto (i,r+2)$$

This preserves the potential S. By moding out this action we obtain an algebra $\tilde{\Pi}(C)$.

Ex: type A_3.





$$(P_3)\begin{cases} \alpha_{12}\alpha_{21} = 0 \\ \alpha_{21}\alpha_{12} - \alpha_{23}\alpha_{32} = 0 \\ \alpha_{32}\alpha_{23} = 0 \end{cases}$$

$$(P_2)\quad \begin{cases} \varepsilon_2 \alpha_{21} = \alpha_{21}\varepsilon_1 \\ \alpha_{12}\varepsilon_2 = \varepsilon_1 \alpha_{12} \\ \varepsilon_2 \alpha_{23} = \alpha_{23}\varepsilon_3 \\ \alpha_{32}\varepsilon_2 = \varepsilon_3 \alpha_{32} \end{cases}$$

In order to get a finite-dimensional algebra we add nilpotency relations on the ε_i:

$$(P_1)_k \qquad \varepsilon_1^k = \varepsilon_2^k = \varepsilon_3^k = 0 \quad (\text{some } k>0)$$

For other types $\underbrace{ADE}_{r=1}$ $\underbrace{BCF_4}_{r=2}$ $\underbrace{G_2}_{r=3}$

$$(P_1)_k \qquad \varepsilon_i^{k\cdot\frac{r}{d_i}} = 0 \ ; \ D = \mathrm{diag}(d_i)$$

minimal symmetrizer of C.

Ex: Type B_2:

$$C = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$r=2$.

$$\begin{tikzcd} 1 \arrow[loop left, "\varepsilon_1"] \arrow[r, shift left, "\alpha_{12}"] & 2 \arrow[l, shift left, "\alpha_{21}"] \arrow[loop right, "\varepsilon_2"] \end{tikzcd}$$

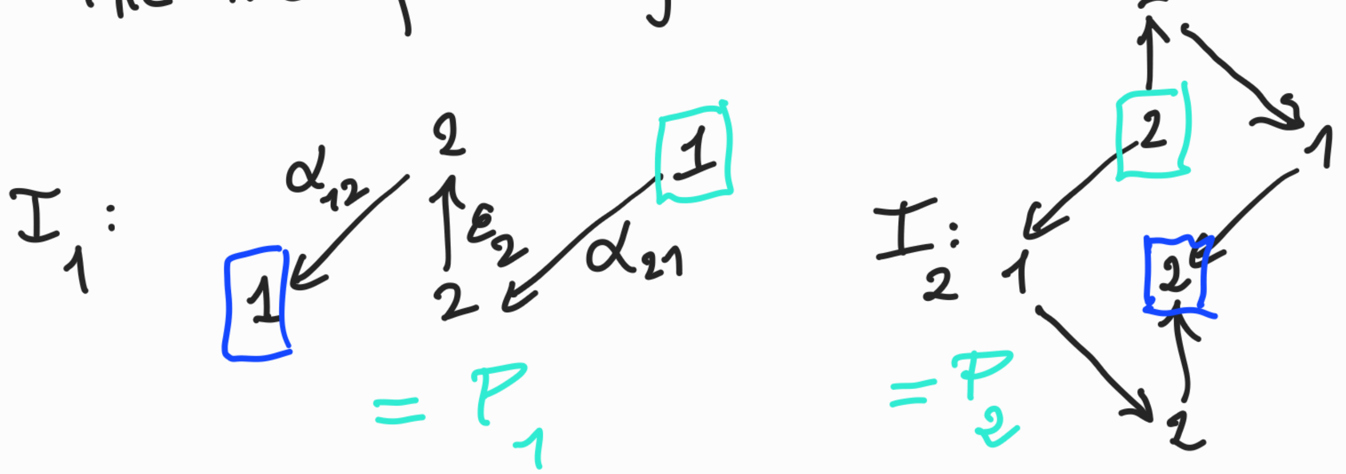
$$(P_1)_k: \quad \varepsilon_1^k = \varepsilon_2^{2k} = 0$$

$$(P2) \quad \varepsilon_1 \alpha_{12} = \alpha_{12}\varepsilon_2^2 \ , \quad \varepsilon_2^2 \alpha_{21} = \alpha_{21}\varepsilon_1$$

$$(P3) \quad \alpha_{12}\alpha_{21} = 0 \qquad \alpha_{21}\alpha_{12}\varepsilon_2 = \varepsilon_2 \alpha_{21}\alpha_{12}$$

$k = 1. \quad \varepsilon_1 = 0 \quad . \quad \varepsilon_2^2 = 0$

The indecomposable injective modules are:



Remarks: ① $Z \quad \mathrm{diag}\left(\frac{r}{d_i}\right)$ is a symmetrizer for ${}^tC$. In GLS we decided to take opposite convention to HL:

$$[HL] \longleftrightarrow [GLS]$$

$$C \longleftrightarrow {}^tC$$

"Langlands duality"

② In [GLS] we work with arbitrary symmetrizable generalized Cartan matrices.

. From now on I will switch to the convention of [GLS].

Def: C symmetrizable gen. Cartan matrix $= (c_{ij})_{1 \leq i,j \leq n}$

$D = \operatorname{diag}(d_i)$, minimal symmetrizer

$d_i \in \mathbb{Z}_{>0}$, $\sum d_i$ minimal

If $c_{ij} < 0$

$g_{ij} = \gcd(c_{ij}, c_{ji})$; $f_{ij} = \dfrac{|c_{ij}|}{g_{ij}}$

. $\Omega \subseteq \{1, 2, \dots, n\}^2$, <u>acyclic</u> orientation :

(i) $\{(i,j),(j,i)\} \cap \Omega \neq \phi \Leftrightarrow c_{ij} < 0$

(ii) if $(i_1, i_2), (i_2, i_3), \dots, (i_t, i_{t+1}) \in \Omega$

then $i_1 \neq i_{t+1}$.

. $Q = (Q_0, Q_1)$ "simple quiver"

$Q_0 = \{1, \dots, n\}$

$Q_1 = \left\{ \alpha_{ij}^{(g)} : j \to i \;\middle|\; (i,j) \in \Omega \; 1 \leq g \leq g_{ij} \right\}$

$\cup \left\{ \varepsilon_i : i \circlearrowleft \;\middle|\; i \in Q_0 \right\}$

. $\overline{Q} = (Q_0, \overline{Q_1})$ obtained by adding an arrow $\alpha_{ji}^{(g)} : i \to j$ for every $\alpha_{ij}^{(g)} \in Q_1$

<u>Def</u>: <u>Algebra $H(C, kD, \Omega)$</u> $k \in \mathbb{Z}_{>0}$ K field

$$H_k = K\overline{Q} / I_k$$

where I_k is the ideal generated by:

$(H_1)_k: \quad \varepsilon_i^{k d_i} = 0 \qquad (i \in Q_0)$

$(H_2): \quad \varepsilon_i^{f_{ji}} \alpha_{ij}^{(g)} = \alpha_{ij}^{(g)} \varepsilon_j^{f_{ij}}$

$(\forall \alpha_{ij}^{(g)} \in Q_1)$

<u>Remarks:</u> (1) H_k is finite-dimensional over K.

(2) If C is symmetric and $k=1$, then

$H_1 = KQ^{\circ}$ ← obtained by removing all ε_i.

More for any k, $H_k \cong K[X]/(X^k) \otimes_K KQ^{\circ}$

(3) In general H_K is similar a "species" where the fields are replaced by truncated polynomial rings.

<u>Def</u> <u>Algebra $\Pi(C, kD)$</u>

$\Pi_k = K\overline{Q} / \overline{I}_k$ where $\overline{I}_k$ is given by:

$(P_1) \quad \varepsilon_i^{k d_i} = 0$

$(P_2) \quad \varepsilon_i^{f_{ji}} \alpha_{ij}^{(g)} = \alpha_{ij}^{(g)} \varepsilon_j^{f_{ij}} \quad (\forall \alpha_{ij}^{(g)} \in \overline{Q}_1)$

(P3) $\forall i \in Q_0$:

$$\sum_{\substack{j \text{ s.t.} \\ c_{ij}<0}} \sum_{g=1}^{g_{ij}} \sum_{f=0}^{f_{ji}-1} \operatorname{sgn}(i,j)\, \varepsilon_i^{f} \alpha_{ij}^{(g)} \alpha_{ji}^{(g)} \varepsilon_i^{f_{ji}-1-f} = 0$$

$$\operatorname{sgn}(i,j) = \begin{cases} 1 & \text{if } (i,j) \in \Omega \\ -1 & \text{if } (i,j) \notin \Omega \end{cases}$$

<u>Note:</u> (P2) and (P3) come from the potential:

$$S(C,\Omega) = \sum_{\substack{i \to j \in \overline{Q}_1 \\ i \neq j}} \sum_{g=1}^{g_{ij}} \operatorname{sgn}(i,j)\, \varepsilon_i^{f_{ij}} \alpha_{ij}^{(g)} \alpha_{ji}^{(g)}$$

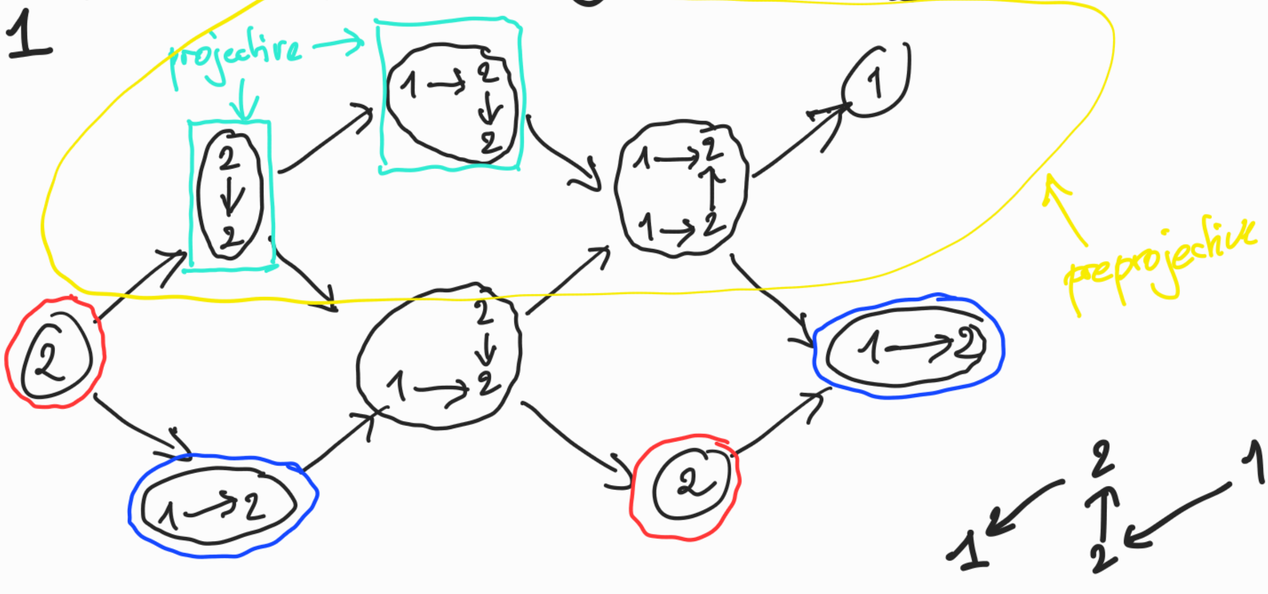
Ex Type C2

$$C = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Omega = (2,1)$$

$$Q: \quad \varepsilon_1 \circlearrowright 1 \xrightarrow{\alpha_{21}} 2 \circlearrowleft \varepsilon_2$$

$$H_1: \quad 1 \xrightarrow{\alpha_{21}} 2 \circlearrowleft \varepsilon_2 \qquad \varepsilon_2^2 = 0$$



Thm (GLS). Π_k is finite-dimensional

$\iff C$ is Cartan (type $A, B, \dots, F_4, G_2$)

In this case Π_k is self-injective.

. In general $\Pi_k\big|_{H_k} \cong \bigoplus_{m\geq 0} \tau^{-m}(H_k)$

AR-translation of H_k

4 Representation theory

Fix $k>0$. H_k, Π_k. $c_i = kd_i$.

For $i \in Q_0$, let $H_i := K[X_i]/(X_i^{c_i})$

. Let $M \in \operatorname{rep}(H_k)$ (resp. $\operatorname{rep}(\Pi_k)$).

$$M = (M_i, \alpha_{ij}^{(g)}, \varepsilon_i)$$

M_i: K-vect. spaces

$\alpha_{ij}^{(g)} \in \operatorname{Hom}_K(M_j, M_i)$, $\varepsilon_i \in \operatorname{End}_K(M_i)$

In particular each M_i is an H_i-module.

Def: M is <u>locally free</u> if $\forall i \in Q_0$, M_i is a free H_i-module.

In this case: $\underline{\operatorname{rk}}(M) = \left(\operatorname{rank}_{H_i}(M_i)\right)_{i\in Q_0}$

4-1 Representations of H_K

Prop: Let $M \in \mathrm{rep}(H_K)$. Then M is locally free iff $(\mathrm{proj}\dim M \leq 1) \Leftrightarrow (\mathrm{inj}\dim M \leq 1)$
$\Leftrightarrow (\mathrm{proj}\dim M < \infty) \Leftrightarrow (\mathrm{inj}\dim M < \infty)$

H_K is an Iwanaga-Gorenstein algebra of dim 1.

$\rightsquigarrow$ $\mathrm{rep}_{l.f.}(H_K)$ is "hereditary" but not abelian, only exact.

Def: M is rigid if $\mathrm{Ext}^1_H(M,M) = 0$.

Thm: (GLS)

- There are finitely many isoclasses of indecomposable locally free rigid modules iff C is a Cartan matrix.
- In that case, $M \longmapsto \underline{\mathrm{rk}}(M)$ gives a bijection with the positive roots $\Delta^+(C)$.
- In general there is a bijection between isoclasses of indec. locally free rigid modules of H_K and the of real Schur roots of (C, Ω).

4-2 Representations of $\Pi_K = \Pi$

Definition: $M \in \mathrm{rep}(\Pi)$. We say that M is $\underline{E\text{-filtered}}$ if it has a filtration where layers are isomorphic to

$E_i :=$ unique loc. free π-module with rank $(0, \dots, 0, 1, 0, \dots, 0)$
$\uparrow$ i

"generalized simples"

This a generalization of "nilpotent modules"

$\rightsquigarrow \quad \mathrm{nil}_E(\pi)$

Clearly: if M is E-filtered, it is locally free.

The converse is false:

<u>Ex:</u> type C_2 minimal symmetrizer.

$$X = \begin{tikzcd} & 2 \arrow[dl, "\alpha_{12}"'] \arrow[dd, "\varepsilon_2"] \\ 1 \arrow[dr, "\alpha_{21}"'] & \\ & 2 \end{tikzcd}$$

is locally free

not E-filtered.

$(c_1 = 1,\ c_2 = 2)$

<u>Def:</u> Let $M \in \mathrm{rep}(\pi)$. Let $i \in Q_0$.

- $\mathrm{fac}_i(M)$: largest quotient module of M supported on vertex i for some k.

We have a s.e.s. $K_i(M) \hookrightarrow M \twoheadrightarrow \mathrm{fac}_i(M)$

- $\mathrm{sub}_i(M)$: ______ submodule of M

$\mathrm{sub}_i(M) \hookrightarrow M \twoheadrightarrow C_i(M)$.

Def: (Crystal module) $M \in \mathrm{rep}(\Pi)$ is a crystal module iff:

- $M = \{0\}$

  or

- $\mathrm{sub}_i(M)$ and $\mathrm{fac}_i(M)$ are free H_i-modules, $K_i(M)$ and $C_i(M)$ are crystal modules.

for every $i \in Q_0$

Rk:

- E_i is crystal.
- If M is a crystal it is E-filtered.

But the converse is false:

Ex: type C_2 minimal symmetriser.

$$Y = \begin{tikzcd} 2 \arrow[r, "\varepsilon_2"] \arrow[d, "\alpha_{12}"] & 2 \\ 1 \arrow[dr, "\alpha_{21}"] & \\ 2 \arrow[r, "\varepsilon_2"'] & 2 \end{tikzcd}$$

not free over H_2

$\mathrm{sub}_1(Y) = \{0\}$

$\mathrm{fac}_1(Y) = \{0\}$

E-filtered.

4-3 Representation varieties

K is alg. closed.

$\underline{r} \in \mathbb{N}^{Q_0}$ a rank vector.

Prop: $\mathrm{rep}_{l.f.}(H, \underline{r})$ is smooth and irreducible.

Let $\mathrm{nil}_E(\pi, \underline{r})$ be the variety of E-filtered π-modules of rank $\underline{r}$. Then

$$\mathrm{rep}_{l.f}(H, \underline{r}) \subset \mathrm{nil}_E(\pi, \underline{r})$$

is an irreducible component.

Thm (GLS)

(i) Every irreducible component of $\mathrm{nil}_E(\pi, \underline{r})$ has dimension $\leqslant \dim \mathrm{rep}_{l.f}(H, \underline{r}) =: d_{\underline{r}}$.

(ii) If Z is an irreducible component of $\mathrm{nil}_E(\pi, \underline{r})$, then

$$\left(\dim Z = d_{\underline{r}} \right) \Longleftrightarrow \left(\begin{array}{l} \text{there exists a dense} \\ \text{open subset of } Z \\ \text{consisting of crystal} \\ \text{modules} \end{array} \right)$$

Let $$\mathrm{Irr}(\pi) := \coprod_{\underline{r} \in \mathbb{N}^{Q_0}} \max \mathrm{Irr}\left(\mathrm{nil}_E(\pi, \underline{r})\right)$$

. $Z \in \mathrm{Irr}(\pi)$ we can define:

. $\mathrm{wt}(Z) = \underline{r}$

. $\varphi_i(Z) = \min \left\{ \varphi_i(M) \mid \begin{array}{l} M \text{ crystal} \\ \text{module on } Z \end{array} \right\}$

$$\varphi_i(M) = \mathrm{rank}_{H_i}\left(\mathrm{sub}_i(M)\right).$$

. $\varepsilon_i(Z) = \varphi_i(Z) - (\mathrm{wt}(Z), \alpha_i)$

- $\tilde{e}_i(Z)$, $\tilde{f}_i(Z)$ $\in \operatorname{Irr}(\pi)$ | defined by some bundle constructions similar to Lusztig

<u>Thm</u> (GLS).

$$(\operatorname{Irr}(\pi), \mathrm{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i) \cong \mathcal{B}(-\infty)$$ "crystal of $U_q^+(\mathfrak{g})$"

<u>Rk:</u> if C is Cartan, more explicit description of $\operatorname{Irr}(\pi)$.