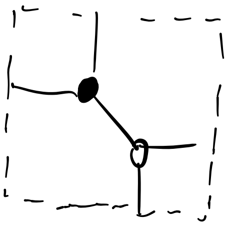
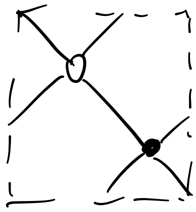
§Dimer models

Defn Let Σ be a closed oriented surface. A <u>dimer model</u> D on Σ is a finite bipartite graph drawn in Σ such that every connected component of $\Sigma \setminus D$ is an open disc.

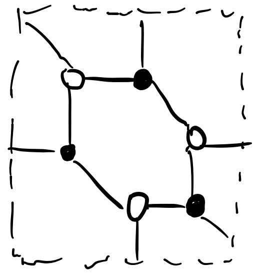
We are most interested in Σ being the torus.

Eg 

2 nodes
3 edges
1 face



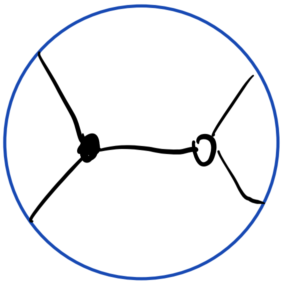
2 nodes
4 edges
2 faces



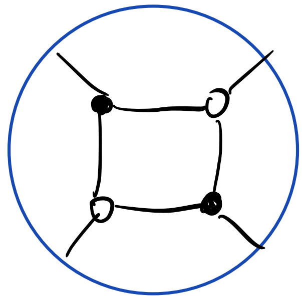
6 nodes
9 edges
3 faces

Defn Let Σ be an oriented surface with boundary. A <u>dimer model</u> D in Σ is a finite bipartite graph in Σ° together with finitely many 'half-edges', connecting nodes of the graph to $\partial\Sigma$. Each $p \in \partial\Sigma$ is incident with at most one half edge, and $\Sigma^\circ \setminus D$ is a union of open discs.

We are most interested in Σ being the disc.

Eg 

2 nodes
1 edge + 4 half-edges
4 faces (all boundary)

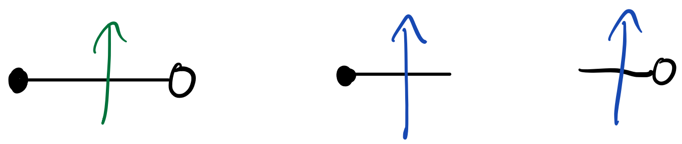


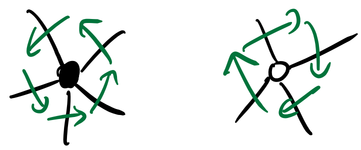
4 nodes
4 edges + 4 half-edges
5 faces (4 boundary)

<u>The dimer quiver</u> $D \rightsquigarrow$ quiver $Q = Q_D$ 'with faces'.

Q_0 = faces of D = conn. components of $\Sigma^\circ \setminus D$.

Q_1 = edges of D, oriented with $\bullet$ on the left:

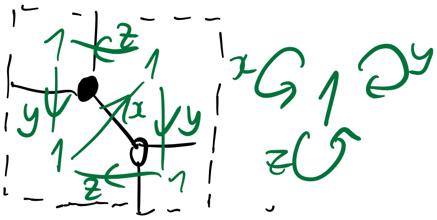


Q_2 'faces' = nodes of D:  distinguished cycles.

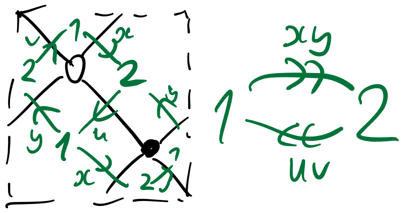
Q has a 'frozen' subquiver F: F_0 = boundary faces of D
F_1 = half-edges of D.

Σ closed $\Rightarrow$ $F = \emptyset$.

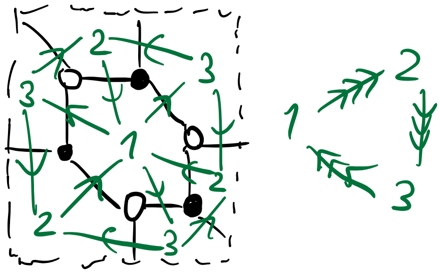
Eg



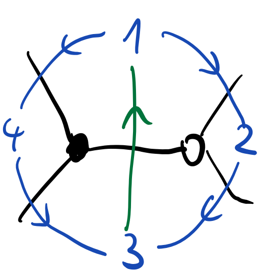
Faces: xyz, xzy

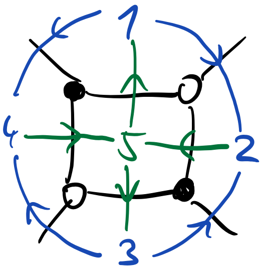


Faces: $vyux$, $vxuy$



6 faces



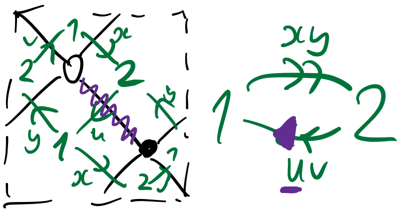


Perfect matchings and the dimer algebra

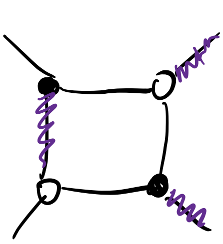
Let Q be a quiver with faces. We assume the 'manifold condition': every $\alpha \in Q_1$ is in the boundary of some face

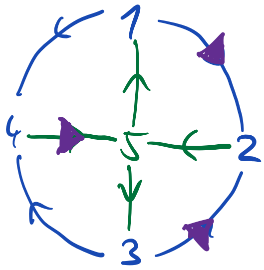
Defn A perfect matching of Q is $\mu \subseteq Q_1$ such that $\forall f \in Q_2$, ∂f contains exactly one arrow in μ.

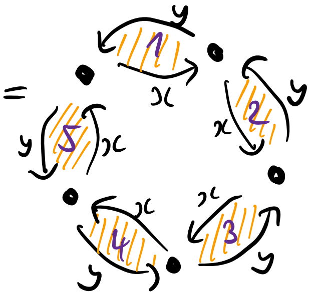
When $Q = Q_D$, a perfect matching of Q is just a perfect matching of D:



Faces: $vy\underline{u}x$, $vx\underline{u}y$





Eg $C_n =$  'circle quiver (with faces)'

A perfect matching of C_n is a choice of x or y for each $i = 1, \dots, n$, i.e. a partition $X \cup Y = \{1, \dots, n\}$.

Let $Z = \mathbb{C}[[t]]$.
Given a perfect matching μ, define a Q-representation V_μ by:

$$V_i = Z \text{ for all } i \in Q_0 \qquad V_\alpha = \begin{cases} t, & \alpha \in \mu \\ 1, & \alpha \notin \mu. \end{cases}$$

'perfect matching module'.

Defn $V \in \operatorname{rep} Q$ is a <u>Q-matrix factorisation</u> (of t) if

1) V_i free over Z for all $i \in Q_0$
2) V_α Z-linear for all $\alpha \in Q_1$
3) $V_p = t \cdot \mathrm{id} : V_i \to V_i$ whenever $p : i \to i$ is the boundary of a face $f \in Q_2$

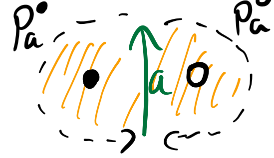
Since Z is an integral domain, (3) plus the manifold condition implies that V_α is injective $\forall\ \alpha \in Q_1$.
If Q is connected, it then follows (by the manifold condition again) that $\operatorname{rank}_Z V_i$ is constant; call this $\operatorname{rank} V$.

Note A perfect matching module is a rank 1 Q-matrix factorisation.

Lem Let $p, q : i \to j$ be paths in Q. If $\exists$ a path $r : i \to j$ such that $rp, rq : i \to i$ both bound a face, then $V_p = V_q$.

Proof $V_r \circ V_p = V_{rp} = t \cdot \mathrm{id} = V_{rq} = V_r \circ V_q$, and V_r is injective.

Special cases: 1) $r = e_i$, i.e. p and q both bound a face 

2)  $\Rightarrow V_{p_a^\bullet} = V_{p_a^\circ}$.

Cor 1) For $Q = C_n$, any $V_\mu \in \operatorname{mod} \Pi$ for $\Pi = \widehat{\mathbb{C}Q} / \overline{(xy - yx)}$.
(complete) preprojective algebra of type $\tilde{A}_{n-1}$.

Exercise Recall μ is equivalent to choosing $\{1, \dots, n\} = X \cup Y$.
Show that, if $\# X = k$, so $\# Y = n-k$, then V_μ is a module for

$$C = \widehat{\mathbb{C}Q} / \overline{(xy - yx, \; y^k - x^{n-k})}.$$

2) For $Q = Q_D$, any $V_\mu \in \operatorname{mod} A$ for

$$A = A_D = \widehat{\mathbb{C}Q} \Big/ \overline{\begin{pmatrix} p - q : \; p, q : i \to j \text{ bound a face} \\ p_a^\bullet - p_a^\circ : \; a \in Q_1 \setminus F_1 \end{pmatrix}}$$ 'dimer algebra'.

(cf. Broomhead's / Beil's ghor algebra – more relations, e.g. $p = q$ when $\exists$ any path r such that pr, qr bound a face).

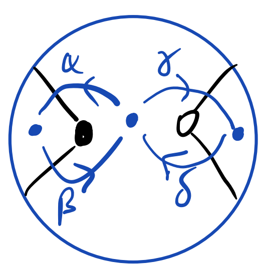
Defn For $Q = Q_D$, define potential $W = \sum_\bullet \circlearrowright - \sum_\circ \circlearrowright$
(linear comb. of cyclic equivalence classes of cycles).

Then $J(Q, F, W) = \widehat{\mathbb{C}Q} / \overline{(\partial_a W : a \in Q_1 \setminus F_1)}$ 'superpotential / frozen Jacobian algebra'

$$\left(p = \alpha_n \cdots \alpha_1 \text{ cycle} \Rightarrow \partial_\alpha p = \sum_{\alpha_i = \alpha} \alpha_{i-1} \cdots \alpha_1 \alpha_n \cdots \alpha_{i+1} \right).$$

Prop Assume D is connected. Then $A \cong J(Q, F, W)$.

Hint $\partial_a W = p_a^\bullet - p_a^\circ$. So enough to show that these imply the extra relations in A when D is connected.

Eg 

$J(Q, F, W) = \widehat{\mathbb{C}Q} \quad (Q_1 = F_1)$

but $A = \widehat{\mathbb{C}Q} / \overline{(\beta\alpha - \delta\gamma)}$.

Rem Σ closed, genus g. Then A_D is 1) infinite-dimensional for $g \geq 1$ (this is good!)
2) Noetherian for $g \leq 1$

$\Rightarrow$ best surface is T^2!

§ Calabi-Yau algebras

Defn Let A be a Noetherian $\mathbb{C}$-algebra, with enveloping algebra $A^{\varepsilon} = A \otimes_{\mathbb{C}} A^{op}$.
(A^{ε}-Mod = A-Bimod).

Say A is d-Calabi-Yau, $d \geq 0$, if

1) $A \in \operatorname{per} A^{\varepsilon}$ (homologically smooth)
2) $\Omega = R\mathrm{Hom}_{A^{\varepsilon}}(A, A^{\varepsilon}) \cong \Sigma^{-d} A$ in $\operatorname{per} A^{\varepsilon}$.

Why?

Thm (Keller) If A is homologically smooth, then $R\mathrm{Hom}_A(\Omega, -)$ is a (right) Serre functor on $D^b(A)$.

($D^b(A)$ = complexes of A-modules with finite dimensional total cohomology.)
Sometimes write $\operatorname{pvd}(A)$ – perfectly valued derived category.

Cor If A is d-Calabi-Yau, then $R\mathrm{Hom}_A(\Omega, -) \cong R\mathrm{Hom}_A(\Sigma^{-d} A, -) \cong \Sigma^d$ is a Serre functor on $D^b(A)$.
Thus (by definition) $D^b(A)$ is a d-Calabi-Yau triangulated category.

Cor Let M, N be finite-dimensional A-modules, A d-CY. Then

$$\mathrm{Ext}^i_A(M,N) = \mathrm{Ext}^{d-i}_A(N,M)^* \quad \forall i \in \mathbb{Z}. \qquad (-)^* = \mathrm{Hom}_{\mathbb{C}}(-, \mathbb{C}).$$

Proof $\mathrm{Ext}^i_A(M,N) = \mathrm{Hom}_{D^b(A)}(M, \Sigma^i N) \underset{\Sigma^d \text{ right Serre functor}}{=} \mathrm{Hom}_{D^b(A)}(\Sigma^i N, \Sigma^d M)^*$

$= \mathrm{Hom}_{D^b(A)}(N, \Sigma^{d-i} M)^* = \mathrm{Ext}^{d-i}_A(N,M)^*$.

Cor A d-CY for $d>0 \Rightarrow \dim_{\mathbb{C}} A = \infty$, or $A=0$.

Proof If $\dim A < \infty$ then $\mathrm{Hom}_A(A,A) = \mathrm{Ext}^d_A(A,A)^* = 0$.

Cor A d-CY $\Rightarrow \mathrm{gldim} A \leq d$.

Proof $\mathrm{Ext}^{d+1}_A(M,N) = \mathrm{Ext}^{-1}_A(N,M)^* = 0$.

Eg Let Π be a preprojective algebra of affine type. Then (Reiten–van den Bergh) Π is 2-CY.

Note 'McKay correspondence' $D^b(\Pi) \simeq D^b(\mathrm{coh} X)$ for X a crepant resolution of a Kleinian (2d Gorenstein) singularity. So $D^b(\mathrm{coh} X)$ is also 2-CY.

Now let A be a Noetherian $\mathbb{C}$-algebra and $e \in A$ idempotent. ($e^2=e$) Write $\underline{A} = A/AeA$, and $D^b_e(A) = \{X \in D^b(A) : eX = 0\}$.

(i.e. an A-module M is in $D_e^b(A)$ iff M is a fin. dim. $\underline{A}$-module).

Defn (A,e) is internally (or relatively) d-Calabi-Yau if

1) $A \in \mathrm{per} A^\varepsilon$ and $\mathrm{gldim} A \leq d$
2) $\exists$ triangle $\Sigma^{-d} A \to \Omega_A \to X \to$ in $D(A^\varepsilon)$ such that $R\mathrm{Hom}_A(X,M) = 0 = R\mathrm{Hom}_{A^{op}}(X,N)$ for any $M \in D_e^b(A)$, $N \in D_e^b(A^{op})$.

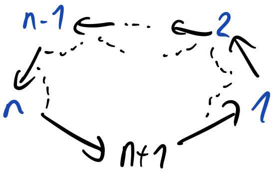
Consequence (1) $\Rightarrow R\mathrm{Hom}_A(\Omega,-)$ is a Serre functor on $D^b(A)$
(2) $\Rightarrow R\mathrm{Hom}_A(\Omega,-) \cong \Sigma^d$ on $D^b_e(A)$.

So if $M \in D_e^b(A)$, $N \in D^b(A)$, then:

$$\mathrm{Hom}_A(M,N) = \mathrm{Hom}_A(N, R\mathrm{Hom}_A(\Omega,M))^*$$
$$= \mathrm{Hom}_A(N, \Sigma^d M)^*.$$

Cor M,N finite-dimensional A-modules, $M \in \underline{A}$-mod. Then

$$\operatorname{Ext}^i_A(M,N) = \operatorname{Ext}^{d-i}_A(N,M)^*$$

Eg $A=$  , $e=\sum_{i=1}^{n} e_i$ is internally $(n+1)$-CY

Note 1) Now there are finite dimensional examples!

2) If (A,e) is int. dCY, and e' is idempotent with $e'e=e=ee'$, then (A,e') is int. dCY.
(standard example: e is a sum of vertex idempotents, e' adds more)

3) If A is dCY, then $(A,0)$ is int. dCY, hence (A,e) is int. dCY for all idempotents e.

4) $(A,1)$ is int. d-CY iff $\operatorname{gldim} A \leq d$.

Thm (P'17) Let (A,e) be int. dCY, and assume A is Noetherian, $\underline{A}=A/AeA$ is finite dimensional. Write $B=eAe$ (boundary algebra). Then:

1) B is g-Iwanaga–Gorenstein: B Noetherian, $\operatorname{injdim}_B B = \operatorname{injdim} B_B = g < \infty$ (in fact $g \leq d$)

$$\Rightarrow GP(B) = \{X \in \operatorname{mod} B : \operatorname{Ext}^{i>0}_B(X,B)=0\}$$

is a Frobenius (exact) category: enough projectives and injectives, projective = injective. $(=\operatorname{proj} B)$

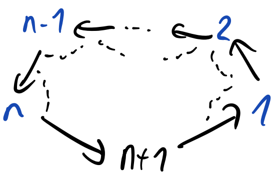
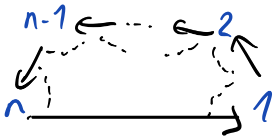
2) $\underline{GP}(B) = GP(B)/\operatorname{proj} B$ is a $(d-1)$-CY triangulated category.

3) $eA \in GP(B)$ is $(d-1)$-cluster-tilting:

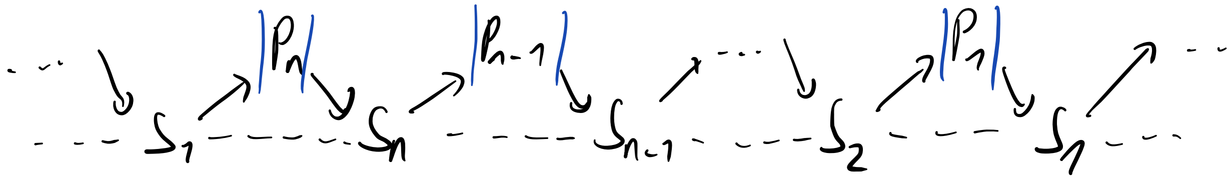
$$\operatorname{add}(eA) = \{X \in GP(B) : \operatorname{Ext}^{1\leq i \leq d-2}_B(X,eA)=0\}$$
$$\big(=\{\text{———}\ \text{"}\ \text{———}\ (eA,X)\ \text{— "}\}\big)$$

4) $A \xrightarrow{\sim} \operatorname{End}_B(eA)^{op}$, $\underline{A} \xrightarrow{\sim} \underline{\operatorname{End}}_B(eA)^{op}$ $\left(:= \operatorname{End}_{\underline{GP}(B)}(eA)^{op}\right)$.

When $d=3$, categories like $GP(B)$ above have applications to cluster algebras (with frozen variables).
(cf. results for dgas: W.K. Yeung, Y. Wu)

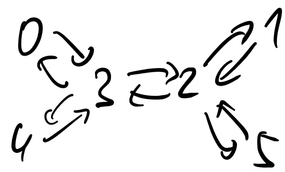
Eg $A=$  $\Rightarrow B=$ , i.e. n cycle $/\mathrm{rad}^2$.

B selfinjective ($=0$-Iwanaga-Gorenstein) $\Rightarrow GP(B)=\mathrm{mod}B$.



$\underline{GP}(B)=$ n-cluster category of type A_1
$eA=B\oplus S_1$ is n-cluster-tilting; $n-1$ mutations $B\oplus S_i$, $2\leq i\leq n$.

Obs $n=2\Rightarrow B=1\rightleftarrows 2$, preprojective algebra of type A_2.
cf. Geiß-Leclerc-Schröer.

Eg 2 Let $A=$ preprojective algebra of affine type. 
$e=e_0$, 0 extending vertex.

A 2-CY $\Rightarrow (A,e)$ int. 2-CY.

$\underline{A}=A/AeA$ is a Dynkin type pre-proj algebra $\Rightarrow$ fin. dim.

$B=eAe\cong Z(A)=R$, a Kleinian singularity

Now: 1) $GP(B)=CM(B)$ is additively finite, with additive generator ($=1$-cluster-tilting object) eA.
2) $\underline{GP}(B)$ is 1-CY, i.e. $\tau\cong\mathrm{id}$.
3) $A\xrightarrow{\sim}\mathrm{End}_B(eA)^{op}\Rightarrow$ AR quiver of $GP(B)$ is preproj. quiver.

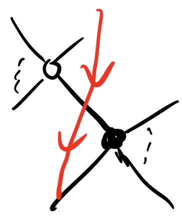
Moreover, A is an NCCR for R, $D^b(A)\simeq D^b(\mathrm{coh}X)$ (2CY) for $X\to\mathrm{Spec}R$ crepant, (VdB, Kapranov-Vasserot)
$\underline{GP}(B)\simeq D_{sg}(R)$ (1CY) (Buchweitz)

§Consistency

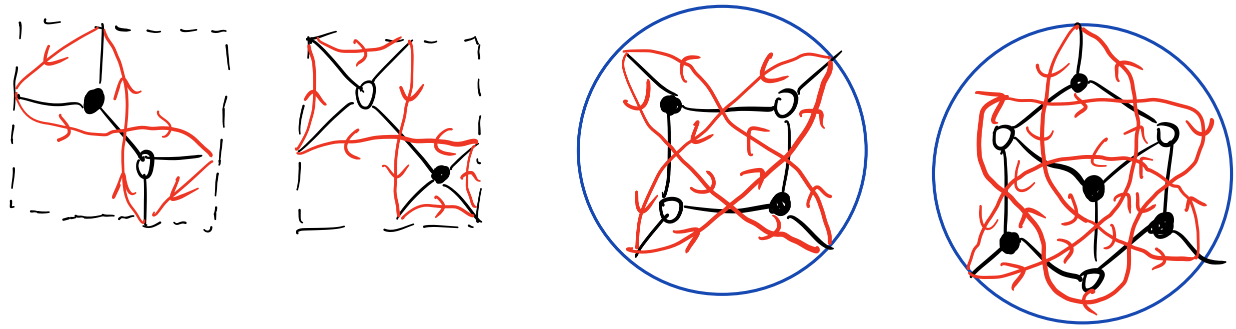
We return to the setting of dimer models: let D be a dimer model on Σ, a surface with or without boundary.

Let $\tilde{\Sigma}$ be the universal cover of Σ, $\tilde{D}$ the lift of D (to an infinite, but periodic, 'dimer model' on $\tilde{\Sigma}$).

D (and $\tilde{D}$) has zig-zag paths (or strands) as follows:



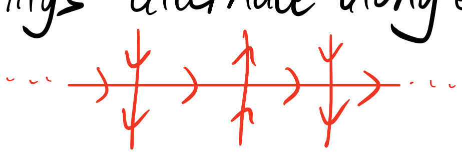
Eg



These strands always satisfy:

1) finitely many strands with finitely many transverse pairwise crossings (modulo deck transformations)

2) signs of crossings alternate along each strand

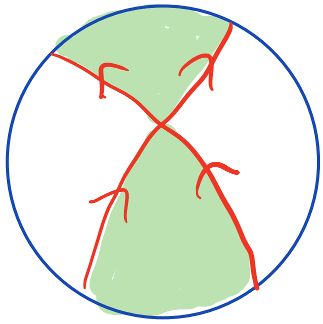
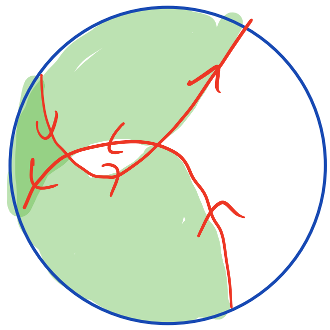


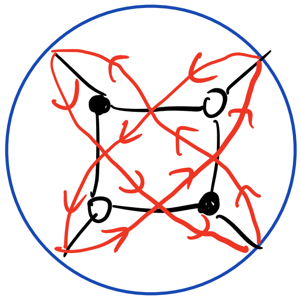
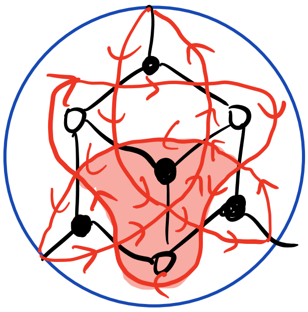
We say D is consistent if

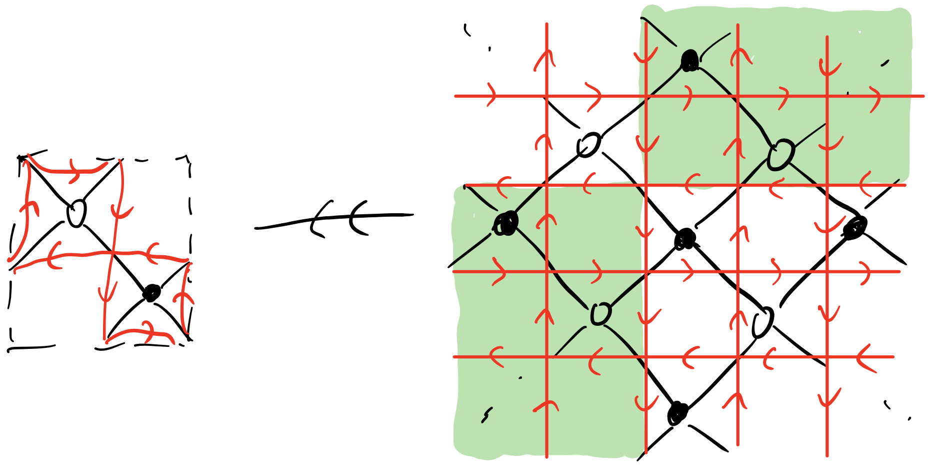
3) strands do not intersect themselves.

4) <u>on $\tilde{D}$</u>, there are no bad lenses

That is, if two strands cross, there are no more crossings following both strands forwards (or backwards).

On disc:  (or )

Examples  ✓  ✗

 ✓

Thm Let $\mathcal{D}$ be a consistent connected dimer model on a surface Σ.
Let A be the dimer algebra, $e = \sum_{i \in F_0} e_i$ frozen/boundary idempotent
(Note $e = 0$ if Σ is closed).

Then (A, e) is int. 3CY if:

1) $\Sigma = T^2$ torus (Broomhead '12)
2) $\Sigma = \Sigma_g$, $g \geq 1$ (Davison '11)

} A is 3CY.

3) $\Sigma = D^2$ disc (P '19+).

Hope True for any surface, without assuming $\mathcal{D}$ is connected.

Proof strategy: $\mathcal{D}$ connected $\Rightarrow A \cong J(Q,F,W)$ frozen Jacobian algebra.

This provides a canonical bimodule complex:

$$0 \to A \otimes \underset{\substack{\uparrow \\ \text{internal vertices} \\ \updownarrow \\ \text{syzygies}}}{(Q_0 \setminus F_0)^\vee} \otimes A \to A \otimes \underset{\substack{\uparrow \\ \text{internal arrows} \\ \updownarrow \\ \text{relations}}}{(Q_1 \setminus F_1)^\vee} \otimes A \to A \otimes \underset{\substack{\uparrow \\ \text{arrows}}}{Q_1} \otimes A \to A \otimes \underset{\substack{\uparrow \\ \text{vertices}}}{Q_0} \otimes A \to A \to 0$$

Exactness of this complex $\Rightarrow (A,e)$ is int. 3CY (Ginzburg / Broomhead if $F=\emptyset$, P '17 in general)

Use consistency to prove exactness.

Key property: 'thinness on the universal cover'
- in $\tilde{A}$ ($=A$ for $\Sigma = D^2$) for any $i,j \in \tilde{Q}_0$ $\exists$ $p_{min}: i \to j$ in $\tilde{Q}$ such that $e_j \tilde{A} e_i \cong \prod_{n \geq 0} \mathbb{C} p_{min} t^n$, t bounding a face. (i.e. $e_j \tilde{A} e_i$ free rank 1 over $Z = \mathbb{C}[[t]]$).

Applications 1) First take $\Sigma = T^2$, $A = A_{\mathcal{D}}$. (see Broomhead)
Choose any vertex $0 \in Q_0$, $e = e_0$.

Then A 3CY (Broomhead) $\Rightarrow (A,e)$ int. 3CY.

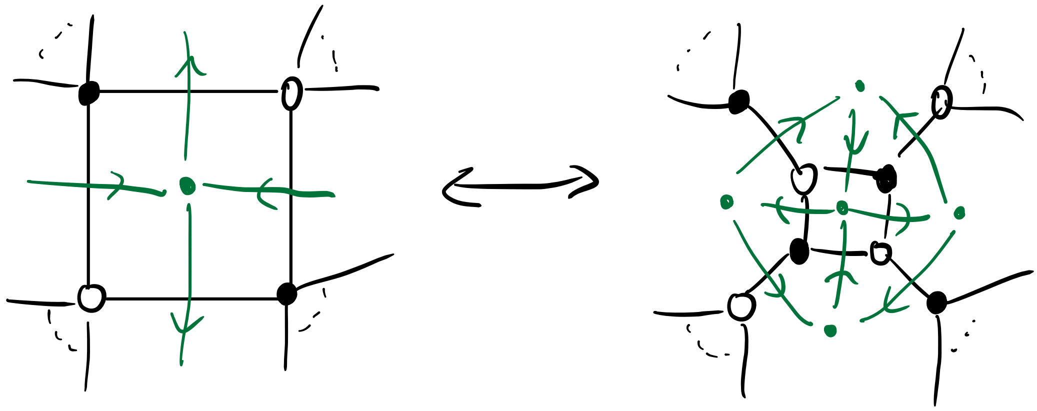
$B = eAe = Z(A) = R$ is a Gorenstein toric 3-fold singularity.

Thm (Gulotta '08) every such R appears this way.

$GP(B) = CM(R)$ admits the 2-cluster-tilting object eA,
$A \cong \mathrm{End}_R(eA)^{op}$ is an NCCR of R.

$GP(B) = D_{sg}(R)$ 2-CY, $D^b(A) \simeq D^b(\mathrm{coh}\, X)$, $X \to \mathrm{Spec}\, R$ crepant resolution.
(Buchweitz) (Bridgeland-King-Reid, Van den Bergh).

Mutation? Some mutations correspond to Seiberg duality (see Vitória '09):



For $D \Leftrightarrow D'$ related by this move, consider $A = A_D$, $A' = A_{D'}$.
Pick $e = e_0$ for 0 different from mutated tile.

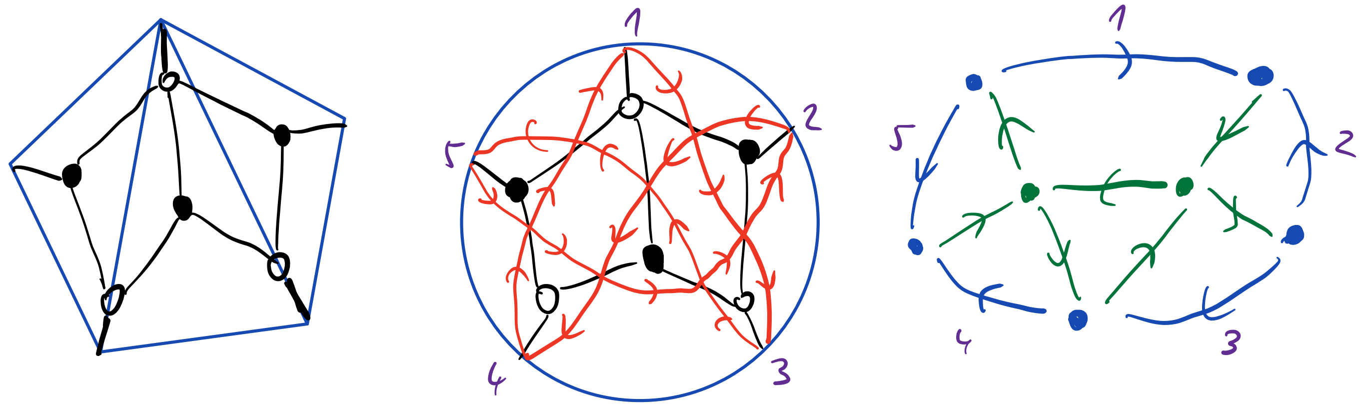
Then $eAe = B \cong B' = eA'e$, $eA, eA' \in GP(B)$ $(= CMR)$ related by mutation.

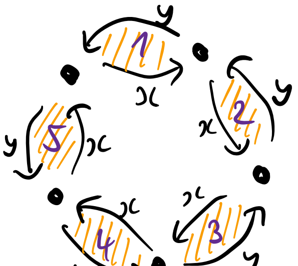
Get two crepant resolutions
$$\begin{tikzcd} X \arrow[rr, leftrightarrow, "\text{flop}"] \arrow[dr] & & X' \arrow[dl] \\ & \operatorname{Spec} R & \end{tikzcd}$$
$D^b(\operatorname{coh} X) \simeq D^b(A) \simeq D^b(A') \simeq D^b(\operatorname{coh} X')$.

2) Now take $\Sigma = D^2$. Then $GP(B)$ is a Frobenius categorification of a cluster algebra structure on a positroid variety in the Grassmannian $\mathrm{Gr}_k^{\wedge}$ (see next time).

Key example: Scott '06, cluster algebra structure on $\mathrm{Gr}_k^{\wedge}$ via Postnikov diagrams = zig-zag paths of consistent dimers in D^2.

$k = 2$:

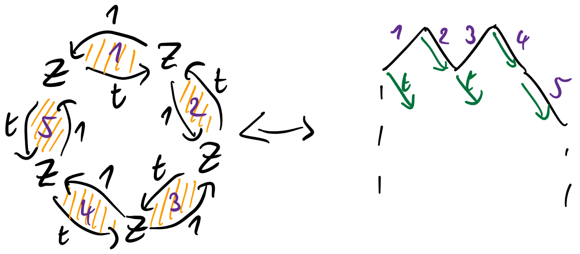


$B =$ 

$\Big/ \; xy = yx, \; y^2 = x^3$

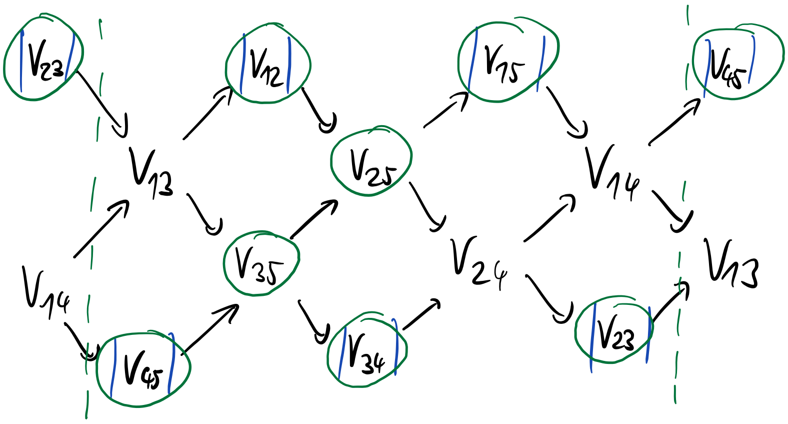
In this case indecomposables in $GP(B)$ are precisely V_μ for μ a matching.
$\Leftrightarrow \{1, \dots, n\} = X \sqcup Y$, $\#X = 2$. Write $V_X := V_\mu$.

Eg $V_{13} =$



$GP(B)$:

$eA = B \oplus V_{25} \oplus V_{35}$.



See Jensen-King-Su, Baur-King-Marsh, Demonet-Luo.

§ Categorification of positroids   w/ İ. Çanakçı, A. King

$Gr_k^n = \{V \subseteq \mathbb{C}^n : \dim V = k\}$ projective variety   'Grassmannian'.

System of projective coordinates $\Delta_I : \widehat{Gr}_k^n \to \mathbb{C}$ ← affine cone   'Plücker coordinates'
indexed by $I \in \binom{n}{k} = \{I \subseteq \{1, \dots, n\} : \# I = k\}$.

$V = \mathrm{rowspan}(M)$, $M = (m_1, \dots, m_n) \in \mathbb{C}^{k \times n} \Rightarrow \Delta_{\{i_1, \dots, i_k\}}(V) = \det(m_{i_1}, \dots, m_{i_k})$,

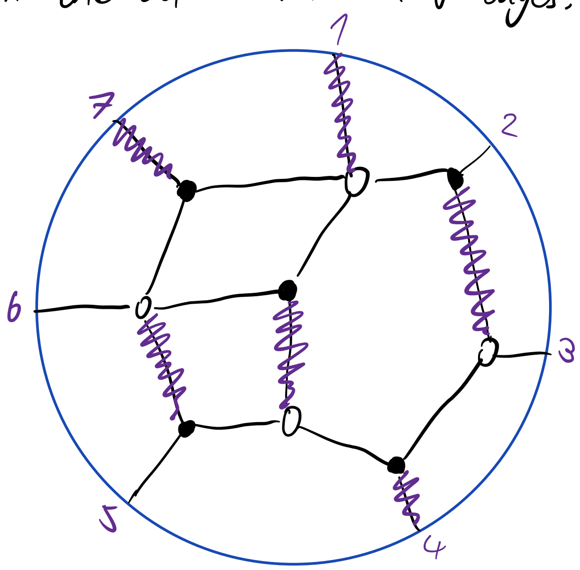
Now fix D, a consistent dimer model in the disc with n half-edges.

A perfect matching μ of D has a boundary value

$$\partial\mu = \{i : i \text{ ⌇⌇⌇}\circ \in \mu\} \cup \{i : i \text{ —}\bullet \notin \mu\}$$

$$\# \partial\mu = k = \# \circ - \# \bullet + \#(\text{—}\bullet)$$

so $\partial\mu \in \binom{n}{k}$.

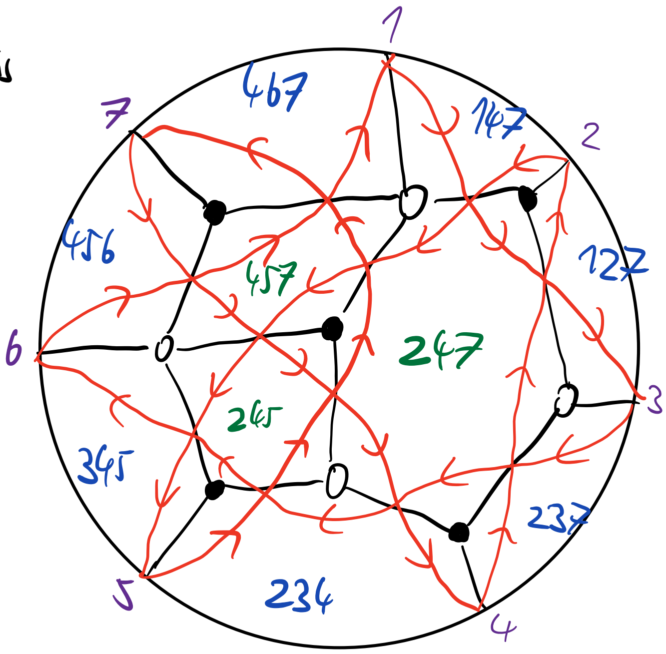


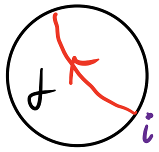
$\partial\mu = 125$

Let $\mathcal{P} = \{\partial\mu : \mu \text{ p.m. of } D\} \subseteq \binom{n}{k}$ 'positroid'

$\Pi = \{V \in \widehat{Gr}_k^n : \Delta_I(V) = 0 \text{ for } I \notin \mathcal{P}\}$ 'closed positroid variety'

<u>Face labels</u> Recall strands = zig-zag paths



f face $\rightsquigarrow I_f = \{ i :$  $\}$

$\# I_f = k$

Let $\mathcal{I} = \{I_f : f \in \partial Q_0\}$ 'necklace'.

Then 1) $\mathcal{I} \subseteq \mathcal{P}$ (not obvious – see later)

2) $\mathcal{I}, \mathcal{P}$ are equivalent data, as is $\pi_D \in S_n$: $i \rightsquigarrow \pi_D(i)$.

Changing D by Seiberg duality does not change this data.

<u>Thm</u> (Oh–Postnikov–Speyer) $\pi_D = \pi_{D'} \Leftrightarrow D$ and D' are linked by a sequence of Seiberg duality moves.

$\Pi^\circ = \{V \in \Pi : \Delta_I(V) \neq 0 \ \forall I \in \mathcal{I}\}$. 'open positroid variety'.

<u>Thm</u> (Galashin–Lam) Let $\mathcal{A}$ be the cluster algebra with invertible frozen variables associated to Q_D, with initial variables $x_f, f \in Q_0$.

Then the map $\mathcal{A} \to \mathbb{C}[\hat{\Pi}^\circ]$ is an isomorphism.

$$x_f \mapsto \Delta_{I_f}$$

<u>Note</u> $\mathcal{A}$ is the cluster algebra categorified by $GP(B)$, $B = eAe$, $A = A_D$ dimer algebra.

Let $C = \mathbb{C}C_n / \begin{pmatrix} xy - yx \\ y^k - x^{n-k} \end{pmatrix}$ circle algebra (Jensen–King–Su).

Let $\Lambda = A, B$ or C.

Prp Λ is a $Z = \mathbb{C}[\![t]\!]$-algebra, and is thin: for primitive idempotents $e_i, e_j \in \Lambda$, $_Z(e_j \Lambda e_i)$ is free of rank 1.
(GKP for A, B, Jensen-King-Su for C).

In particular, Λ is free and finitely generated over Z.
Write $CM(\Lambda) = \{ M \in \operatorname{mod} \Lambda : {}_Z M \text{ is free and f.g.} \}$.

Note 1) $CM(C) = GP(C)$. $CM(B) \supseteq GP(B)$, often strict.
2) A, C (but not B) given by quivers with faces.
In this case, thinness $\Leftrightarrow$ every indecomposable projective $P_f = \Lambda e_f$ is a rank 1 Q-matrix factorisation.

$B = eAe \Rightarrow$ restriction functor $e : CM(A) \to CM(B)$.

Prp (GKP) $\exists$ a canonical algebra map $C \to B$ such that the restriction functor $\rho : CM(B) \to CM(C)$ is fully faithful.

Claim Face labels are explained algebraically by the fact that

$$\rho(eP_f) \cong V_{I_f} \in CM(C)$$

Prp (GKP) Let Q be a quiver with faces such that $H^1(|Q|) = 0$. Then every rank 1 Q-matrix factorisation V is isomorphic to a perfect matching module.

Proof $t \in \mathbb{C}[\![t]\!]$ is prime, so $\mu = \{ \alpha \in Q_1 : V_\alpha \text{ not invertible} \}$ is a perfect matching. We claim $V \cong V_\mu$.

For each $\alpha \in Q_1$, $\exists\, \lambda_\alpha \in Z^\times$ s.t. $V_\alpha = \lambda_\alpha t^{\varepsilon}$, and if p bounds a face then $\prod_{\alpha \in p} \lambda_\alpha = 1$.

Thus $(\lambda_\alpha)_{\alpha \in Q_1}$ is a cochain in $Z^1(|Q|; Z^\times)$. Since $H^1(|Q|; Z^\times) = 0$, $(\lambda_\alpha)_{\alpha \in Q_1}$ is the boundary of $(\eta_i)_{i \in Q_0} \in (Z^\times)^{Q_0}$

$$\lambda_\alpha = \frac{\eta_{h\alpha}}{\eta_{t\alpha}} \text{ for all } \alpha \in Q_1.$$

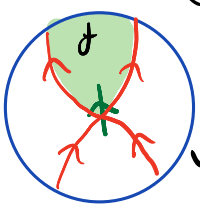
Then
$$\begin{tikzcd} Z \arrow[r, "\lambda_\alpha t^\varepsilon"] & Z \\ Z \arrow[u, "\partial_{t\alpha}\, 2"] \arrow[r, "t^\varepsilon"] \arrow[ur, phantom, "\circlearrowleft"] & Z \arrow[u, "2\, \partial_{h\alpha}"'] \end{tikzcd}$$
gives an isomorphism $V_\mu \xrightarrow{\sim} V$.

<u>Cor</u> $\forall j \in Q_0$, $P_j \cong V_\mu$ for some perfect matching μ.

<u>Obs</u> $p(eV_\mu) \cong V_{\partial\mu}$. ($\partial\mu \rightsquigarrow$ matching $\{x_i : i \in \partial\mu\} \cup \{y_i : i \notin \partial\mu\}$ of C_n)

So just need to identify μs from projectives, calculate boundary value.

<u>Thm</u> (GKP) $P_j \cong V_\mu$ for μ the 'downstream wedge' matching (Muller–Speyer '17).

$\mu = \{\alpha :$  $\}$. $\overset{\Longrightarrow}{\scriptstyle MS}$ $\partial\mu = I_j$.

So face labels are categorified by projective A-modules.

<u>Cor</u> $\mathcal{I} \subseteq \mathcal{P}$: all face labels are boundary values.

Proof of theorem is via calculation of projective resolution of all V_μ.

<u>The twist</u> For fixed D, get map $\underline{\mu} : (\mathbb{C}^\times)^{Q_0} \to \Pi^0$ by
$$\Delta_I(\underline{\mu}(x)) = \sum_{\mu : \partial\mu = I} \mu(x), \quad \mu(x) = \prod_{\alpha \in \mu} \frac{x_{h\alpha}}{x_{t\alpha}}. \quad \text{(Postnikov)}$$

Get map $c : \Pi^0 \dashrightarrow (\mathbb{C}^\times)^{Q_0}$ by $V \mapsto (\Delta_{I_j}(V))_{j \in Q_0}$.

The composition $(\mathbb{C}^\times)^{Q_0} \xrightarrow{\underline{\mu}} \Pi^0 \overset{c}{\dashrightarrow} (\mathbb{C}^\times)^{Q_0}$ is not an iso, but

(Muller–Speyer '17) $\exists$ $tw : \Pi^0 \to \Pi^0$ such that $(\mathbb{C}^\times)^{Q_0} \xrightarrow{\underline{\mu}} \Pi^0 \xrightarrow{tw} \Pi^0 \overset{c}{\dashrightarrow} (\mathbb{C}^\times)^{Q_0}$ is.

Consider the function $CC: CM(B) \to \mathbb{C}[(\mathbb{C}^\times)^{Q_0}]$ given by:

$$CC(V) = x^{[FV]} \sum_{N \leq GV} x^{-[N]}$$

where • $FV = \mathrm{Hom}_B(eA, V)$, $GV = \mathrm{Ext}^1_B(eA, V) \in \mathrm{mod}\, A$. $(A \cong \mathrm{End}_B(eA)^{op})$

• $[X]$ = class of X in $K_0(\mathrm{mod}\, A) \cong \bigoplus_{j \in Q_0} \mathbb{Z}[P_j]$.

• count infinite families of submodules by $\chi(\mathrm{Gr}_{\underline{\dim} N}(GV))$.

Rem CC restricts to a cluster character on $GP(B)$ (JKS '16, P '19$^+$).

$$\overline{CC}(V) = CC(V)\Big|_{x_j = \Delta_{I_j}} \in \mathbb{C}[\Pi^\circ] \quad \text{for } x_j = x^{[P_j]}.$$

Remark $CC(eP_j) = x_j$, so $\overline{CC}(eP_j) = \Delta_{I_j}$; cf. $\rho(eP_j) \cong V_{I_j}$.

Thm Let $V \in CM(B)$ such that $\rho(V) \cong V_I$, $I \in \binom{n}{k}$.
(equiv. $V = eV_\mu$ for μ p.m. of Q_D).

Then $\Delta_I \circ tw = \dfrac{\overline{CC}(\Omega V)}{\overline{CC}(PV)}$ where $0 \to \Omega V \to PV \to V \to 0$, and PV is projective.

Slogan: twist $= \Omega$ ('syzygy').

Key step: $G\Omega V = \underline{\mathrm{Hom}}_B(eA, V) = FV/F'V$ for $F'V = \{eA \to V$ factoring over proj $B\}$

so $\{N \leq G\Omega V\} \longleftrightarrow \{F'V \leq M \leq FV\}$

Prop $\{F'V \leq M \leq FV\} \longleftrightarrow \{\mu : \partial_\mu = I\}$.

Sketch proof of Thm 1) Thinness $\Rightarrow$ all Ms are r.k. 1 Q-rps, hence p.m. modules.
2) $eF'V = eFV = V$, so $eM = V$
$\Rightarrow \rho(eM) \cong V_I \Rightarrow$ bdy value must be I.

3) For $N \in \mathcal{G}\Omega V$: $0 \to F'V \to M \to N \to 0 \Rightarrow [N] = [M] - [F'V]$

$$0 \to \Omega V \to PV \to V \to 0 \rightsquigarrow 0 \to F\Omega V \to FPV \to F'V \to 0$$
$$\Rightarrow [F'V] = [FPV] - [F\Omega V]$$

So $[F\Omega V] - [N] = [FPV] - [M]$

Hence $$CC(\Omega V) = \sum_{F'V \leq M \leq FV} x^{[FPV]-[M]}$$
$$= CC(PV) \sum_{F'V \in M \in FV} x^{-[M]}$$

4) Using proj. res. of p.m. module $M = V/\mu$, show that
$$\mu(x) = x^{-[M]}.$$